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K_0 of Exchange Rings with Stable Range 1*

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Abstract. A ring R is called weakly generalized abelian (for short, WGA-ring) if for each idempotent e in R, there exist idempotents f, g, h in R such that $eR \cong fR \oplus gR$ and $(1-e)R \cong fR \oplus hR$, while gR and hR have no isomorphic nonzero summands. By an example we will show that the class of generalized abelian rings (for short, GA-rings) introduced in [10] is a proper subclass of the class of WGA-rings. We will prove that, for an exchange ring R with stable range 1, $K_0(R)$ is an ℓ-group if and only if R is a WGA-ring.

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1. Introduction

First of all, let us recall a longstanding open problem about regular rings ([9], p.200 or [6], Open Problem 27, p.347):

If R is a unit-regular ring, is $K_0(R)$ torsion-free and unperforated?

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For general unit-regular rings, Goodearl gave a negative answer by constructing a concrete unit-regular ring R whose $K_0(R)$ has nontrivial torsion part ([8, Theorem 5.1]). Then the fundamental problem was to state which classes of regular rings has torsion-free K_0-groups. Indeed, we now have known that there exist some special classes of regular rings have torsion-free K_0-groups, including regular rings satisfying general comparability ([6, Theorem 8.16]), N^*-complete regular rings ([7, Theorem 2.6]), and right $\aleph_0$-continuous regular rings ([2, Theorem 2.13]). The latest result is that the K_0-group of every semiartinian unit-regular ring is torsion-free ([3, Theorem 1]).

Recently, the first author and Qin [10] extended this study to a more general setting, that of exchange rings. Our main technical tool for studying the torsion freeness of $K_0(R)$ is motivated by the following result from ordered algebra ([4, Theorem 3.7]): For abelian groups, being torsion-free is equivalent to being lattice-orderable. So we introduce the class of GA-rings. We say that a ring R is a *GA-ring* if for each idempotent e in R, eR and $(1-e)R$ have no isomorphic nonzero summands. We denote by **GAERS-1** the class of generalized abelian exchange rings with stable range 1. We proved in (Lu and Qin, Theorem 5.3) that, for any ring $R \in$ **GAERS-1**, $K_0(R)$ is always an archimedean ℓ-group.

In this note, we will consider the following more general problem:

Under what condition, $K_0(R)$ of an exchange ring with stable range 1 is torsion-free?

In order to establish a more complete result, we introduce the class of WGA-rings. By an example we will show that the class of GA-rings is a proper subclass of the class of WGA-rings. In particular, we will prove that, for an exchange ring R with stable range 1, $K_0(R)$ is an ℓ-group if and only if R is a WGA-ring.

2. Preliminaries

In this section, we simply review some basic definitions and some well known results about rings and modules, K_0-groups, and ℓ-groups. The reader is referred to [1] for the general theory of rings and modules, to [11] for the basic properties of K_0-groups, and to [4] for the general theory of ℓ-groups.

Rings and modules: Throughout, all rings are associative with identity and all modules are unitary right R-modules. For a ring R, we denote by $FP(R)$ the class of all finitely generated projective R-modules. A ring R is said to be *directly finite* if for $x, y \in R$, $xy = 1$ implies $yx = 1$. A ring R is said to be *stably finite* if all matrix rings $M_n(R)$ over R are directly finite for any positive integers n; this is equivalent to the condition that, for $K \in FP(R)$, $K \oplus R^m \cong R^m$ implies $K = 0$. A ring R is said to have *stable range 1* if for any $a, b \in R$ satisfying $aR + bR = R$, there exists $y \in R$ such that $a + by \in U(R)$ (the group of all units of R). Clearly if a ring R has stable range 1, then R is stably finite. Following [12], we say that a ring R is an *exchange ring* if for every R-module A_R and any decompositions $A = B \oplus C = (\bigoplus_{i\in I} A_i)$ with $B \cong R_R$ as right R-modules, there exist submodules $A_i^{'} \subseteq A_i$ for each $i \in I$ such that $A = B \oplus (\bigoplus_{i\in I} A_i^{'})$. The class of exchange rings is quite large. It includes all semiregular rings, all clean rings, all π-regular rings and all C^*-algebras with real rank zero.

K_0**-groups**: Let R be a ring. Two modules $A, B \in FP(R)$ are *stably isomorphic* if $A \oplus nR_R \cong B \oplus nR_R$ for some positive integer n. We denote by $[A]$ the stable isomorphism class of A, and by $K_0(R)^+$ the set of all stable isomorphism classes on $FP(R)$. The set $K_0(R)^+$, endowed with the operation $[A] + [B] = [A \oplus B]$, is a monoid with zero element [0] (for short, 0). By formally adjoining additive inverses for the elements of $K_0(R)^+$, we embed $K_0(R)^+$ in an abelian group, the K_0*-group* of R, denoted $K_0(R)$. In particular, every element of $K_0(R)$ has the form $[A] - [B]$ for suitable $A, B \in FP(R)$. According to ([6], Chapter 15), there is a natural way to make $K_0(R)$ into a pre-order abelian group with order-unit, as follows: $K_0(R)^+$ is a cone, i.e., an additively closed subset of $K_0(R)$ such that $0 \in K_0(R)^+$. Then, it can determines a pre-order on $K_0(R)$ by the following rule: For any $x, y \in K_0(R)$, $x \leq y$ if and only if $y - x \in K_0(R)^+$. We refer to the pre-order on $K_0(R)$ determined by this cone as the *natural pre-order* on $K_0(R)$.

ℓ**-groups**: Let L be a partially ordered set. If for any $x, y \in L$, the set of upper bounds of x and y has a least element z, z is called the *least upper bound* of x and y and is written $z = x \vee y$. The *greatest lower bound* w of x and y is defined similarly and is written $w = x \wedge y$. If every pair of elements has a least upper bound, L is called an *upper semilattice*, and if every pair of elements has a greatest lower bound, L is called a *lower semilattice*. If L is both an upper semilattice and a lower semilattice, then L is called a *lattice*.

A *partially ordered abelian group* G is an abelian group that is also a partially ordered set such that for any $a, b, c \in G$, $c+a+d \leq c+b+d$ whenever $a \leq b$. We will denote by G^+ the set $\{a \in G : a \geq 0\}$, and is usually called the *positive cone* of G. Two elements $a, b \in G$ are said to be *orthogonal* if $a \wedge b$ exists in G and $a \wedge b = 0$. A partially ordered abelian group G is an ℓ*-group* if the underlying order endows G with structure of lattice. In view of ([4], Proposition 3.5), every ℓ-group is torsion-free. The following standard of ℓ-groups is necessary for our present paper: A partially ordered abelian group G is an ℓ-group if and only if for all $g \in G$, there exist $a, b \in G$ such that $a \wedge b = 0$ and $g = a - b$ ([4], Proposition 4.3).

3. Main Result and Its Proof

In order to prove the main result of this paper, we need several lemmas. Let us first state the main definition of this paper.

Definition 1. *A ring R is called a WGA-ring if for any idempotent e in R, there exist idempotents f, g, h in R such that $eR \cong fR \oplus gR$ and $(1-e)R \cong fR \oplus hR$, while gR and hR have no isomorphic nonzero summands.*

From Definition 1, we easily see that every GA-ring is a WGA-ring. But the converse does not hold in general. It follows that the class of GA-rings is a proper subclass of the class of WGA-rings. Consider the following examples.

Example 2.
(1) A ring R is *connected* if it has no nontrivial idempotents. Clearly every connected ring is a WGA-ring. In particular, every local ring is a WGA-ring.
(2) For a ring R, we denote by $Lat\ (R_R)$ the lattice of all right ideals of R. The ring R is *distributive* if the lattice $Lat(R_R)$ is a distributive lattice, i.e., for any $I, J, K \in Lat(R_R)$, $I \cap (J + K) = (I \cap J) + (I \cap K)$; this is equivalent to the condition that $I + (J \cap K) = (I + J) \cap (I + K)$. A direct computation shows that, for a distributive ring R, all idempotents in R commute each other. Further we have that every distributive ring is abelian, so is a WGA-ring.
(3) Let $\mathbb{Z}$ be the ring of integers, and let

$$R = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ \mathbb{Z}_2 & \mathbb{Z}_2 \end{pmatrix}, \text{ where } \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}.$$

Clearly R is a unit-regular ring. Observe that all nontrivial idempotents in R are as follows:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

By a direct computation, R is indeed an WGA-ring. In view of ([10, Remark 3.2]), for regular rings, being abelian is equivalent to be generalized abelian. So R is clearly not a GA-ring. It follows that the class of GA-rings is indeed a proper subclass of the class of WGA-rings.

For a ring R, we denote by $Idem(R)$ the set of all idempotents in R. Recall that $e, f \in Idem(R)$ are called *orthogonal* if $ef = fe = 0$. We now define a relation on $Idem(R)$, as follows: For $e, f \in Idem(R)$, $f \leq e$ if and only if there exists $g \in Idem(R)$ such that $e = f + g$, and f and g are orthogonal. A short computation shows that the relation $\leq$ is actually a partial order on $Idem(R)$, and $f \leq e$ if and only if $f = ef = fe$.

Lemma 3. *The following conditions are equivalent for a ring R:*
(1) *R is a WGA-ring.*
(2) *For any two orthogonal idempotents e_1 and e_2 in R, there exist idempotents f, g, h in R such that $e_1R \cong fR \oplus gR$ and $e_2R \cong fR \oplus hR$, while gR and hR have no isomorphic nonzero summands.*

Proof.
(2)⇒(1) is trivial.
(1)⇒(2) Let e_1, e_2 be two orthogonal idempotents in R, and suppose e_1 and e_2 do not satisfy (2); then for any idempotents f, g, h in R satisfying $e_1R \cong fR \oplus gR$ and $e_2R \cong fR \oplus hR$, gR and hR have isomorphic nonzero summands. Since e_1 and e_2 are orthogonal, $e_2 \leq 1 - e_1$, so there exists some idempotent e_0 in R such that $1 - e_1 = e_2 + e_0$, and e_2 and e_0 are orthogonal. Then we have

$$(1 - e_1)R = (e_2 + e_0)R = e_2R + e_0R = e_2R \oplus e_0R.$$

So $(1 - e_1)R = fR \oplus hR \oplus e_0R$. It follows that gR and $hR \oplus e_0R$ also have isomorphic nonzero summands for any idempotents f, g, h, e_0 in R, so e_1R and $(1 - e_1)R$ can not satisfy (2), which contradicts the assumption. ■

Lemma 4. *The following conditions are equivalent for a ring R with stable range 1:*

(1) *R is a WGA-ring.*

(2) *For any $e \in Idem(R)$, there exist idempotents f, g, h in R such that $[eR] = [fR] + [gR]$ and $[(1-e)R] = [fR] + [hR]$, while $[gR] \wedge [hR] = 0$ in $K_0(R)^+$.*

(3) *For any two orthogonal idempotents e_1 and e_2 in R, there exist idempotents f, g, h in R such that $[e_1R] = [fR] + [gR]$ and $[e_2R] = [fR] + [hR]$, while $[gR] \wedge [hR] = 0$ in $K_0(R)^+$.*

(4) *For any two orthogonal idempotents e_1 and e_2 in R and any positive integers m and n, there exist idempotents f, g, h in R such that $[e_1R] = [fR] + [gR]$ and $[e_2R] = [fR] + [hR]$, while $m[gR] \wedge n[hR] = 0$ in $K_0(R)^+$.*

Proof.
(1)⇒(2) Clearly R is stably finite, so, in view of ([6, Proposition 15.3]), the natural pre-order on $K_0(R)$ is a partial order. In particular, for any $A \in PF(R)$, we have

$$A \neq 0 \text{ if and only if } [A] > 0 \text{ in } K_0(R)^+.$$

Now, given any two orthogonal idempotents e_1, e_2 in R, by assumption, there exist idempotents f, g, h in R such that $e_1R \cong fR \oplus gR$ and $e_2R \cong fR \oplus hR$, while gR and hR have no isomorphic nonzero summands. So $[e_1R] = [fR] + [gR]$ and $[e_2R] = [fR] + [hR]$. Clearly 0 is a lower bound of $[gR]$ and $[hR]$. Suppose $0 < [A] \leq [gR] \wedge [hR]$. Then, by Evans' Cancellation Theorem ([5], Theorem 2), A must be a common nonzero summand of gR and hR, which contradicts (1). It follows that 0 is the greatest lower bound of $[gR]$ and $[hR]$ in $K_0(R)^+$. So $[gR] \wedge [hR] = 0$.
(2)⇒(3) is clear by Lemma 3.
(3)⇒(4) Given any two positive integers m and n, we set $k = \max\{m, n\}$ and $s = 2k$. Notice that $[gR] \wedge [hR]$ exists in $K_0(R)^+$, so we have

$$s([gR]\wedge[hR]) = 2k[gR]\wedge\big\{(2k-1)[gR]+[hR]\big\}\wedge\cdots\wedge\{[gR]+(2k-1)[hR]\}\wedge 2k[hR].$$

Then we further have

$$0 \leq m[gR] \wedge n[hR] \leq k[gR] \wedge k[hR] \leq s([gR] \wedge [hR]) = 0.$$

It follows that $m[eR] \wedge n[fR]$ exists in $K_0(R)^+$, and $m[eR] \wedge n[fR] = 0$.
(4)⇒(1) is clear by way of contradiction. ■

In order to prove the main result, we also need the following two lemmas.

Lemma 5. *Let R be a ring, and let e be an idempotent in R. If $eR \cong A \oplus B$ for some $A, B \in FP(R)$, then there exist idempotents α, β in R such that α and β are orthogonal, and $\alpha R \cong A$ and $\beta R \cong B$.*

Proof. Let α and β be the projections on A and B respectively. Notice that $End_R(eR) \cong eRe \subseteq R$, and that $e : eR \to eR$ is clearly an R-homomorphism of eR to itself, so $e = \alpha + \beta$. Clearly α and β are orthogonal, and we have

$$A \cong \alpha(eR) = \alpha(\alpha + \beta)R = (\alpha^2 + \alpha\beta)R = \alpha R.$$

Similarly, we also have $\beta R \cong B$, as desired. ■

Lemma 6. *Let R be a ring. If R is a WGA-ring then so is $\bigoplus_{i=1}^{n} R$ for any positive integers n.*

Proof. By a simple induction on n, it suffices to show that $R \oplus R$ is also a WGA-ring.

Suppose that $(e_1, e_1^{'})$ and $(e_2, e_2^{'})$ are two orthogonal idempotents in $R \oplus R$. Then e_1 and e_2, $e_1^{'}$ and $e_2^{'}$ are respectively orthogonal idempotents in R. For e_1, e_2, since R is a WGA- ring, there exist idempotents f, g, h in R such that

$$e_1R \cong fR \oplus gR, \quad \text{and} \quad e_2R \cong fR \oplus hR,$$

while gR and hR have no isomorphic nonzero summands. Similarly, for $e_1^{'}, e_2^{'}$, there also exist idempotents $f^{'}, g^{'}, h^{'}$ in R such that

$$e_1^{'}R \cong f^{'}R \oplus g^{'}R \quad \text{and} \quad e_2^{'}R \cong f^{'}R \oplus h^{'}R,$$

while $g^{'}R$ and $h^{'}R$ have no isomorphic nonzero summands. So we have

$$(e_1, e_1^{'})(R \oplus R) \cong (f, f^{'})(R \oplus R) \oplus (g, g^{'})(R \oplus R)$$

and

$$(e_2, e_2^{'})(R \oplus R) \cong (f, f^{'})(R \oplus R) \oplus (h, h^{'})(R \oplus R).$$

Notice that gR and hR, and $g^{'}R$ and $h^{'}R$ have no isomorphic nonzero summands, respectively. So $(g, g^{'})(R \oplus R)$ and $(h, h^{'})(R \oplus R)$ have no isomorphic nonzero summands. It follows that $R \oplus R$ is also a WGA-ring. ■

We are now in a position to prove the main result of this paper.

Theorem 7. *Let R be an exchange ring with stable range 1. The following conditions are equivalent:*
(1) *R is a WGA-ring.*
(2) *$K_0(R)$ is an ℓ-group with respect to the natural pre-order on $K_0(R)$.*

Proof.
(1)⇒(2) Clearly since R is stably finite, the natural pre-order on $K_0(R)$ is actually a partial order. First, if R contains no nontrivial idempotents, then the conclusion is clear. Now, given any $x \in K_0(R)$, in view of ([13, Corollary 2.2]), there exists a complete set of pairwise orthogonal idempotents $e_1, e_2, \cdots, e_k$ in R and a set of nonnegative integers $n_1, n_2, \cdots, n_k$ such that

$$x = n_1[e_1R] + \cdots + n_s[e_sR] - n_{s+1}[e_{s+1}R] - \cdots - n_k[e_kR].$$

Then we have

$$x = [n_1(e_1R) \oplus \cdots \oplus n_s(e_sR)] - [n_{s+1}(e_{s+1}R) \oplus \cdots \oplus n_k(e_kR)].$$

Now, set

$$A = n_1(e_1R) \oplus \cdots \oplus n_s(e_sR), \quad \text{and} \quad B = n_{s+1}(e_{s+1}R) \oplus \cdots \oplus n_k(e_kR).$$

By Lemma 6, we see that the following ring

$$S := \bigoplus_{i=1}^{k} n_iR.$$

is also a WGA-ring. Further we set

$$\widetilde{e_1} = \big(\underbrace{e_1, \cdots, e_1}_{n_1}, \cdots, \underbrace{e_s, \cdots, e_s}_{n_s}, 0, \cdots, 0\big) \text{ correspond to } A$$

and

$$\widetilde{e_2} = \big(0, \cdots, 0, \underbrace{e_{s+1}, \cdots, e_{s+1}}_{n_{s+1}}, \cdots, \underbrace{e_k, \cdots, e_k}_{n_k}\big) \text{ correspond to } B.$$

Then $\widetilde{e_1}$ and $\widetilde{e_2}$ are two orthogonal idempotents in S. So there exist idempotents $\widetilde{f}, \widetilde{g}, \widetilde{h}$ in S such that

$$A = \widetilde{e_1}S \cong \widetilde{f}S \oplus \widetilde{g}S \text{ and } B = \widetilde{e_2}S \cong \widetilde{f}S \oplus \widetilde{h}S,$$

while $\widetilde{g}S$ and $\widetilde{h}S$ have no isomorphic nonzero summands. Notice that every S-module is clearly an R-module. So $\widetilde{f}S, \widetilde{g}S, \widetilde{h}S \in FP(R)$. Notice that S is an exchange ring with stable range 1. So $[\widetilde{g}S] \wedge [\widetilde{h}S] = 0$ in $K_0(S)^+$. Then we have

$$x = [A] - [B] = [\widetilde{g}S] - [\widetilde{h}S], \text{ while } [\widetilde{g}S] \wedge [\widetilde{h}S] = 0 \text{ in } K_0(R)^+.$$

So, in view of ([4, Proposition 4.3]), $K_0(R)$ is an ℓ-group with respect to the natural pre-order on $K_0(R)$.

(2)⇒(1) Given any idempotent e in R, let $x = [eR] - [(1-e)R]$. Then $x \in K_0(R)$. Since $K_0(R)$ is an ℓ-group, we write

$$[A] = [eR] \wedge [(1-e)R] \text{ for some } A \in FP(R)$$

and

$$[B] = [eR] - [A], \quad \text{and} \quad [C] = [(1-e)R] - [A].$$

Then we have

$$[B] \wedge [C] = ([eR] - [A]) \wedge ([(1-e)R] - [A]) = ([eR] \wedge [(1-e)R]) - [A] = 0.$$

By Evans' Cancellation Theorem ([5, Theorem 2]), we further have

$$eR \cong A \oplus B, \text{ and } (1-e)R \cong A \oplus C.$$

By Lemma 5, there exist idempotents f_1, f_2, g, h in R such that

$$f_1R \cong A, gR \cong B, f_2R \cong A \quad \text{and} \quad hR \cong C$$

Thus

$[eR]$=$[f_1R]$+$[gR]$, and $[(1-e)R] = [f_2R]+[hR]$, while $[gR]\wedge[hR] = [B]\wedge[C] = 0$. So by Lemma 4, R is a WGA-ring. ■

According to the knowledge of ordered algebra, for an abelian group, being torsion-free is equivalent to being lattice-orderable. So Theorem 7 establishes a complete description for the torsion freeness of the K_0-groups of exchange rings with stable range 1.

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