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On Convergence of Vector-Valued Weak Amarts and Pramarts*

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Abstract. A sequence (X_n) of random elements in Banach space $\mathbb{E}$ is called essentially (weakly) tight if and only if for every $\varepsilon > 0$ there exists a (weakly) compact subset K of $\mathbb{E}$ such that $\mathbb{P}(\bigcap_{n\in\mathbb{N}} [X_n \in K]) > 1 - \varepsilon$. The main aim of this note is to give some (weakly) almost sure convergence results for $\mathbb{E}$ - valued weak amarts and pramarts in terms of their essential (weak) tightness.

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0. Introduction

The usual notion of uniform tightness is frequently used in probability theory (cf. [1]). By the Prokhorov's theorem, every sequence of random elements in Polish spaces which converges in distribution is uniformly tight. The notion of essential tightness used in the note is rather stronger than the usual uniform one. More precisely, in [7] Krupa and Zieba proved that an L^1-bounded strong amart in Banach spaces converges almost surely (a.s.) if and only if it is essentially tight. Here we shall apply the approach and another due to Davis et al [4] and Bouzar [2] to extend the main convergence results of these authors for amarts to weak amarts and pramarts of Pettis integrable functions in Banach spaces without the

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Radon-Nikodym property. Namely, after recalling some fundamental notations and definitions in the next section, we shall present in Sec. 2 the main results concerning (weak) a.s. convergence of weak amarts and pramarts. Finally, we shall give in Sec. 3 some related comparison examples.

1. Notations and Definitions

Throughout the note, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $(\mathcal{F}_n)$ a nondecreasing sequence of complete subσ-field of $\mathcal{F}$ with $\mathcal{F}_n \uparrow \mathcal{F}$. By $\mathbb{T}$ we denote the directed set of all bounded stopping times for $(\mathcal{F}_n)$. Then it is known (cf. [11]) that $(\mathcal{F}_n)$ and $\mathcal{F}$ induce the correspondent directed net $(\mathcal{F}_\tau, \tau \in \mathbb{T})$ of complete subσ-fields of $\mathcal{F}$, where each $\mathcal{F}_\tau = \{A \in \mathcal{F} : A \cap \{\tau = n\} \in \mathcal{F}_n$ for all $n \in \mathbb{N}\}$. Further, let $\mathbb{E}$ be a (real) Banach space and $\mathbb{E}^*$ its topological dual. A subset S of $\mathbb{E}^*$ is said to be total or norming, resp. if and only if $\langle x^*, x\rangle = 0$ for every $x^* \in S$ implies $x = 0$ or for every $x \in \mathbb{E}$ we have $\|x\| = \sup\{|\langle x^*, x\rangle| : x^* \in B(\mathbb{E}^*)\}$, resp., where $B(\mathbb{E}^*)$ is the closed unit ball of $\mathbb{E}^*$. It is easily checked that if S is norming, then it is also total. Now let $M(\mathbb{E})$ stand for the space of all strong $\mathcal{F}$-measurable elements $X : \Omega \to \mathbb{E}$. Such an X is said to be *Bochner integrable*, write $X \in P^1(\mathbb{E})$, or *Pettis integrable*, write $X \in P^1(\mathbb{E})$, resp. if $E(\|X\|) = \int_\Omega (\|X\|)d\mathbb{P} < \infty$ or $F(\|X\|) = \sup\{E(|\langle x^*, X\rangle|) : x^* \in B(\mathbb{E}^*)\} < \infty$ resp. Unless otherwise specified, from now on, we shall consider only the sequences (X_n) in $P^1(\mathbb{E})$ such that each X_n is strongly $\mathcal{F}_n$-measuable and the Pettis $\mathcal{F}_q$-conditional expectation $E_q(X_n)$ of X_n exists for every $1 \le q \le n$. Thus by the Pettis's measurability theorem, we can suppose in the note, without any loss of generality, that $\mathbb{E}$ is separable. However, it should be noted that even in the case $\mathbb{E} = \ell_2$, an $X \in P^1(\mathbb{E})$ would fail to have the Pettis $\mathcal{A}$-conditional expectation for some subσ-field of $\mathcal{F}$. For more information, the reader is referred to [13].

Now let recall that a sequence (X_n) in $P^1(\mathbb{E})$ is said to be

a) a (weak) *strong amart* (cf. [5]) if and only if the net $(E(X_\tau), \tau \in \mathbb{T})$ of Pettis intergrals converges (weakly) strongly in $\mathbb{E}$, where $X_\tau(\omega) = X_{\tau(\omega)}(\omega)$ for every $\omega \in \Omega$ and $\tau \in \mathbb{T}$.

b) a *uniform amart* in $L^1(\mathbb{E})$ if and only if for every $\varepsilon > 0$ there exists $p \in \mathbb{N}$ such that for all $\sigma, \tau \in \mathbb{T}$ with $\tau \ge \sigma \ge p$, we have

$$E(\|E_\sigma(X_\tau) - X_\sigma\|) < \varepsilon,$$

where $E_\sigma(X)$ denotes the Bochner $\mathcal{F}_\sigma$-conditional expectation of $X \in L^1(\mathbb{E})$.

It is known that $\dim\mathbb{E} < \infty$ if and only if every $\mathbb{E}$-valued strong amart is a uniform one. For other comparison examples, the reader is referred to the last Sec. 3. Especially, Krupa and Zieba [7] proved that an L^1-*bounded strong amart converges a.s.* to some $X \in L^1(\mathbb{E})$ *if and only if it is essentially tight*, i.e. for every $\varepsilon > 0$ there exists a compact subset K of $\mathbb{E}$ such that $\mathbb{P}\big(\bigcap_{n\in\mathbb{N}} [X_n \in K]\big) > 1 - \varepsilon$.

The main aim of the note is to extend the result to weak amarts and pramarts (cf. [5]) in $P^1(\mathbb{E})$. As a consequence, several versions of the Ito-Nisio theorem (cf. [6]) are established for pramarts (X_n) with $\liminf_{\tau \in \mathbb{T}} \inf E(\|X_\tau\|) < \infty$.

2. Main Results

Since the note deals with (weak) a.s. convergence of adapted sequences in $P^1(\mathbb{E})$, it is useful to remember that by the Mackey theorem ([12, IV. 3. 3, p. 132]), the closed convex subsets of $\mathbb{E}^*$ are the same for the weak-star topology $\sigma(\mathbb{E}^*, \mathbb{E})$ and for the Mackey topology $\tau(\mathbb{E}^*, \mathbb{E})$ of $\mathbb{E}$, i.e. the topology of uniform convergence on all weakly compact convex cirled subsets of $\mathbb{E}$. Thus by the separability of $\mathbb{E}$, the space $\mathbb{E}^*$ is also separable for the weak-star topology $\sigma(\mathbb{E}^*, \mathbb{E})$, hence so is for the Mackey topology $\tau(\mathbb{E}^*, \mathbb{E})$. The following result gives a little more information of $\tau(\mathbb{E}^*, \mathbb{E})$.

Lemma 2.1. *Let S be a total subset of $\mathbb{E}^*$. Then there exists a sequence (x_n^*) of S such that the countable collection D of all linear combinations with rational coefficients of elements of (x_n^*) is dense in E^* for the Mackey topology $\tau(\mathbb{E}^*, \mathbb{E})$. Consequently, the subset $D_1 = D \cap B(\mathbb{E}^*)$ is dense in $B(\mathbb{E}^*)$ equipped with the Mackey topology $\tau(\mathbb{E}^*, \mathbb{E})$ inherated from $\mathbb{E}^*$, hence also norming.*

Proof. Let S be as given in the lemma. Then the first conclusion of D follows directly from Lemma III. 31 and III. 32 in ([3, p. 81]). Hence D_1 is naturally dense in $B(\mathbb{E}^*)$ equipped with the Mackey topology $\tau(\mathbb{E}^*, \mathbb{E})$ inherated from $\mathbb{E}^*$. Now let $x \in \mathbb{E}$ and $x^* \in B(\mathbb{E}^*)$. By the density of D_1 in $B(\mathbb{E}^*)$, it follows that there exists a sequence (y_n^*) of D_1 which converges to x^* in $\tau(\mathbb{E}^*, \mathbb{E})$. Consequently, $(\langle y_n^*, x\rangle)$ converges to $\langle x^*, x\rangle$. Thus

$$|\langle x^*, x\rangle| = \lim_{n\to\infty} |\langle y_n^*, x\rangle| \le \sup\{|\langle e, x\rangle| : e \in D_1\} \le \|x\|.$$

By taking the supremum over $x^* \in B(\mathbb{E}^*)$, one obtains

$$\|x\| = \sup\{|\langle e, x\rangle| : e \in D_1\}.$$

In other words, D_1 is norming. This completes the proof. ∎

Before going to the next lemma, it is useful to recall that a sequence (X_n) of $M(\mathbb{E})$ is said to be converging scalarly a.s. to an $X \in M(\mathbb{E})$ if and only if for every $x^* \in \mathbb{E}^*$ the scalar sequence $(\langle x^*, X_n\rangle)$ converges a.s. to $\langle x^*, X\rangle$. By Lemma 2.1, we see that if (X_n) converges scalarly a.s. to both X and X' simultaneously, then $X = X'$ a.s. Indeed, by the countability of the set D, associated with $S = \mathbb{E}^*$, given in Lemma 2.1, there exists a subset Ω_0 of Ω with $\mathbb{P}(\Omega_0) = 1$ such that for every $e \in D$ and $\omega \in \Omega_0$ we have $\langle e, X(\omega)\rangle = \langle e, X'(\omega)\rangle$. But since the subset D_1 of D is norming, as shown in Lemma 2.1, it follows from the above equality that $X(\omega) = X'(\omega)$ for every $\omega \in \Omega_0$. Beside the remark, it is also reasonable to say that an $X \in M(\mathbb{E})$ is *a scalar cluster point* of (X_n) a.s. if and only if for every $x^* \in \mathbb{E}^*$ the scalar function $\langle x^*, X\rangle$ is a cluster point of the scalar sequence $(\langle x^*, X_n\rangle)$ a.s. Consequently, if X is a weak cluster point of (X_n) a.s., then it is a scalar cluster point of (X_n) a.s.. Using the latter notion and the first lemma, we intend to prove the next one which is also needed in the sequel.

Lemma 2.2. *Let (X_n) be a sequence in $M(\mathbb{E})$. Suppose that for every $x^* \in \mathbb{E}^*$ the sequence $(\langle x^*, X_n\rangle)$ converges a.s. to some $\Phi(x^*) \in M(\mathbb{R})$ and $X \in M(\mathbb{E})$ is a scalar cluster point of (X_n) a.s. Then (X_n) converges scalarly a.s. to X. Consequently, (X_n) converges (weakly) a.s. if and only if the sequence $(X_n(\omega))$ is relatively (weakly) compact in $\mathbb{E}$ a.s.*

Proof. Let $(X_n), \Phi(.)$ and X be as given in the lemma. Given any but fixed $x^* \in \mathbb{E}^*$, by the cluster point approximation Theorem 1.2.1 ([5, p. 11]), there exists a sequence $(\tau_n(x^*))$ of $\mathbb{T}$ with each $\tau_n(x^*) \geq n$ such that the sequence $(\langle x^*, X_{\tau_n(x^*)}\rangle)$ converges to $\langle x^*, X\rangle$ a.s. On the other hand, as the sequence $\langle x^*, X_n\rangle$ converges a.s. to $\Phi(x^*)$, by ([8, Lemma 3]), the sequence $(\langle x^*, X_{\tau_n(x^*)}\rangle)$ converges also to $\Phi(x^*)$ a.s. It follows that $\Phi(x^*) = \langle x^*, X\rangle$ a.s. Thus the sequence $\langle x^*, X_n\rangle$ converges itself to $\langle x^*, X\rangle$ a.s. Therefore by definition, the sequence (X_n) converges scalarly a.s. to X. This proves the first conclusion of the lemma. To see its consequence, it is useful to note that the set $S = \mathbb{E}^*$ is naturally total. Thus by the previous lemma, there exists a countable set D, associated with S in the sense of Lemma 2.1, which is dense in $\mathbb{E}^*$ for the Mackey topology $\tau(\mathbb{E}^*, \mathbb{E})$. Suppose first that both $(X_n(\omega))$ is relatively weakly compact and $X(\omega)$ is a scalar cluster point of $(X_n(\omega))$ a.s. in $\mathbb{E}$. Then by the Krein-Smulian theorem and the first conclusion of the lemma, it follows that there exists a subset Ω_0 of Ω with $\mathbb{P}(\Omega_0) = 1$ satisfying

a) the closed cirled convex hull $\widehat{co}\{X_n(\omega)\}$ of $(X_n(\omega))$ is a weakly compact subset of $\mathbb{E}$ for every $\omega \in \Omega_0$, noting that by the Mazur's theorem, the weak and the strong closure of a convex subset in $\mathbb{E}$ are the same.

b) $(\langle x^*, X_n(\omega)\rangle)$ converges to $\langle x^*, X(\omega)\rangle$ for every $\omega \in \Omega_0$ and $x^* \in D$.

Consequently, by the $\tau(\mathbb{E}^*, \mathbb{E})$-density of D in $\mathbb{E}^*$ and the properties (a) and (b), for every $x^* \in \mathbb{E}^*$ and $\omega \in \Omega_0$ the sequence $(\langle x^*, X_n(\omega)\rangle)$ converges to $\langle x^*, X(\omega)\rangle$. In other words, (X_n) converges weakly a.s. to X. In the second case, when $(X_n(\omega))$ is relatively compact a.s. in $\mathbb{E}$, we proceed exactly as in the first situation to conclude that $(X_n(\omega))$ converges weakly to $X(\omega)$ for every $\omega \in \Omega_0$. However by the a.s. relative compactness of (X_n) and the Mazur's theorem, one can conclude more that, in the case, $\widehat{co}\{X_n(\omega)\}$ is even compact for every $\omega \in \Omega_0$. Then $(X_n(\omega))$ converges in norm to $(X_n(\omega))$ for each $\omega \in \Omega_0$, since the weak and the norm topology coincide on $\widehat{co}\{X_n(\omega)\}$. This completes the proof, noting that the necessity of the condition in both cases is trival. ∎

In order to present the first convergence result for weak amarts (X_n) in $P^1(\mathbb{E})$, it is worth remaking that Pettis intergrable random elements in $\mathbb{E}$ are not necessarily Bochner integrable. Therefore it is resonable to impose on (X_n) the following weaker conditions.

Definition 2.3. *A sequence (X_n) in $P_1(\mathbb{E})$ is said to be*

a) *σ-bounded (cf.[9]) if and only if there exists a nondecreasing sequence (B_n) of events adapted to $(\mathcal{F}_n)$ with $\lim_{n\to\infty} \mathbb{P}(B_n) = 1$ and such that restricted to each B_k sequence X_n is L^1 -bounded;*

b) *scalarly σ-bounded if and only if for each $x^* \in \mathbb{E}^*$ the sequence $(\langle x^*, X_n\rangle)$ is σ-bounded;*

c) *essentially weakly tight if and only if for every $\varepsilon > 0$ there exists a weakly compact subset K of $\mathbb{E}$ such that $\mathbb{P}\big(\bigcap_{n\in\mathbb{N}}[X_n \in K]\big) > 1 - \varepsilon$.*

Proposition 2.4. *Let (X_n) be a scalarly σ-bounded weak amart in $P^1(\mathbb{E})$. Suppose that (X_n) has a scalar cluster point $X \in M(\mathbb{E})$ a.s. Then (X_n) converges also scalarly a.s. to X. Moreover, if (X_n) is essentially weakly tight, then (X_n) converges weakly a.s.*

Proof. Let (X_n) be as given in the proposition. Then for any but fixed $x^* \in \mathbb{E}^*$ there exists a nondecreasing sequence $(B_n(x^*))$ of events adapted to $(\mathcal{F}_n)$ with $\lim_{n\to\infty} \mathbb{P}(B_n(x^*)) = 1$ such that restricted to each $B_k(x^*)$, the scalar sequence $(\langle x^*, X_n\rangle)$ is L^1-bounded. Therefore by the restriction Theorem 5.2.9 ([5, p. 186]) and the amart convergence Theorem 1.2.5 ([5, p. 11]), restricted to each $B_k(x^*)$, the real-valued amart $(\langle x^*, X_n\rangle)$ converges a.s. Thus by taking the pieces, one can conclude that the sequence $(\langle x^*, X_n\rangle)$ converges a.s. to the resulting random variable $\Phi(x^*)$. Consequently, if $X \in M(\mathbb{E})$ is a scalar cluster point of (X_n) a.s., then by Lemma 2.2, the sequence (X_n) converges scalarly a.s. to X. It proves the first conclusion of the proposition. Now suppose more that (X_n) is essentially weakly tight. Then it is clear that the sequence $(X_n(\omega))$ is relatively weakly compact a.s. This with the second part of Lemma 2.2 guarantees that (X_n) converges weakly a.s. to the weak cluster point X of (X_n) which completes the proof. ∎

Lemma 2.5. *Let (X_n) be a sequence in $M(\mathbb{E})$. Then every of the following conditions implies the next one.*

(a) *(X_n) converges a.s.;*

(b) *(X_τ) converges in distribution;*

(c) *(X_n) is eseentially tight.*

Proof. Suppose first that (a) is satisfied. Then by ([8, Lemma 3]), the sequence (X_{τ_n}) converges a.s., hence in distribution for every sequence (τ_n) of $\mathbb{T}$ with each $\tau_n \geq n$. However as the convergence in distribution is metrizable, we can conclude that the same net (X_τ) converges also in distribution. It proves (b). Next, suppose that (b) is satisfied. Then by ([8, Theorem 1]), the sequence (X_n) converges essentialy in distribution to a probability measure υ on Borel sets of $\mathbb{E}$, i.e. for every υ-continuity subset A of $\mathbb{E}$, we have

$$\lim_{n\to\infty} \mathbb{P}\big(\bigcup_{j\geq n}[X_j \in A]\big) = \upsilon(A) = \lim_{n\to\infty} \mathbb{P}(\bigcap_{j\geq n}[X_j \in A]).$$

Then by Theorem 2.2 [7], (X_n) is essentially tight. It proves (c) and the lemma. ∎

The next result is an essential extension of Theorem 4.1 [7].

Proposition 2.6. *Let (X_n) be a scalarly σ-bounded weak amart in $P^1(\mathbb{E})$. Then the following conditions are equivalent*
(a) *(X_n) converges a.s.;*
(b) *(X_n) is essentially tight.*

Proof.
(a) $\Rightarrow$ (b) is a consequence of the previous lemma. To see the converse implication (b) $\Rightarrow$ (a) it is useful to note first that by (b), the sequence (X_n) has a weak cluster point $X \in M(\mathbb{E})$. Thus we can proceed exactly as in the first part of the proof of Proposition 2.4 to conclude that for every $x^* \in \mathbb{E}^*$ the sequence $(\langle x^*, X_n\rangle)$ converges a.s. to $\langle x^*, X\rangle$. Finally, we apply the consequence of Lemma 2.2 to get (b) $\Rightarrow$ (a), noting that the essential tightness of (X_n) implies its a.s. relative compactness. ■

Now let recall that a sequence (X_n) in $L^1(\mathbb{E})$ is said to be a pramart (cf. [5]) if and only if for every $\varepsilon > 0$ there exists $p \in \mathbb{N}$ such that for all $\sigma, \tau \in \mathbb{T}$ with $\tau \geq \sigma \geq p$ we have

$$\mathbb{P}(\|E_\sigma(X_\tau) - X_\sigma\| > \varepsilon) < \varepsilon.$$

It is clear that every uniform amart is a pramart. The related examples given at the end allow one to distinguish pramarts from weak amarts in Banach spaces. Here we are in the position to give some versions of the Ito-Nisio theorem [6] for pramarts in $L^1(\mathbb{E})$ which contain the main convergence results of Davis et al. [4] and Bouzar [2].

Proposition 2.7. *Let (X_n) be a pramart in $L^1(\mathbb{E})$ satisfying*

$$\inf_{\tau \in \mathbb{T}} E(\|X_\tau\|) > \infty.$$

Then the following conditions are equivalent:
(a) *(X_n) converges a.s. to some element of $L^1(\mathbb{E})$;*
(b) *(X_n) is essentially tight;*
(c) *(X_n) is essentially weakly tight;*
(d) *(X_n) has a weak cluster point a.s.;*
(e) *there exist a total set S and some $X \in M(\mathbb{E})$ such that $\langle x^*, X\rangle$ is a cluster point of $(\langle x^*, X_n\rangle)$ a.s. for every $x^* \in S$.*

Proof. Since the first implication $(a) \Rightarrow (b)$ follows from Lemma 2.5 and the next implication $(b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e)$ are true in general, it remains to prove only the last one $(e) \Rightarrow (a)$. To see this, let (X_n) be as given in the proposition. Then by Remark 5 and Theorem 6 [10], (X_n) has a unique Riesz-Talagrand decomposition: $X_n = M_n + P_n$ where (M_n) is a uniformly integrable martingale and (P_n) goes to zero a.s. Now let both S and X be as given in (e). Then by the cluster approximation theorem 1.2.4 ([5, p. 11]), for every $x^* \in S$ there exists a. sequence $(\tau_n(x^*))$ with each $\tau_n(x^*) \geq n$ and such that the sequence $(\langle x^*, X_{\tau_n(x^*)}\rangle)$

converges a.s. to $\langle x^*, X\rangle$. On the other hand, by the Riesz-Talagrand decomposition of (X_n), it follows that the sequence $(\langle x^*, P_n\rangle)$ converges to zero a.s., hence by ([8, Lemma 3]), so does the sequence $(\langle X^*, P_{\tau_n(x^*)}\rangle)$. Consequently, the real-valued sequence $(\langle x^*, M_{\tau_n(x*)}\rangle)$ converges a.s. just to $\langle x^*, X\rangle$. But as a uniformly integrable real-valued martingale, the sequence $(\langle x^*, M_n\rangle)$ converges itself a.s. It turns out that $(\langle x^*, M_n\rangle)$ must converge just to its cluster point $\langle x^*, X\rangle$ a.s. This with the recent martingale limit theorem due to Davis et al. [4] (see also [5, Theorem 5.3.27, p. 209]) guarantees that (M_n) converges to X a.s., hence so does the pramart (X_n), by its Riesz-Talagrand decomposition. The proof is thus complete. ∎

3. Comparison Examples

To distinguish Proposition 2.4 from the next ones, we start with the following example.

Example 3.1. Let $\mathbb{E}$ be a separable Banach space with $\dim\mathbb{E} = \infty$. Then there exists an L_∞-bounded weak amart which is neither a strong amart nor a pramart.

Indeed, by the assumption, there exists a sequence (x_n) in the unit ball of $\mathbb{E}$ which converges weakly, but it disconverges in norm. Thus if we set each $X_n = x_n$ then (X_n) is really a weak amart, noting that the net $(E(X_\tau))$ converges weakly to the same weak limit as (x_n). However, the sequence $(E(X_n) = x_n)$ does not converge in norm. Consequently, it cannot be a strong amart. Further as for all $q < n$, we have

$$\|E_q(X_n) - X_q\| = \|x_n - x_q\|,$$

so by the disconvergence of (x_n) in norm, the sequence (X_n) cannot be a pramart either. The example also shows that one can apply neither Proposition 2.6 nor Proposition 2.7 to (X_n), since (X_n) is neither essentially tight nor a pramart.

The next example distinguishes Propsition 2.6 from the next one.

Example 3.2. There exists an L^1-bounded weak amart in $\mathbb{E} = \ell_2$ which is essentially tight, but it is not a pramart.

Indeed, choose

$$\Omega = [0,1]; a_0 = 0; a_n = \sum_{j=1}^{n} 2^{-(j+1)};\ \mathcal{F}_n = \sigma-\{[a_{j-1}, a_i) : 1 \le j \le n\}; \mathcal{F} = \bigvee_{n=1}^{\infty} \mathcal{F}_n$$

and $\mathbb{P}$ the Lebesgue measure restricted to $\mathcal{F}$. Further, set $X_l = 0$ and $X_n = e_n a_{n-1} 1_{[a_{n-1}, a_n)}$ for every $n \ge 2$, where (e_n) is the usual basis for ℓ_2. It is easily seen that (X_n) is an L^1-bounded weak amart which is essentially tight, since $a_m \uparrow \frac{1}{2}$ as $m \uparrow \infty$ and for every $m \ge 1$ we have

$$\mathbb{P}(\bigcap_{n=1}^{\infty} [X_n \in K_m]) \ge \frac{1}{2} + a_m,$$

where K_m is the compact subset of ℓ_2 given by

$$K_m = \{0\} \cup \{e_j : 1 \leq j \leq m\}.$$

Thus one can apply Proposition 2.6 to conclude that (X_n) converges a.s. (to zero). On the other hand, one can never apply Proposition 2.7, since (X_n) is not a pramart. Indeed, for all $q < n$ we have

$$\mathbb{P}(\|E_q(X_n) - X_q\| > \frac{1}{2}) > \frac{1}{2}.$$

The following final example not only distinguishes Proposition 2.7 from the others, but it also shows that one cannot apply any known pramart convergence results to it, except Proposition 2.7.

Example 3.3. There exists a nonnegative pramart (X_n) with $\inf_{\tau \in \mathbb{T}} E(X_\tau) = 0$ and $E(X_n) \uparrow \infty$ as $n \to \infty$. Consequently, it cannot be an amart.

Indeed, let $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\mathcal{F}_n)$ be as given in the previous example. Further, let define each $X_n = 3^n 1_{[a_{n-l}, a_n)}$ and

$$\tau_n = \begin{cases} n, & \omega \notin [a_{n-1}, a_n), \\ n+1, & \omega \in [a_{n-1}, a_n). \end{cases}$$

Then it is easily seen that (X_n) is a nonnegative pramart with $E(X_n) \uparrow \infty$ as $n \uparrow \infty$ and

$$0 \leq \inf_{\tau \in \mathbb{T}} E(X_\tau) \leq \sup_{n \geq 1} E(X_{\tau_n}) = 0$$

since each $X_{\tau_n} = 0$. Therefore one cannot apply any known pramart convergence result to the example, except Proposition 2.7.

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