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The Quantum Double of a Dual Andruskiewitsch-Schneider Algebra Is a Tame Algebra*

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Abstract. In this paper, we study the representation theory of the quantum double $\mathcal{D}(\Gamma_{n,d})$. We give the structure of projective modules of $\mathcal{D}(\Gamma_{n,d})$ at first. By this, we give the Ext-quiver (with relations) of $\mathcal{D}(\Gamma_{n,d})$ and show that $\mathcal{D}(\Gamma_{n,d})$ is a tame algebra.

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1. Introduction

In this paper, k is an algebraically closed field of characteristic 0 and an algebra is a finite dimensional associative k-algebra with identity element.

Although the quantum doubles of finite dimension Hopf algebras are important, not very much is known about their representations in general. A complete list of simple modules of the quantum doubles of Taft algebras is given by Chen in [2]. He also gives all indecomposable modules for the quantum double of a special Taft algebra in [3]. From this, we can deduce immediately that the quantum double of this special Taft algebra is tame. The authors of [7] study

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the representation theory of the quantum doubles of the duals of the generalized Taft algebras in detail. They describe all simple modules, indecomposable modules, quivers with relations and AR-quivers of the quantum doubles of the duals of the generalized Taft algebras explicitly and show that these quantum doubles are tame.

The structures of basic Hopf algebras of finite representation type are gotten in [11]. In fact, the authors of [11] show that basic Hopf algebras of finite representation type and monomial Hopf algebras (see [4]) are the same. But for the structure of tame basic Hopf algebras, we know little. In [10], the author gives the structure theorem for tame basic Hopf algebras in the graded case. In order to study tame basic Hopf algebras or generally tame Hopf algebras, we need more examples of tame Hopf algebras.

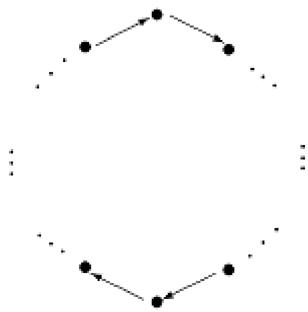
The Andruskiewitsch-Schneider algebra is a kind of generalization of generalized Taft algebra and of course Taft algebra. Therefore, it is natural to ask the following question: whether is the quantum double of dual Andruskiewitsch-Schneider algebra a tame algebra? In this paper, we give an affirmative answer. As a consequence, we give some new examples of tame Hopf algebra.

Our method is direct. Explicitly, we firstly give the structure of projective modules of the Drinfel Double of a dual Andruskiewitsch-Schneider algebra by direct computations. Then using this, we get its Ext-quiver with relations which will help us to get the desired conclusion.

2. Main Results

In this section, we will study the Drinfeld Double $\mathcal{D}(\Gamma_{n,d})$, which is a generalization of [7], of $(A(n,d,\mu,q))^{*cop}$. Our main result is to give the structure of projective modules of $\mathcal{D}(\Gamma_{n,d})$. By this, we give the Ext-quiver (with relations) of $\mathcal{D}(\Gamma_{n,d})$ and show that $\mathcal{D}(\Gamma_{n,d})$ is a tame algebra. This section relays heavily on [7] and we refer the reader to this paper.

The algebra $\Gamma_{n,d} := kZ_n/J^d$ with $d|n$ is described by quiver and relations. The quiver is a cycle,



with n vertices $e_0, \dots, e_{n-1}$. We shall denote by γ_i^m the path of length m starting at the vertex e_i. The relations are all paths of length $d \geqslant 2$.

We give the Hopf structure on $\Gamma_{n,d}$. We fix a primitive d-th root of unity q

and a $\mu \in k$.

$$\Delta(e_t) = \sum_{j+l=t} e_j \otimes e_l + \alpha_t^0 - \beta_t^0, \quad \Delta(\gamma_t^1) = \sum_{j+l=t} e_j \otimes \gamma_l^1 + q^l \gamma_j^1 \otimes e_l + \alpha_t^1 - \beta_t^1$$
$$\varepsilon(e_t) = \delta_{t0}, \ \ \varepsilon(\gamma_t^1) = 0, \ \ S(e_t) = e_{-t}, \ \ S(\gamma_t^1) = -q^{t+1}\gamma_{-t-1}^1$$

where

$$\alpha_t^s = \mu \sum_{l=s+1}^{d-1} \sum_{i+j=t} \frac{q^{jl}(s)!_q}{l!_q(d-l+s)!_q} \gamma_i^l \otimes \gamma_j^{d+s-l},$$
$$\beta_t^s = \sum_{l=s+1}^{d-1} \sum_{i+j+d=t} \frac{q^{jl}(s)!_q}{l!_q(d-l+s)!_q} \gamma_i^l \otimes \gamma_j^{d+s-l}.$$

Proposition 2.1. *With above comultiplication, counit and antipode, $\Gamma_{n,d}$ is a Hopf algebra.*

Proof. We only prove that Δ is an algebra morphism. The other axioms of Hopf algebras can be proved easily from this. In order to do it, it is enough to prove that, for $s, t \in \{0, \cdots, n-1\}$,

$$\Delta(e_s)\Delta(e_t) = \Delta(\delta_{st}e_t), \quad \Delta(\gamma_s^1 e_t) = \Delta(\gamma_s^1)\Delta(e_t), \quad \Delta(e_t\gamma_s^1) = \Delta(e_t)\Delta(\gamma_s^1).$$

We have

$$\begin{aligned}\Delta(e_s)\Delta(e_t) &= \Big(\sum_{j+l=s} e_j \otimes e_l + \alpha_s^0 - \beta_s^0\Big)\Big(\sum_{j+l=t} e_j \otimes e_l + \alpha_t^0 - \beta_t^0\Big) \\ &= \Big(\sum_{j+l=s} e_j \otimes e_l\Big)\Big(\sum_{j+l=t} e_j \otimes e_l\Big) + \Big(\sum_{j+l=s} e_j \otimes e_l\Big)\alpha_t^0 \\ &- \Big(\sum_{j+l=s} e_j \otimes e_l\Big)\beta_t^0 + \alpha_s^0\Big(\sum_{j+l=t} e_j \otimes e_l\Big) - \beta_s^0\Big(\sum_{j+l=t} e_j \otimes e_l\Big) + r\end{aligned}$$

where $r = \alpha_s^0\alpha_t^0 - \alpha_s^0\beta_t^0 - \beta_s^0\alpha_t^0 + \beta_s^0\beta_t^0$ and clearly $r \in J^d \otimes kZ_n + kZ_n \otimes J^d$. Thus $r = 0$. Note that in $\alpha_t^0 = \mu \sum_{l=1}^{d-1} \sum_{i+j=t} \frac{q^{jl}}{l!_q(d-l)!_q} \gamma_i^l \otimes \gamma_j^{d-l}$ every component, say $\gamma_i^l \otimes \gamma_j^{d-l}$, the end point of γ_i^l is e_{i+l} and that of γ_j^{d-l} is e_{j+d-l}. Thus

$$(e_m \otimes e_n)(\gamma_i^l \otimes \gamma_j^{d-l}) \neq 0$$

implies $m + n = i + l + j + d - l = t + d$. Similarly, $(\gamma_i^l \otimes \gamma_j^{d-l})(e_m \otimes e_n) \neq 0$ implies $m + n = t$. Therefore, if $s \neq t + p$,

$$\Big(\sum_{j+l=s} e_j \otimes e_l\Big)\alpha_t^0 = 0, \quad \beta_s^0\Big(\sum_{j+l=t} e_j \otimes e_l\Big) = 0$$

and if $s \neq t$

$$\Big(\sum_{j+l=s} e_j \otimes e_l\Big)\beta_t^0 = 0, \quad \alpha_s^0\Big(\sum_{j+l=t} e_j \otimes e_l\Big) = 0$$

Thus if $s \neq t+p$ and $s \neq t$, $\Delta(e_s)\Delta(e_t) = 0$.

If $s = t+p$,

$$\begin{aligned}
\Delta(e_s)\Delta(e_t) &= \Big(\sum_{j+l=s} e_j \otimes e_l\Big)\alpha_t^0 - \beta_s^0\Big(\sum_{j+l=t} e_j \otimes e_l\Big) \\
&= \alpha_t^0 - \beta_s^0 \\
&= \mu \sum_{l=1}^{d-1} \sum_{i+j=t} \frac{q^{jl}}{l!_q (d-l)!_q} \gamma_i^l \otimes \gamma_j^{d-l} \\
&\quad - \mu \sum_{l=1}^{d-1} \sum_{i+j+d=t+d} \frac{q^{jl}}{l!_q (d-l)!_q} \gamma_i^l \otimes \gamma_j^{d-l} \\
&= 0.
\end{aligned}$$

If $s = t$, $\Delta(e_s)\Delta(e_t) = \sum_{j+l=s} e_j \otimes e_l - (\sum_{j+l=s} e_j \otimes e_l)\beta_t^0 + \alpha_t^0(\sum_{j+l=t} e_j \otimes e_l) = \sum_{j+l=s} e_j \otimes e_l - \beta_t^0 + \alpha_t^0 = \Delta(e_t)$.

Thus, in a word, $\Delta(e_s)\Delta(e_t) = \Delta(\delta_{st} e_t)$.

Next, let us show that $\Delta(\gamma_s^1 e_t) = \Delta(\gamma_s^1)\Delta(e_t)$.

$$\begin{aligned}
\Delta(\gamma_s^1)\Delta(e_t) &= \Big(\sum_{j+l=s} (e_j \otimes \gamma_l^1 + q^l \gamma_j^1 \otimes e_l) + \alpha_s^1 - \beta_s^1\Big)\Big(\sum_{j+l=t} e_j \otimes e_l + \alpha_t^0 - \beta_t^0\Big) \\
= \Big(\sum_{j+l=s} e_j \otimes \gamma_l^1 &+ q^l \gamma_j^1 \otimes e_l\Big)\Big(\sum_{j+l=t} e_j \otimes e_l\Big) + \Big(\sum_{j+l=s} e_j \otimes \gamma_l^1 + q^l \gamma_j^1 \otimes e_l\Big)\alpha_t^0 \\
- \Big(\sum_{j+l=s} e_j \otimes \gamma_l^1 &+ q^l \gamma_j^1 \otimes e_l\Big)\beta_t^0 + \alpha_s^1\Big(\sum_{j+l=t} e_j \otimes e_l\Big) - \beta_s^1\Big(\sum_{j+l=t} e_j \otimes e_l\Big).
\end{aligned}$$

Similar to the computation of $\Delta(e_s)\Delta(e_t) = \Delta(\delta_{st} e_t)$, if $s \neq t$,

$$\Big(\sum_{j+l=s} e_j \otimes \gamma_l^1 + q^l \gamma_j^1 \otimes e_l\Big)\beta_t^0 = 0, \quad \alpha_s^1 v(\sum_{j+l=t} e_j \otimes e_l\Big) = 0$$

and if $s \neq t+p$,

$$\Big(\sum_{j+l=s} e_j \otimes \gamma_l^1 + q^l \gamma_j^1 \otimes e_l\Big)\alpha_t^0 = 0, \quad \beta_s^1\Big(\sum_{j+l=t} e_j \otimes e_l\Big) = 0.$$

Thus if $s \neq t$ and $s \neq t+p$, $\Delta(\gamma_s^1)\Delta(e_t) = 0$.

If $s = t+p$,

$$\Delta(\gamma_s^1)\Delta(e_t) = \Big(\sum_{j+l=s} e_j \otimes \gamma_l^1 + q^l \gamma_j^1 \otimes e_l\Big)\alpha_t^0 - \beta_s^1(\sum_{j+l=t} e_j \otimes e_l)$$

$$= \sum_{j+l=s}(e_j\otimes\gamma_l^1+q^l\gamma_j^1\otimes e_l)\Big(\mu\sum_{l=1}^{d-1}\sum_{i+j=t}\frac{q^{jl}}{l!_q(d-l)!_q}\gamma_i^l\otimes\gamma_j^{d-l}\Big)-\beta_s^1$$
$$=\mu\sum_{l=1}^{d-1}\sum_{i+j=t}\frac{q^{jl}}{l!_q(d-l)!_q}(\gamma_i^l\otimes\gamma_j^{d-l+1}+q^{j+d-l}\gamma_i^{l+1}\otimes\gamma_j^{d-l})-\beta_s^1$$
$$=\mu\sum_{l=2}^{d-1}\sum_{i+j=t}\Big(\frac{q^{jl}}{l!_q(d-l)!_q}+q^{j(l-1)}q^{j+d-l+1}\frac{1}{(l-1)!_q(d-l+1)!_q}\Big)\gamma_i^l\otimes\gamma_j^{d-l+1}-\beta_s^1$$
$$=\mu\sum_{l=2}^{d-1}\sum_{i+j=t}\frac{q^{jl}}{l!_q(d-l+1)!_q}\gamma_i^l\otimes\gamma_j^{d-l+1}-\beta_s^1$$
$$=0.$$

If $s=t$,

$$\begin{aligned}\Delta(\gamma_s^1)\Delta(e_t)=&\Big(\sum_{j+l=s}e_j\otimes\gamma_l^1+q^l\gamma_j^1\otimes e_l\Big)\\&-\Big(\sum_{j+l=s}e_j\otimes\gamma_l^1+q^l\gamma_j^1\otimes e_l\Big)\beta_t^0+\alpha_t^1\Big(\sum_{j+l=t}e_j\otimes e_l\Big)\\=&\Big(\sum_{j+l=s}e_j\otimes\gamma_l^1+q^l\gamma_j^1\otimes e_l\Big)-\beta_t^1+\alpha_t^1=\Delta(\gamma_t^1)\end{aligned}$$

where the second equality can be gotten by a similar computation in the case of $s=t+p$.

Therefore, we have $\Delta(\gamma_s^1e_t)=\Delta(\gamma_s^1)\Delta(e_t)$. The equality $\Delta(e_t\gamma_s^1)=\Delta(e_t)\Delta(\gamma_s^1)$ can be gotten similarly. ■

By [11], we know that the most typical examples of basic Hopf algebras of finite representation type are Taft algebras and the dual of $A(n,d,\mu,q)$, which as an associative algebra is generated by two elements g and x with relations

$$g^n=1,\quad x^d=\mu(1-g^d),\quad xg=qgx$$

with comultiplication Δ, counit ε, and antipode S given by

$$\Delta(g)=g\otimes g,\quad \Delta(x)=1\otimes x+x\otimes g$$
$$\varepsilon(g)=1,\quad \varepsilon(x)=0$$
$$S(g)=g^{-1},\quad S(x)=-xg^{-1}.$$

We call this Hopf algebra the Andruskiewitsch-Schneider algebra. If $\mu=0$, then it is the so-called generalized Taft algebra (see [8]). If $\mu=0$ and $d=n$, then clearly it is the usual Taft algebra.

Lemma 2.2. *As a Hopf algebra,* $(\Gamma_{n,d})^{*cop} \cong A(n,d,\mu,q)$ *by* $\widehat{\gamma_1^0} \mapsto G$, $\widehat{\gamma_0^1} \mapsto X$ *and*

$$\Delta(\gamma_i^m) = \left(\sum_{s+t=i, v+l=m} \binom{m}{v}_q q^{vt} \gamma_s^v \otimes \gamma_t^l \right) + \alpha_i^m - \beta_i^m.$$

Proof. It is a direct computation. ∎

We always denote $\sum_{s+t=i,v+l=m} \binom{m}{v}_q q^{vt}\gamma_s^v \otimes \gamma_t^l$ by M_i^m.

Lemma 2.3.

$$(id \otimes \Delta)M_l^m = \sum_{m_1+m_2+m_3=m, l_1+l_2+l_3=l} \frac{q^{m_1(l_2+l_3)+m_2l_3} m!_q}{(m_1)!_q(m_2)!_q(m_3)!_q} \gamma_{l_1}^{m_1} \otimes \gamma_{l_2}^{m_2} \otimes \gamma_{l_3}^{m_3},$$

$$(id \otimes \Delta)\alpha_l^m = \mu \sum_{m_1+m_2+m_3=d+m, l_1+l_2+l_3=l} \frac{q^{m_1(l_2+l_3)+m_2l_3} m!_q}{(m_1)!_q(m_2)!_q(m_3)!_q} \gamma_{l_1}^{m_1} \otimes \gamma_{l_2}^{m_2} \otimes \gamma_{l_3}^{m_3},$$

$$(id \otimes \Delta)\beta_l^m = \mu \sum_{m_1+m_2+m_3=d+m, l_1+l_2+l_3+d=l} \frac{q^{m_1(l_2+l_3)+m_2l_3} m!_q}{(m_1)!_q(m_2)!_q(m_3)!_q} \gamma_{l_1}^{m_1} \otimes \gamma_{l_2}^{m_2} \otimes \gamma_{l_3}^{m_3}.$$

Proposition 2.4. *The Drinfeld double* $\mathcal{D}(\Gamma_{n,d})$ *is described as follows: as a coalgebra, it is* $(\Gamma_{n,d})^{*cop} \otimes \Gamma_{n,d}$. *We write the basis elements* $G^iX^j\gamma_l^m$, *with* $i,l \in \{0,1,\dots,n-1\}$, $0 \le j,m \le d-1$. *The following relations determined the algebra structure completely:*

$$G^n = 1, \quad X^d = \mu(1-G^d), \quad GX = q^{-1}XG$$

the product of elements γ_l^m *is the usual product of paths, and*

$$\gamma_l^m G = q^{-m} G \gamma_l^m \tag{$*1$}$$

in particular $e_lG = Ge_l$ *since* $e_l = \gamma_l^0$ *by the definition of* γ_l^m, *and*

$$\gamma_l^m X = \begin{cases} q^{-m}X\gamma_{l+1}^m - q^{-m}(m)_q\gamma_{l+1}^{m-1} + q^{l+1-m}(m)_qG\gamma_{l+1}^{m-1} & \text{if } m \geqslant 1 \\ X\gamma_{l+1}^0 - \frac{\mu}{(d-1)!_q}(\gamma_{l+1}^{d-1} - \gamma_{l+1-d}^{d-1}) + \frac{\mu q^{l+1}}{(d-1)!_q}G(\gamma_{l+1}^{d-1} - \gamma_{l+1-d}^{d-1}) & \text{if } m = 0. \end{cases} \tag{$*2$}$$

Proof. We only prove equality $(*1)$, $(*2)$. For equality $(*1)$, by the definition of Drinfeld double,

$$\begin{aligned} \gamma_l^m G &= (1 \otimes \gamma_l^m)(\widehat{\gamma_1^0} \otimes 1) \\ &= \sum_{m_1+m_2+m_3=m, l_1+l_2+l_3=l} C_{l_1,l_2,l_3}^{m_1,m_2,m_3} \widehat{\gamma_1^0}(S^{-1}(\gamma_{l_3}^{m_3})?\gamma_{l_1}^{m_1}) \otimes \gamma_{l_2}^{m_2} && \text{(I)} \\ &\quad + \mu \sum_{m_1+m_2+m_3=d+m, l_1+l_2+l_3=l} C_{l_1,l_2,l_3}^{m_1,m_2,m_3} \widehat{\gamma_1^0}(S^{-1}(\gamma_{l_3}^{m_3})?\gamma_{l_1}^{m_1}) \otimes \gamma_{l_2}^{m_2} && \text{(II)} \\ &\quad - \mu \sum_{m_1+m_2+m_3=d+m, l_1+l_2+l_3+d=l} C_{l_1,l_2,l_3}^{m_1,m_2,m_3} \widehat{\gamma_1^0}(S^{-1}(\gamma_{l_3}^{m_3})?\gamma_{l_1}^{m_1}) \otimes \gamma_{l_2}^{m_2} && \text{(III)} \end{aligned}$$

where $C^{m_1,m_2,m_3}_{l_1,l_2,l_3} = \frac{q^{m_1(l_2+l_3)+m_2l_3}m!_q}{(m_1)!_q(m_2)!_q(m_3)!_q}$. By observation, we can find the following results.

For term (I), $\widehat{\gamma_1^0}(S^{-1}(\gamma_{l_3}^{m_3})?\gamma_{l_1}^{m_1}) \neq 0$ only if $l_1 = 1, l_3 = n-1, l_2 = l, m_1 = 0, m_3 = 0, m_2 = m$. In this case $C^{m_1,m_2,m_3}_{l_1,l_2,l_3} = q^{-m}$. Thus $(I) = q^{-m}\widehat{\gamma_1^0} \otimes \gamma_l^m = q^{-m}G\gamma_l^m$. In a similar way, we can find $(II) = 0$ and $(III) = 0$. Thus ($*$1) is proved.

For equality ($*$2),

$$\begin{aligned}
\gamma_l^m X &= (1\otimes\gamma_l^m)(\widehat{\gamma_0^1}\otimes 1)\\
&= \sum_{m_1+m_2+m_3=m, l_1+l_2+l_3=l} C^{m_1,m_2,m_3}_{l_1,l_2,l_3}\widehat{\gamma_0^1}(S^{-1}(\gamma_{l_3}^{m_3})?\gamma_{l_1}^{m_1})\otimes\gamma_{l_2}^{m_2} \qquad \text{(I)}\\
&\quad + \mu \sum_{m_1+m_2+m_3=d+m, l_1+l_2+l_3=l} C^{m_1,m_2,m_3}_{l_1,l_2,l_3}\widehat{\gamma_0^1}(S^{-1}(\gamma_{l_3}^{m_3})?\gamma_{l_1}^{m_1})\otimes\gamma_{l_2}^{m_2} \qquad \text{(II)}\\
&\quad - \mu \sum_{m_1+m_2+m_3=d+m, l_1+l_2+l_3+d=l} C^{m_1,m_2,m_3}_{l_1,l_2,l_3}\widehat{\gamma_0^1}(S^{-1}(\gamma_{l_3}^{m_3})?\gamma_{l_1}^{m_1})\otimes\gamma_{l_2}^{m_2} \qquad \text{(III)}
\end{aligned}$$

For term (I), it is easy to find that there are only three cases satisfying $\widehat{\gamma_0^1}(S^{-1}(\gamma_{l_3}^{m_3})?\gamma_{l_1}^{m_1}) \neq 0$. They are

(1): $l_1 = 0,\ l_2 = l+1,\ l_3 = n-1,\ m_1 = 0,\ m_2 = m-1,\ m_3 = 1$
(2): $l_1 = 0,\ l_2 = l+1,\ l_3 = n-1,\ m_1 = 0,\ m_2 = m,\ m_3 = 0$
(3): $l_1 = 0,\ l_2 = l+1,\ l_3 = n-1,\ m_1 = 1,\ m_2 = m-1,\ m_3 = 0.$

For case (1), it is straightforward to prove that

$$C^{m_1,m_2,m_3}_{l_1,l_2,l_3}\widehat{\gamma_0^1}(S^{-1}(\gamma_{l_3}^{m_3})?\gamma_{l_1}^{m_1})\otimes\gamma_{l_2}^{m_2} = -q^{-m}(m)_q 1\otimes\gamma_{l+1}^{m-1}.$$

For case (2), we have

$$C^{m_1,m_2,m_3}_{l_1,l_2,l_3}\widehat{\gamma_0^1}(S^{-1}(\gamma_{l_3}^{m_3})?\gamma_{l_1}^{m_1})\otimes\gamma_{l_2}^{m_2} = -q^{-m}\widehat{\gamma_0^1}\otimes\gamma_{l+1}^m = -q^{-m}X\gamma_{l+1}^m.$$

For case (3), we have

$$\begin{aligned}
&C^{m_1,m_2,m_3}_{l_1,l_2,l_3}\widehat{\gamma_0^1}(S^{-1}(\gamma_{l_3}^{m_3})?\gamma_{l_1}^{m_1})\otimes\gamma_{l_2}^{m_2}\\
&= -q^{l+1-m}(m)_q\widehat{\gamma_1^0}\otimes\gamma_{l+1}^{m-1} = -q^{l+1-m}(m)_q G\gamma_{l+1}^{m-1}.
\end{aligned}$$

For term (II), there are also three possible cases such that $\widehat{\gamma_0^1}(S^{-1}(\gamma_{l_3}^{m_3})?\gamma_{l_1}^{m_1}) \neq 0$. They are

(1): $l_1 = 0,\ l_2 = l+1,\ l_3 = n-1,\ m_1 = 0,\ m_2 = d+m-1,\ m_3 = 1$
(2): $l_1 = 0,\ l_2 = l+1,\ l_3 = n-1,\ m_1 = 0,\ m_2 = d+m,\ m_3 = 0$
(3): $l_1 = 0,\ l_2 = l+1,\ l_3 = n-1,\ m_1 = 1,\ m_2 = d+m-1,\ m_3 = 0.$

Thus if $m \geqslant 1$, we have that $\gamma_{l_2}^{m_2} \in J^d$ which is zero by the definition of $\Gamma_{n,d}$. Thus $\widehat{\gamma_0^1}(S^{-1}(\gamma_{l_3}^{m_3})?\gamma_{l_1}^{m_1})\otimes\gamma_{l_2}^{m_2} \neq 0$ only if $m = 0$.

Assume $m=0$. For case (1),

$$\mu C^{m_1,m_2,m_3}_{l_1,l_2,l_3}\widehat{\gamma^1_0}(S^{-1}(\gamma^{m_3}_{l_3})?\gamma^{m_1}_{l_1})\otimes\gamma^{m_2}_{l_2}=\frac{-\mu}{(d-1)!_q}\gamma^{d-1}_{l+1}.$$

For case (2),

$$\mu C^{m_1,m_2,m_3}_{l_1,l_2,l_3}\widehat{\gamma^1_0}(S^{-1}(\gamma^{m_3}_{l_3})?\gamma^{m_1}_{l_1})\otimes\gamma^{m_2}_{l_2}=0.$$

For case (3),

$$\mu C^{m_1,m_2,m_3}_{l_1,l_2,l_3}\widehat{\gamma^1_0}(S^{-1}(\gamma^{m_3}_{l_3})?\gamma^{m_1}_{l_1})\otimes\gamma^{m_2}_{l_2}=\frac{\mu q^{l+1}}{(d-1)!_q}G\gamma^{d-1}_{l+1}.$$

For term (III), there are also three cases which we need to consider.

(1): $l_1=0,\ l_2=l+1-d,\ l_3=n-1,\ m_1=0,\ m_2=d+m-1,\ m_3=1$
(2): $l_1=0,\ l_2=l+1-d,\ l_3=n-1,\ m_1=0,\ m_2=d+m,\ m_3=0$
(3): $l_1=0,\ l_2=l+1-d,\ l_3=n-1,\ m_1=1,\ m_2=d+m-1,\ m_3=0.$

If $m\geqslant 1$, we also have term $(III)=0$. Assume $m=0$. For case (1),

$$\mu C^{m_1,m_2,m_3}_{l_1,l_2,l_3}\widehat{\gamma^1_0}(S^{-1}(\gamma^{m_3}_{l_3})?\gamma^{m_1}_{l_1})\otimes\gamma^{m_2}_{l_2}=\frac{-\mu}{(d-1)!_q}\gamma^{d-1}_{l+1-d}.$$

For case (2),

$$\mu C^{m_1,m_2,m_3}_{l_1,l_2,l_3}\widehat{\gamma^1_0}(S^{-1}(\gamma^{m_3}_{l_3})?\gamma^{m_1}_{l_1})\otimes\gamma^{m_2}_{l_2}=0.$$

For case (3),

$$\mu C^{m_1,m_2,m_3}_{l_1,l_2,l_3}\widehat{\gamma^1_0}(S^{-1}(\gamma^{m_3}_{l_3})?\gamma^{m_1}_{l_1})\otimes\gamma^{m_2}_{l_2}=\frac{\mu q^{l+1}}{(d-1)!_q}G\gamma^{d-1}_{l+1-d}.$$

■

In order to study the structure of projective modules of $\mathcal{D}(\Gamma_{n,d})$, we first decompose $\mathcal{D}(\Gamma_{n,d})$ into a direct sum of algebras $\Gamma_0,\cdots,\Gamma_{n-1}$ and study each of these algebras. Our method is from [7]. This method were used by several authors, see [13, 14].

Proposition 2.5. *The elements $E_u:=\frac{1}{n}\sum_{i,j\in Z_n}q^{-i(u+j)}G^ie_j$, for $u\in Z_n$, are central orthogonal idempotents, and $\sum_{u\in Z_n}E_u=1$. Therefore,*

$$\mathcal{D}(\Gamma_{n,d})\cong\bigoplus_{u\in Z_n}\mathcal{D}(\Gamma_{n,d})E_u.$$

Proof. We only prove that $E_uX=XE_u$, the others are easy.

$$\begin{aligned}E_uX&=\frac{1}{n}\sum_{i,j\in Z_n}q^{-i(u+j)}G^ie_jX\\&=\frac{1}{n}\sum_{i,j\in Z_n}q^{-i(u+j)}G^i\Big[Xe_{j+1}-\frac{\mu}{(d-1)!_q}(\gamma^{d-1}_{j+1}-\gamma^{d-1}_{j+1-d})\\&\quad+\frac{\mu q^{j+1}}{(d-1)!_q}G(\gamma^{d-1}_{j+1}-\gamma^{d-1}_{j+1-d})\Big].\end{aligned}$$

Note that

$$\begin{aligned}
&\sum_{j\in Z_n} \frac{q^{-i(u+j)}\mu}{(d-1)!_q}(\gamma_{j+1}^{d-1}-\gamma_{j+1-d}^{d-1})\\
&=\sum_{j\in Z_n} q^{-i(u+j)}\frac{\mu}{(d-1)!_q}\gamma_{j+1}^{d-1}-\sum_{j\in Z_n} q^{-i(u+j)}\frac{\mu}{(d-1)!_q}\gamma_{j+1-d}^{d-1}\\
&=\sum_{j\in Z_n} q^{-i(u+j)}\frac{\mu}{(d-1)!_q}\gamma_{j+1}^{d-1}-\sum_{l\in Z_n} q^{-i(u+l+d)}\frac{\mu}{(d-1)!_q}\gamma_{l+1}^{d-1}\\
&=\frac{\mu}{(d-1)!_q}\sum_{j\in Z_n}(q^{-i(u+j)}-q^{-i(u+j+d)})\gamma_{j+1}^{d-1}\\
&=0.
\end{aligned}$$

Similarly,

$$\sum_{j\in Z_n} \frac{q^{-i(u+j)}\mu q^{j+1}}{(d-1)!_q}G(\gamma_{j+1}^{d-1}-\gamma_{j+1-d}^{d-1})=0.$$

Thus

$$\begin{aligned}
E_uX&=\frac{1}{n}\sum_{i,j\in Z_n} q^{-i(u+j)}G^iXe_{j+1}\\
&=\frac{1}{n}\sum_{i,j\in Z_n} q^{-i(u+j)}q^{-i}XG^ie_{j+1}\\
&=X\frac{1}{n}\sum_{i,j\in Z_n} q^{-i(u+j+1)}G^ie_{j+1}\\
&=XE_u.
\end{aligned}$$

■

Define $\Gamma_u := \mathcal{D}(\Gamma_{n,d})E_u$. The above propositions tell us that we need to study Γ_u. We now define some idempotents inside Γ_u, which are not central, but we will use them to describe a basis for Γ_u.

Proposition 2.6. *Set* $E_{u,j} = \sum_{v=0}^{\frac{n}{d}-1} e_{j+vd}E_u$, *for* $j \in Z_d$. *Then* $E_{u,j}E_{u,l} = \delta_{jl}E_{u,j}$ *and* $\sum_{j=0}^{d-1} E_{u,j} = E_u$. *We also have* $E_{u,j} = E_{u,j'}$ *if and only if* $j \equiv j'$ (mod d).

Moreover, the following relations hold within Γ_u*:*

$$GE_{u,j} = q^{u+j}E_{u,j} = E_{u,j}G, \quad XE_{u,j} = E_{u,j-1}X$$

$$\gamma_l^m E_{u,j} = \begin{cases} E_{u,j+m}\gamma_l^m & \text{if } \ l \equiv j \pmod d \\ 0 & \text{otherwise.}\end{cases}$$

Proof. We only prove that $XE_{u,j} = E_{u,j-1}X$. The proof of other equalities is the same with that of Proposition 2.7 in [7].

$$
\begin{aligned}
E_{u,j-1}X &= \sum_{v=0}^{\frac{n}{d}-1}(e_{j-1+vd}X)E_u \\
&= \sum_{v=0}^{\frac{n}{d}-1}\Big[Xe_{j+vd} - \frac{\mu}{(d-1)!_q}\big(\gamma_{j+vd}^{d-1} - \gamma_{j+(v-1)d}^{d-1}\big) \\
&\quad + \frac{\mu q^{j+vd}}{(d-1)!_q}G(\gamma_{j+vd}^{d-1} - \gamma_{j+(v-1)d}^{d-1})\Big]E_u \\
&= \sum_{v=0}^{\frac{n}{d}-1} Xe_{j+vd}E_u = XE_{u,j}.
\end{aligned}
$$

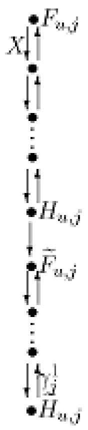
■

We can now describe a basis for Γ_u and a grading on Γ_u, as follows: $\Gamma_u = \bigoplus_{s=1-d}^{d-1}(\Gamma_u)_s$ with $(\Gamma_u)_s = \text{span}\,\{X^t\gamma_j^m E_{u,j} | j \in Z_n, 0 \le m, t \le d-1, m-t = s\}$. This is a sum of eigenspaces for G: if $yE_{u,j}$ is an element in $(\Gamma_u)_s$, we have that

$$G \cdot yE_{u,j} = q^{s+j+u}yE_{u,j}, \quad yE_{u,j} \cdot G = q^{j+u}yE_{u,j}.$$

Now set $F_{u,j} := \gamma_j^{d-1}E_{u,j}$ for $j \in Z_n$. If j is an element in Z_n, we shall denote its representative modulo d in $\{1, \ldots, d\}$ by $< j >$ and its representative modulo d in $\{0, \ldots, d-1\}$ by $< j >^-$.

Proposition 2.7. *The module $\Gamma_u F_{u,j}$ has the following form:*



where $H_{u,j} := X^{<2j+u-1>-1}F_{u,j}$, $\widetilde{F}_{u,j} := X^{<2j+u-1>^-}F_{u,j}$ and $\widetilde{H}_{u,j} := X^{d-1}F_{u,j}$. In this diagram, $\downarrow$ represents the action of X and $\uparrow$ represents the action of the suitable arrow up to a nonzero scalar; the basis vectors are eigenvectors for the action of G.

Note that when $2j + u - 1 \equiv 0 (mod d)$, the single arrow does not occur, the module is simple, and we have $H_{u,j} = \widetilde{H}_{u,j} = X^{d-1}F_{u,j}$ and $\widetilde{F}_{u,j} = F_{u,j}$.

In order to prove this proposition, we require the following lemma:

Lemma 2.8.
(1) $X^m F_{u,j} = 0$ *if* $m \geqslant d$;
(3) *Assume* $m < d$, *the element* $\gamma^b_{d+j-m-1} X^m F_{u,j}$ *is equal to*

$$q^{-\frac{b(2m-b+1)}{2}} \frac{(m)!_q}{(m-b)!_q} \prod (q^{2j+u-1-(m-t+1)-1}) X^{m-b} F_{u,j}$$

if $b < m$ *and is* 0 *otherwise.*

Proof.
(1): It is enough to prove that $X^d F_{u,j} = 0$. In fact, $X^d F_{u,j} = \mu(1 - G^d) F_{u,j} = \mu(F_{u,j} - (q^{d-1+u+j})^d F_{u,j}) = \mu(F_{u,j} - F_{u,j}) = 0$.
(2): This is proved by induction on b. When $b = 1$, we take the arrow on the left across X, using the relation in Proposition 5.4.

$$\gamma^1_{j+d-1-m} X^m \gamma^{d-1}_j E_{u,j} = \big(q^{-1} X \gamma^1_{j-m+d} - q^{-1}\gamma^0_{j-m+d} + q^{j-m-1+d} G \gamma^0_{j-m+d}\big) X^{m-1} \gamma^{d-1}_j E_{u,j}$$
$$= \big(q^{-1} X \gamma^1_{j-m+d} - q^{-1}(1 - q^{2(j-m+d)+u}) \gamma^0_{j-m+d}\big) X^{m-1} \gamma^{d-1}_j E_{u,j}. \qquad (*)$$

We want to compute that what $\gamma^0_{j-m+d} X^{m-1} \gamma^{d-1}_j$ is.

If $m = 1$, clearly $\gamma^0_{j-m+d} X^{m-1} \gamma^{d-1}_j = X^{m-1} \gamma^{d-1}_j$.

If $m > 1$,

$$\begin{aligned} &\gamma^0_{j-m+d} X^{m-1} \gamma^{d-1}_j \\ &= \Big[X \gamma^0_{j-m+d+1} - \frac{\mu}{(d-1)!_q} \big(\gamma^{d-1}_{j-m+d+1} - \gamma^{d-1}_{j-m+1}\big) \\ &\quad + \frac{\mu q^{j-m+d+1}}{(d-1)!_q} G \big(\gamma^{d-1}_{j-m+d+1} - \gamma^{d-1}_{j-m+1}\big)\Big] X^{m-2} \gamma^{d-1}_j. \end{aligned}$$

Since $m - 2 < d - 1$, by induction on the length of path, we know

$$\gamma^{d-1}_l X^{m-2} \gamma^{d-1}_j = 0 \quad for \;\; l \in Z_n.$$

Thus

$$\begin{aligned} \gamma^0_{j-m+d} X^{m-1} \gamma^{d-1}_j &= X \gamma^0_{j-m+d+1} X^{m-2} \gamma^{d-1}_j \\ &= \cdots = X^{m-1} \gamma^0_{j+d-1} \gamma^{d-1}_j \\ &= X^{m-1} \gamma^{d-1}_j. \end{aligned}$$

Therefore,

$$\begin{aligned} (*) &= q^{-1} X \gamma^1_{j-m+d} X^{m-1} \gamma^{d-1}_j E_{u,j} - q^{-1}(1 - q^{2(j-m+d)+u}) X^{m-1} \gamma^{d-1}_j E_{u,j} \\ &= q^{-1} X (q^{-1} X \gamma^1_{j-m+d+1} - q^{-1}(1 - q^{2(j-m+d)+u+2}) \gamma^0_{j-m+d-1}) X^{m-2} \gamma^{d-1}_j E_{u,j} \\ &\quad - q^{-1}(1 - q^{2(j-m+d)+u}) X^{m-1} \gamma^{d-1}_j E_{u,j} \\ &= q^{-2} X^2 \gamma^1_{j-m+d+1} X^{m-2} \gamma^{d-1}_j E_{u,j} \\ &\quad + (-q^{-2} - q^{-1} + q^{2j-2m+u-1} + q^{2j-2m+u}) X^{m-1} \gamma^{d-1}_j E_{u,j} \\ &= \cdots \\ &= q^{-m} X^m \gamma^1_{j+d-1} \gamma^{d-1}_j E_{u,j} + \sum_{p=1}^{m} (-q^{-p} + q^{2j-2m+u+p-2}) X^{m-1} \gamma^{d-1}_j E_{u,j} \\ &= q^{-m} (m)_q (q^{2j-m+u-1} - 1) X^{m-1} \gamma^{d-1}_j E_{u,j}. \end{aligned}$$

■

Proof of Proposition 2.7. Here we apply Lemma 2.8 with $b = 1$ to see that the element $\gamma_{d+j-m-1}X^m F_{u,j}$ is equal to 0 if $m \equiv 2j + u - 1 \pmod{d}$ and is a nonzero multiple of $X^{m-1}F_{u,j}$ otherwise. ■

Definition 2.1. *We define permutations of the indices in Z_n by $\sigma_u(j) := d + j - \langle 2j + u - 1\rangle$. Note that the arrow going up from $H_{u,j}$ in the diagram in Proposition 2.7 is $\gamma^1_{\sigma_u(j)}$.*

Remark. We can easily see that $\sigma_u(j) = j$ if and only if $\langle 2j + u - 1\rangle = d$, and that if $\langle 2j + u - 1\rangle \neq d$, then $\sigma_u^2(j) = j + d$ and so σ_u has order $\frac{2n}{d}$.

We shall now define large modules (and we will see later that they are a full set of representatives of the indecomposable projective modules).

Lemma 2.9. *If $\langle 2j+u-1\rangle \neq d$, then there exists an element $K_{u,j}$ homogeneous of degree $d - \langle 2j + u - 1\rangle^- - 1$ such that $H_{u,j} = \gamma^1_{\sigma_u(j)-1}K_{u,j}$.*

Proof. Consider $H_{u,j} = X^{\langle 2j+u-1\rangle -1}\gamma_j^{d-1}E_{u,j}$. We first take one arrow across X; there exist scalars $\alpha_1, \ldots , \alpha_{2j+u-2}$ such that

$$\begin{aligned} H_{u,j} &= X^{\langle 2j+u-1\rangle -1}\gamma_j^{d-1}E_{u,j} \\ &= qX^{\langle 2j+u-1\rangle -2}\gamma^1_{j+d-3}X\gamma_j^{d-2}E_{u,j} + \alpha_1 X^{\langle 2j+u-1\rangle -2}\gamma_j^{d-2}E_{u,j} \\ &= q^2X^{\langle 2j+u-1\rangle -3}\gamma^1_{j+d-4}X^2\gamma_j^{d-2}E_{u,j} + (\alpha_1+\alpha_2)X^{<2j+u-1>-2}\gamma_j^{d-2}E_{u,j} \end{aligned}$$

where the third equality follows the following computation

$$\begin{aligned} &\gamma^1_{j+d-4}X^2 = (q^{-1}X\gamma^1_{j+d-3} - q^{-1}\gamma^0_{j+d-3} + q^{j+d-4}G\gamma^0_{j+d-3})X \\ &= q^{-1}X(q^{-1}X\gamma^1_{j+d-2} - q^{-1}\gamma^0_{j+d-2} + q^{j+d-3}G\gamma^0_{j+d-3}) \\ &- q^{-1}\Big[X\gamma^0_{j+d-2} - \frac{\mu}{(d-1)!_q}(\gamma^{d-1}_{j+d-2} - \gamma^{d-1}_{j-2}) + \frac{\mu q^{j+d-1}}{(d-1)!_q}G(\gamma^{d-1}_{j+d-2} - \gamma^{d-1}_{j-2})\Big] \\ &+q^{j+d-4}G\Big[X\gamma^0_{j+d-2} - \frac{\mu}{(d-1)!_q}(\gamma^{d-1}_{j+d-2} - \gamma^{d-1}_{j-2}) + \frac{\mu q^{j+d-1}}{(d-1)!_q}G(\gamma^{d-1}_{j+d-2} - \gamma^{d-1}_{j-2})\Big] \end{aligned}$$

and,

$$\begin{aligned} X^{\langle 2j+u-1\rangle -3}\gamma^1_{j+d-4}X^2\gamma_j^{d-2}E_{u,j} = X^{<2j+u-1>-3}(&q^{-2}X^2\gamma^1_{j+d-2} - q^{-1}X\gamma^0_{j+d-2} \\ &+ q^{j+d-4}GX\gamma^0_{j+d-2} - q^{-2}X\gamma^0_{j+d-2} \\ &+ q^{j+d-4}XG\gamma^0_{j+d-2})\gamma_j^{d-2}E_{u,j}. \end{aligned}$$

Repeat the above process, we have that

$$\begin{aligned} H_{u,j} &= \cdots \\ &= q^{2j+u-2}\gamma^1_{\sigma_u(j)-1}X^{<2j+u-1>-1}\gamma_j^{d-2}E_{u,j} \\ &\quad + (\alpha_1 + \cdots + \alpha_{2j+u-2})X^{<2j+u-1>-2}\gamma_j^{d-2}E_{u,j}. \end{aligned}$$

We now repeat the process on the second term of the last identity, and continue until there is an arrow in front of all the terms; so there exist scalars β_i', β_i'' and β_i such that the following identities hold:

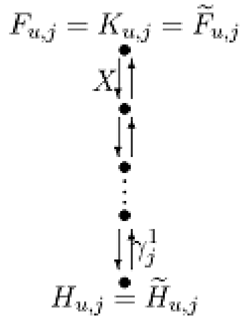
$$\begin{aligned}
H_{u,j} &= q^{2j+u-2}\gamma^1_{\sigma_u(j)-1}X^{<2j+u-1>-1}\gamma_j^{d-2}E_{u,j} + \beta_1' X^{<2j+u-1>-2}\gamma_j^{d-2}E_{u,j}\\
&= q^{2j+u-2}\gamma^1_{\sigma_u(j)-1}X^{<2j+u-1>-1}\gamma_j^{d-2}E_{u,j}\\
&\quad + \beta_1'(q^{2j+u-3}\gamma^1_{\sigma_u(j)-1}X^{<2j+u-1>-2}\gamma_j^{d-3}E_{u,j} + \beta_2'' X^{<2j+u-1>-3}\gamma_j^{d-3}E_{u,j})\\
&= q^{2j+u-2}\gamma^1_{\sigma_u(j)-1}X^{<2j+u-1>-1}\gamma_j^{d-2}E_{u,j} + \beta_1\gamma^1_{\sigma_u(j)-1}X^{<2j+u-1>-2}\gamma_j^{d-3}E_{u,j}\\
&\quad + \beta_2' X^{<2j+u-1>-3}\gamma_j^{d-3}E_{u,j}\\
&= \cdots\\
&= q^{2j+u-2}\gamma^1_{\sigma_u(j)-1}X^{<2j+u-1>-1}\gamma_j^{d-2}E_{u,j}\\
&\quad + \gamma^1_{\sigma_u(j)-1}\sum_{p=2}^{<2j+u-1>}\beta_{p-1}X^{<2j+u-1>-p}\gamma_j^{d-p-1}E_{u,j}\\
&= \gamma^1_{\sigma_u(j)-1}K_{u,j}
\end{aligned}$$

with $K_{u,j}$ nonzero and homogeneous of degree $d- < 2j+u-1 > -1 = d- < 2j+u-1 >^{-} -1$. ■

Definition 2.2. *If $< 2j+u-1 >= d$, set $K_{u,j} = F_{u,j}$. Note that it is homogeneous of degree of $d-1$.*

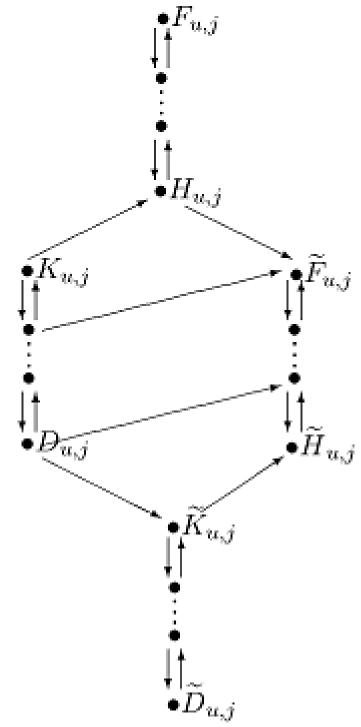
Now consider $\Gamma_u K_{u,j}$. The following result is immediate:

Proposition 2.10. *Assume that $< 2j+u-1 >= d$. Define $L(u,j) := \Gamma_u K_{u,j}$. Then $L(u,j)$ has the following structure:*



In the case $< 2j+u-1 >\neq d$, we obtain the following structure:

Proposition 2.11. *Assume that $< 2j+u-1 >\neq d$. Then module $\Gamma_u K_{u,j}$ has the following structure:*



where $D_{u,j} := X^{d-\langle 2j+u-1\rangle^{-}-1}K_{u,j}, \widetilde{K}_{u,j} = X^{d-\langle 2j-u+1\rangle}$ *and* $\widetilde{D}_{u,j} := X^{d-1}K_{u,j}$. *As before,* $\downarrow$ *denotes the action of* X *and* $\uparrow$ *the action of suitable arrow up to a nonzero scalar.* $\nearrow$ *also denotes the action of suitable arrow up to a nonzero scalar.*

To prove this, we need some preliminaries.

Lemma 2.12. *For* $s < d$, *we have*

$$\gamma^0_{\sigma_u(j)-s}X^{s-1}K_{u,j} = X^{s-1}\gamma^0_{\sigma_u(j)-1}K_{u,j}.$$

Proof.

$$\gamma^0_{\sigma_u(j)-s}X^{s-1}K_{u,j} = \Big[X\gamma^0_{\sigma_u(j)-s+1} - \frac{\mu}{(d-1)!_q}(\gamma^{d-1}_{\sigma_u(j)-s+1} - \gamma^{d-1}_{\sigma_u(j)-s+1-d}) + \frac{\mu q^{\sigma_u(j)-s+1}}{(d-1)!_q}G(\gamma^{d-1}_{\sigma_u(j)-s+1} - \gamma^{d-1}_{\sigma_u(j)-s+1-d})\Big]X^{s-2}K_{u,j}.$$

Using formula $(*2)$ in Proposition 2.4 again and again, it is easy to find that for any path of length $d-1$, we have

$$\gamma^{d-1}_i X^{s-2}K_{u,j} = \sum c_{jl}X^j\gamma^l_m K_{u,j}$$

with $l \geqslant 2$. But, by Lemma 2.9 and Proposition 2.7, $\gamma^l_m K_{u,j} = 0$ for $l \geqslant 2$. Thus

$$\gamma^0_{\sigma_u(j)-s}X^{s-1}K_{u,j} = X\gamma^0_{\sigma_u(j)-s+1}X^{s-2}K_{u,j}.$$

Repeat this process, we get our desire conclusion. ∎

Lemma 2.13. *We have that* $\gamma^t_{\sigma_u(j)-s-1}X^sK_{u,j} = \sum_{b=s-t+1}^{s} q^{-b}c_{b,s}\gamma^{b-s+t-1}_{\sigma_u(j)-b}X^b H_{u,j} + c_{s-t,s}X^{s-t}K_{u,j}$ *where* $c_{s,s} = 1,\ c_{b,s} = 0$ *if* $b < 0$*, and* $c_{b,s} = \zeta_s \cdots \zeta_{b+1}$ *with* $\zeta_s = (s)_q q^{-s}(q^{-(2j+u-1)-s} - 1)$ *if* $0 \le b \le s-1$.

Proof. The proof is by induction on t, and we write out the case $t = 1$ here:

$$\begin{aligned}
&\gamma^1_{\sigma_u(j)-s-1}X^sK_{u,j}\\
&= q^{-1}X\gamma^1_{\sigma_u(j)-s}X^{s-1}K_{u,j} + (-q^{-1} + q^{\sigma_u(j)-s-1}G)\gamma^0_{\sigma_u(j)-s}X^{s-1}K_{u,j}\\
&= q^{-1}X\gamma^1_{\sigma_u(j)-s}X^{s-1}K_{u,j} + (-q^{-1} + q^{\sigma_u(j)-s-1}G)X^{s-1}\gamma^0_{\sigma_u(j)-1}K_{u,j}\\
&= q^{-1}X\gamma^1_{\sigma_u(j)-s}X^{s-1}K_{u,j} + (-q^{-1} + q^{\sigma_u(j)-s-1}G)X^{s-1}K_{u,j}\\
&= q^{-1}X\gamma^1_{\sigma_u(j)-s}X^{s-1}K_{u,j} - (q^{-1} - q^{-2j-u+1-2s})X^{s-1}K_{u,j}\\
&= q^{-1}X(q^{-1}X\gamma^1_{\sigma_u(j)-s+1} - q^{-1}\gamma^0_{\sigma_u(j)-s+1} + q^{\sigma_u(j)-s}G\gamma^0_{\sigma_u(j)-s+1})X^{s-2}K_{u,j}\\
&\quad - (q^{-1} - q^{-2j-u+1-2s})X^{s-1}K_{u,j}\\
&= q^{-2}X^2\gamma^1_{\sigma_u(j)-s+1}X^{s-2}K_{u,j} - q^{-1}X(q^{-1} - q^{-2j-u+1-2s+2})X^{s-2}K_{u,j}\\
&\quad - (q^{-1} - q^{-2j-u+1-2s})X^{s-1}K_{u,j}\\
&= q^{-2}X^2\gamma^1_{\sigma_u(j)-s+1}X^{s-2}K_{u,j}\\
&\quad - (q^{-1} + q^{-2} - q^{-2j-u+1-2s} - q^{-2j-u+1-2s+1})X^{s-1}K_{u,j}\\
&= \cdots\\
&= q^{-s}X^s\gamma^1_{\sigma_u(j)-1}K_{u,j} + \zeta_sX^{s-1}K_{u,j}.
\end{aligned}$$

∎

Lemma 2.14. *For* $0 \le s \le d- < 2j+u-1 >^- -1$*, the elements* $X^sK_{u,j}$ *and* $X^s\widetilde{F}_{u,j}$ *are linearly independent.*

Proof. The proof of this lemma is same to the proof of Lemma 2.18 in [7]. For completeness, we write it out.

Assume that $\alpha X^sK_{u,j} + \beta X^s\widetilde{F}_{u,j} = 0$ with $0 \le s \le d- < 2j+u-1 >^- -1$. Multiply by $\gamma^{s+1}_{\sigma_u(j)-1-s}$. Using Lemma 2.8, $\gamma^{s+1}_{\sigma_u(j)-1-s}X^s\widetilde{F}_{u,j}$ is a multiple of $(q^{2j+u-1-(s+<2j+u-1>-(s+1)+1)} - 1)X^{<2j+u-1>}F_{u,j}$ which is zero.

On the other hand, by using Lemma 2.13, $\gamma^{s+1}_{\sigma_u(j)-1-s}X^sK_{u,j}$ is equal to $\sum_{b=0}^{s} q^{-b}c_{b,s}\ \gamma^b_{\sigma_u(j)-b}X^bH_{u,j}$. Now $\zeta_p = 0$ if and only if $p \equiv -2j-u+1\ (mod\ d)$. But if $0 \le b+1 \le p \le s \le d- < 2j+u-1 >^- -1$, we have $\zeta_p \ne 0$ and therefore $c_{b,s} \ne 0$ for all b, s with $0 \le b+1 \le p \le s \le d- < 2j+u-1 >^- -1$.

So multiplying the identity $\alpha X^sK_{u,j} + \beta X^s\widetilde{F}_{u,j} = 0$ by $\gamma^{s+1}_{\sigma_u(j)-1-s}$ gives a nonzero multiple of $\alpha\gamma^{s+1}_{\sigma_u(j)-1-s}X^sK_{u,j}$ with $\gamma^{s+1}_{\sigma_u(j)-1-s}X^sK_{u,j}$ nonzero, so $\alpha = 0$ and therefore $\beta = 0$. ∎

Proof of Proposition 2.11. We apply Lemma 2.13 with $t = d-1 = s$ to see that $\gamma^{d-1}_{\sigma_u(j)-d}X^{d-1}K_{u,j}$ is a nonzero multiple of $\widetilde{F}_{u,j}$:

If $b \leq d - \langle 2j+u-1\rangle^- - 1$, then $c_{b,d-1} = 0$;

If $b \geqslant d - \langle 2j+u-1\rangle^- + 1$, then $\gamma^{b-1}_{\sigma_u(j)-b}X^b H_{u,j} = 0$;

Finally, if $b = d - \langle 2j+u-1\rangle^-$, then $\gamma^{d-1}_{\sigma_u(j)-d}X^{d-1}K_{u,j}$ is a nonzero multiple of $X^{\langle 2j+u-1\rangle}\gamma^{d-1}_j E_{u,j}$.

Thus, in particular, $X^s K_{u,j} \neq 0$ for all $s = 0, \ldots, d-1$. The rest of this structure follows this and Lemma 2.14. ■

By Proposition 2.11, it is easy to find all possible submodules of $\Gamma_u K_{u,j}$.

Corollary 2.15. *When $\langle 2j+u-1\rangle \neq d$, the module $\Gamma_u K_{u,j}$ has exactly two composition series:*

$$\Gamma_u K_{u,j} \supset \Gamma_u F_{u,j} + \Gamma_u \widetilde{D}_{u,j} \supset \Gamma_u F_{u,j} \supset \Gamma_u \widetilde{F}_{u,j} \supset 0$$

and

$$\Gamma_u K_{u,j} \supset \Gamma_u F_{u,j} + \Gamma_u \widetilde{D}_{u,j} \supset \Gamma_u \widetilde{D}_{u,j} \supset \Gamma_u \widetilde{F}_{u,j} \supset 0. \qquad ■$$

When $\langle 2j+u-1\rangle \neq d$, we also define $L(u,j) := \Gamma_u K_{u,j}/(\Gamma_u F_{u,j} + \Gamma_u \widetilde{D}_{u,j})$. By this corollary, it is a simple module of dimension $d - \langle 2j+u-1\rangle^-$, and the composition factors of the composition series in this corollary are $L(u,j)$, $L(u,\sigma_u(j))$, $L(u,\sigma_u^{-1}(j))$ and $L(u,j)$.

To decompose Γ_u into a sum of indecomposable modules, we find modules isomorphic to the $\Gamma_u K_{u,j}$ inside Γ_u.

Lemma 2.16. *If $0 \leq h \leq d - < 2j+u-1 >^- -1$, then right multiplication by X^h induces an isomorphism $\Gamma_u K_{u,j} \xrightarrow{\cong} \Gamma_u K_{u,j}X^h$ of Γ_u-modules.*

Proof. The proof is very similar to the proof of Lemma 2.22 in [7].

In order to prove this lemma, we must show that right multiplication by X^h embeds $\Gamma_u K_{u,j}$ into $\Gamma_u K_{u,j}X^h$. Thus it is enough to prove for h maximal. For this, we only need to check that $\widetilde{H}_{u,j}X^h \neq 0$ and $\widetilde{D}_{u,j}X^h \neq 0$. Since, by Proposition 2.11, $\widetilde{H}_{u,j}$ equals the multiplication of $\widetilde{D}_{u,j}$ and some arrows, $\widetilde{H}_{u,j}X^h \neq 0$ implies $\widetilde{D}_{u,j}X^h \neq 0$. Therefore, we only need to consider $\widetilde{H}_{u,j}X^{d-<2j+u-1>^- -1}$.

To compute this, we need a relation similar to that in Lemma 2.8:

$$\begin{aligned} &X^{d-1}\gamma^{d-1}_j E_{u,j} X^b \\ &= q^{-\frac{b(2(d-1)-b+1)}{2}} \frac{(d-1)!_q}{(d-1-b)!_q} \prod_{t=1}^{b} (q^{2j+(d-1)+u+t} - 1) X^{d-1}\gamma^{d-1-b}_{j+b} E_{u,j+b}. \end{aligned}$$

This formula has been given in [7] (proof of Lemma 2.22). It is easy to see that it is also true for our case. Using this relation, we see that $\widetilde{H}_{u,j}X^{d-<2j+u-1>^- -1}$ is a nonzero multiple of

$$\prod_{t=1}^{d-<2j+u-1>^{-}-1} (q^{2j-1+u+t}-1)X^{d-1}\gamma_{j+d-<2j+u-1>^{-}-1}^{<2j+u-1>^{-}}E_{u,-j-u}$$

which is nonzero. ∎

We now can decompose Γ_u into a sum of indecomposable modules. Note that if $< 2j+u-1 >\neq d$, the module $\Gamma_u K_{u,j}$ has dimension $2d$ while if $< 2j+u-1 >= d$, it has dimension d.

Theorem 2.17. *Γ_u decomposes into a direct sum of indecomposable modules in the following way:*

$$\Gamma_u = \bigoplus_{j\in Z_n} \bigoplus_{h=0}^{d-<2j+u-1>^{-}-1} \Gamma_u K_{u,j} X^h.$$

Proof. We first prove the sum is direct. The sums over h are direct since the summands are in different right G-eigenspaces. The outer sum is also direct because the summands have non-isomorphic socle: the socle of

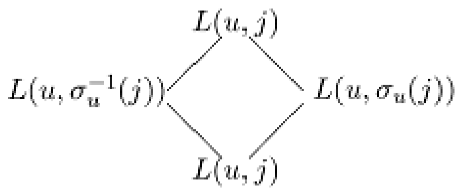
$$\bigoplus_{h=0}^{d-<2j+u-1>^{-}-1} \Gamma_u K_{u,j} X^h \text{ is } \bigoplus_{h=0}^{d-<2j+u-1>^{-}-1} L(u,j)X^h$$

with $L(u,j)X^h \cong L(u,j)$.

Equality follows from dimension counting. For example, when d is odd, $\dim\bigoplus_{j\in Z_n}\bigoplus_{h=0}^{d-<2j+u-1>^{-}-1} \Gamma_u K_{u,j}X^h = \frac{n}{d}(d{\cdot}d+2d\frac{d(d-1)}{2}) = nd+nd(d-1) = nd^2$ and thus $n\dim\bigoplus_{j\in Z_n}\bigoplus_{h=0}^{d-<2j+u-1>^{-}-1} \Gamma_u K_{u,j}X^h = n^2d^2 = dim\mathcal{D}(\Gamma_{n,d})$. Thus $\Gamma_u = \bigoplus_{j\in Z_n}\bigoplus_{h=0}^{d-<2j+u-1>^{-}-1}\Gamma_u K_{u,j}X^h$. ∎

Corollary 2.18. *Set $P(u,j) = \Gamma_u K_{u,j}$ for all u, j. The modules $P(u,j)$ are projective, and they represent the different isomorphism classes of projective $\mathcal{D}(\Gamma_{n,d})$-modules when u and j vary in Z_n.*

When $< 2j+u-1 >\neq d$, their structure is

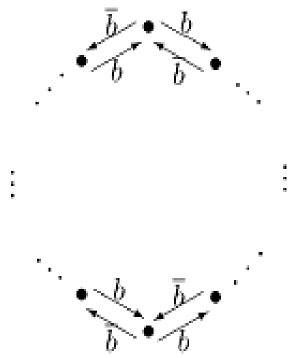


When $< 2j+u-1 >= d$, $P(u,j) = L(u,j)$ is simple.

Moreover, the $L(u,j)$ represent all the isomorphism classes of simple $\mathcal{D}(\Gamma_{n,d})$-modules when u and j vary in Z_n. Those of dimension d are also projective, and there are $\frac{n^2}{d}$ projective simple.

With these preparations, we can give the quiver with relations of $\mathcal{D}(\Gamma_{n,d})$.

Theorem 2.19. *The quiver of* $\mathcal{D}(\Gamma_{n,d})$ *has* $\frac{n^2}{d}$ *isolated vertices which correspond to the simple projective modules, and* $\frac{n(d-1)}{2}$ *copies of the quiver*



with $\frac{2n}{d}$ *vertices and* $\frac{4n}{d}$ *arrows. The relations on this quiver are* bb, $\overline{b}\overline{b}$ *and* $b\overline{b}-\overline{b}b$. *The vertices in this quiver correspond to the simple modules* $L(u,j), L(u,\sigma_u(j))$, $\cdots, L\big(u,\sigma_u^{\frac{2n}{d}-1}(j)\big)$.

Proof. The proof is same to that of Theorem 2.25 in [7]. ∎

An algebra A is said to be of *finite representation type* provided there are finitely many non-isomorphic indecomposable A-modules. A is of *tame type* or A is a *tame* algebra if A is not of finite representation type, whereas for any dimension $d > 0$, there are finite number of A-$k[T]$-bimodules M_i which are free as right $k[T]$-modules such that all but a finite number of indecomposable A-modules of dimension d are isomorphic to $M_i \otimes_{k[T]} k[T]/(T-\lambda)$ for $\lambda \in k$. The following conclusion is our main aim.

Theorem 2.20. $\mathcal{D}(\Gamma_{n,d})$ *is a tame algebra.*

Proof. By Theorem 2.19, we know that $\mathcal{D}(\Gamma_{n,d})$ is a special biserial algebra (for definition, see [6]) and thus it is tame or of finite representation type (see II.3.1 of [6]).

Given a quiver Γ, we associate with Γ the following quiver Γ_s called the *separated quiver* of Γ. If $\{1,\dots,n\}$ are the vertices of Γ, then the vertices of Γ_s are $\{1,\dots,n,1',\dots,n'\}$. For each arrow $\overset{i}{\cdot} \longrightarrow \overset{j}{\cdot}$ in Γ, we have by definition an arrow $\overset{i}{\cdot} \longrightarrow \overset{j'}{\cdot}$ in Γ_s. It is known that for a finite quiver Q, path algebra kQ is of finite representation type if and only if the underlying graph $\overline{Q}$ of Q is one of finite Dynkin diagrams: A_n, D_n, E_6, E_7, E_8.

Clearly, the separated quiver of the quiver drawn in Theorem 2.19 is not a union of finite Dynkin diagrams. Indeed, it has two components $\widetilde{A}_{\frac{2n}{d}-1}$ (Euclidean diagram). Let J denote the Jacobson radical of $\mathcal{D}(\Gamma_{n,d})$. Since the separated quiver of $\mathcal{D}(\Gamma_{n,d})$ is not a union of finite Dynkin diagrams and the quivers of $\mathcal{D}(\Gamma_{n,d})$ and $\mathcal{D}(\Gamma_{n,d})/J^2$ are identical, Theorem 2.6 in Chapter X of [1] implies $\mathcal{D}(\Gamma_{n,d})/J^2$ is not of finite representation type. Thus $\mathcal{D}(\Gamma_{n,d})$ is not of finite representation type and $\mathcal{D}(\Gamma_{n,d})$ is tame. ■

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