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Weakly *d*-Koszul Modules

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Abstract. Let A be a d-Koszul algebra and $M \in gr(A)$, we show that M is a weakly d-Koszul module if and only if $\mathcal{E}(G(M)) = \oplus_{n\geq 0} \mathrm{Ext}\,_A^n(G(M), A_0)$ is generated in degree 0 as a graded $E(A)$-module. Moreover, relations among weakly d-Koszul modules, d-Koszul modules and Koszul modules are discussed. We also show that the Koszul dual of a weakly d-Koszul module M: $\mathcal{E}(M) = \oplus_{n\geq 0} \mathrm{Ext}\,_A^n(M, A_0)$ is finitely generated as a graded $E(A)$-module.

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1. Introduction

This paper is a continuation work of [9]. The concept of weakly d-Koszul module, which is a generalizaion of d-Koszul module, is firstly introduced in [9]. This class of modules resemble classical d-Koszul modules in the way that they admits a tower of d-Koszul modules. It is well known that both Koszul modules and d-Koszul modules are pure and they have many nice homological properties. From [9], we know that although weakly d-Koszul modules are not pure, they have many perfect properties similar to d-Koszul modules.

Using Koszul dual to characterize Koszul modules is another effective aspect. For Koszul and d-Koszul modules, we have the following well known results from [4] and [6].

- *Let A be a Koszul algebra and $M \in gr_s(A)$. Then M is a Koszul module if and only if the Koszul dual $\mathcal{E}(M) = \oplus_{n\geq 0} Ext\,_A^n(M, A_0)$ is generated in degree 0 as a graded $E(A)$-module.*

- *Let A be a d-Koszul algebra and $M \in gr_s(A)$. Then M is a d-Koszul module if and only if the Koszul dual $\mathcal{E}(M) = \oplus_{n\geq 0}Ext\,^n_A(M, A_0)$ is generated in degree 0 as a graded $E(A)$-module.*

It is a pity that we cannot get the similar result for weakly d-Koszul module though it is a generalizaion of d-Koszul module. We only have a necessary condition for weakly d-Koszul modules (see [9]):

- *Let M be a weakly d-Koszul module with homogeneous generators being of degrees d_0 and d_1 ($d_0 < d_1$). Then $\mathcal{E}(M)$ is generated in degrees 0 as a graded $E(A)$-module.*

One of the aims of this paper is to get a similar equivalent description for weakly d-Koszul modules. In order to do this, we cite the notion of the *associated graded module of a module,* denoted by $G(M)$, the formal definition will be given later. If we replace the weakly d-Koszul module M by $G(M)$, we can get the similar result:

- *Let A be a d-Koszul algebra and $M \in gr(A)$. Then M is a weakly d-Koszul module if and only if $\mathcal{E}(G(M)) = \oplus_{n\geq 0}Ext\,^n_A(G(M), A_0)$ is generated in degree 0 as a graded $E(A)$-module.*

From this point of view, weakly d-Koszul modules have a close relation between classical d-Koszul modules and Koszul modules.

It is well known that to determine whether the Koszul dual $\mathcal{E}(M)$ is finitely generated or not is very difficult in general. In this paper, we show that $\mathcal{E}(M)$ is finitely generated as a graded $E(A)$-module for a weakly d-Koszul module M, which is an application of Theorem 2.5 [9] and another main result of this paper.

The paper is organized as follows. In Sec. 2, we introduce some easy definitions and notations which will be used later. In Sec. 3, we investigate the relations between weakly d-Koszul modules and d-Koszul modules. Moreover, we construct a lot of classical d-Koszul and Koszul modules from a given weakly d-Koszul module. As we all know, using Koszul dual to characterize Koszul modules is another effective aspect. For weakly d-Koszul modules, we prove that M is a weakly d-Koszul module if and only if $\mathcal{E}(G(M)) = \oplus_{n\geq 0}\mathrm{Ext}\,^n_A(G(M), A_0)$ is generated in degree 0 as a graded $E(A)$-module. In the last section, we show that the Koszul dual of a weakly d-Koszul module M: $\mathcal{E}(M) = \oplus_{n\geq 0}\mathrm{Ext}\,^n_A(M, A_0)$ is finitely generated as a graded $E(A)$-module.

We always assume that $d \geq 2$ is a fixed integer in this paper.

2. Notations and Definitions

Throughout this paper, $\mathbb{F}$ denotes a field and $A = \bigoplus_{i\geq 0} A_i$ is a graded $\mathbb{F}$-algebra such that (a) A_0 is a semi-simple Artin algebra, (b) A is generated in degree zero and one; that is, $A_i \cdot A_j = A_{i+j}$ for all $0 \leq i, j < \infty$, and (c) A_1 is a finitely generated $\mathbb{F}$-module. The graded Jacobson radical of A, which we denote by J, is $\bigoplus_{i\geq 1} A_i$. We are interested in the category $Gr(A)$ of graded A-modules, and its full subcategory $gr(A)$ of finitely generated modules. The morphisms in these categories, denoted by $\mathrm{Hom}_{Gr(A)}(M, N)$, are the A-module maps of degree zero. We denote by $Gr_s(A)$ and $gr_s(A)$ the full subcategory of $Gr(A)$ and $gr(A)$

respectively, whose objects are generated in degree s. An object in $Gr_s(A)$ or $gr_s(A)$ is called a *pure* A-module.

Endowed with the Yoneda product, $\text{Ext}\,^*_A(A_0, A_0) = \bigoplus_{i\geq 0} \text{Ext}\,^i_A(A_0, A_0)$ is a graded algebra which is usually called Yoneda-Ext-algebra of A. Let M and N be finitely generated graded A-modules. Then

$$\text{Ext}\,^*_A(M,N) = \bigoplus_{i\geq 0} \text{Ext}\,^i_A(M,N)$$

is a graded left $\text{Ext}\,^*_A(N,N)$-module. For simplicity, we write $E(A) = \text{Ext}\,^*_A(A_0, A_0)$, and $\mathcal{E}(M) = \text{Ext}\,^*_A(M, A_0)$ which is a graded $E(A)$-module, usually called the Koszul dual of M.

Form [6], we know that the Koszul $\mathcal{E}(M)$ of a graded module M is bigraded; that is, if $[x] \in \text{Ext}^n_A(M, A_0)_s$, we denote the degrees of $[x]$ as (n, s), call the first degree *ext*-degree and the second degree *shift*-degree.

For the sake of convenience, we introduce a function $\delta : \mathbb{N}\times\mathbb{Z} \to \mathbb{Z}$ as follows. For any $n \in \mathbb{N}$ and $s \in \mathbb{Z}$,

$$\delta(n,s) = \begin{cases} \frac{nd}{2} + s, & \text{if } n \text{ is even,} \\ \frac{(n-1)d}{2} + 1 + s, & \text{if } n \text{ is odd.} \end{cases}$$

When $s = 0$, we usually write $\delta(n, 0) = \delta(n)$, as introduced in some other literatures before.

Definition 2.1.[6] *A graded algebra $A = \bigoplus_{i\geq 0} A_i$ is called a d-Koszul algebra if the trivial module A_0 admits a graded projective resolution*

$$\mathbf{P} : \quad \cdots \to P_n \to \cdots \to P_1 \to P_0 \to A_0 \to 0,$$

such that P_n is generated in degree $\delta(n)$ for all $n \geq 0$. In particular, A is a Koszul algebra when $d = 2$.

Definition 2.2. *Let A be a d-Koszul algebra. For $M \in gr(A)$, we call M a d-Koszul module if there exists a graded projective resolution*

$$\mathbf{Q} : \quad \cdots \to Q_n \xrightarrow{f_n} \cdots \to Q_1 \xrightarrow{f_1} Q_0 \xrightarrow{f_0} M \to 0,$$

and a fixed integer s such that for each $n \geq 0$, Q_n is generated in degree $\delta(n, s)$.

From the definition above, it is easy to see that d-Koszul modules are pure since Q_0 is pure. Similarly, when $d = 2$, d-Koszul module is just the Koszul module introduced in [4].

Definition 2.3. *Let A be a d-Koszul algebra. We say that $M \in gr(A)$ is a weakly d-Koszul module if there exists a minimal graded projective resolution of M:*

$$\mathbf{Q} : \cdots \to Q_i \xrightarrow{f_i} \cdots \longrightarrow Q_1 \xrightarrow{f_1} Q_0 \xrightarrow{f_0} M \to 0,$$

such that for $i, k \geq 0$, $J^k \ker f_i = J^{k+1}Q_i \cap \ker f_i$ if i is even and $J^k \ker f_i = J^{k+d-1}Q_i \cap \ker f_i$ if i is odd.

We usually call kerf_{n-1} the n^{th} syzygy of M, which is sometimes written as $\Omega^n(M)$. From Definitions 2.2 and 2.3, we can get the following easy Proposition.

Proposition 2.4. *Let A be a d-Koszul algebra and $M \in gr(A)$. Then we have the following statements.*
(1) *If M is a d-Koszul module, then M is a weakly d-Koszul module,*
(2) *Let M be pure. Then M is a d-Koszul module if and only if M is a weakly d-Koszul module.*

Proof. It is routine to check. ■

Our definition of weakly d-Koszul modules agrees with the definition of weakly Koszul modules introduced in [11] when $d = 2$. Theorem 4.3 in [11] proved that M is a weakly Koszul module if and only if $\mathcal{E}(M)$ is a Koszul $E(A)$-module. We will show that M is a weakly d-Koszul module if and only if $G(M)$ is a d-Koszul A-module, where $d > 2$ in the following section.

3. The Relations Between Weakly d-Koszul Modules and Classical d-Koszul and Koszul Modules

In this section, we will investigate the relations between weakly d-Koszul modules and classical d-Koszul and Koszul modules. To do this, we construct classical d-Koszul and Koszul modules from the given weakly d-Koszul modules. We also provide a criteria theorem for a finitely generated graded module to be a weakly d-Koszul module in terms of the *associated graded module* of it and the Koszul dual of M.

Let A be a graded $\mathbb{F}$ algebra and $M \in gr(A)$, we can get another graded module, denoted by $G(M)$, called the *associated graded module of* M as follows:

$$G(M) = M/JM \oplus JM/J^2M \oplus J^2M/J^3M \oplus \cdots .$$

Similarly, we can define $G(A)$ for a graded algebra.

Proposition 3.1. *Let A be a graded $\mathbb{F}$-algebra and $M \in gr(A)$. Then*
(1) *$G(A) \cong A$ as a graded $\mathbb{F}$-algebra,*
(2) *$G(M)$ is a finitely generated graded A-module,*
(3) *If M is pure, then $G(M) \cong M$ as a graded A-module.*

Proof. By the definition, $G(A)_i = J_i/J_{i+1} = A_i$ for all $i \geq 0$ since the graded $\mathbb{F}$-algebra $A = A_0 \oplus A_1 \oplus \cdots$ is generated in degrees 0 and 1. Now the first assertion is clear. For the second assertion, by (1), we only need to prove that $G(M)$ is a graded $G(A)$-module. We define the module action as follows:

$$\mu : G(A) \otimes G(M) \longrightarrow G(M)$$

via

$$\mu((a + J^iA) \otimes (m + J^jM)) = a \cdot m + J^{i+j-1}M$$

for all $a + J^iA \in G(A)$ and $m + J^jM \in G(M)$. It is easy to check that μ is well-defined and under μ, $G(M)$ is a graded $G(A)$-module. The proof of the third assertion is similar to (1) and we omit it. ∎

Lemma 3.2. *Let $0 \to K \to M \to N \to 0$ be a split exact sequence in $gr(A)$, where A is a d-Koszul algebra. Then M is a d-Koszul module if and only if K and N are both d-Koszul modules.*

Proof. It is obvious that we have the following commutative diagram with exact rows and columns since $0 \to K \to M \to N \to 0$ is a split exact sequence,

$$\begin{array}{ccccccccc} & & \vdots & & \vdots & & \vdots & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & P_2 & \longrightarrow & P_2 \oplus Q_2 & \longrightarrow & Q_2 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & P_1 & \longrightarrow & P_1 \oplus Q_1 & \longrightarrow & Q_1 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & P_0 & \longrightarrow & P_0 \oplus Q_0 & \longrightarrow & Q_0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

where $\mathbf{P}$, $\mathbf{P} \oplus \mathbf{Q}$ and $\mathbf{Q}$ are the minimal graded projective resolutions of K, M and N respectively. It is evident that $\mathbf{P} \oplus \mathbf{Q}$ is generated in degree s if and only if both $\mathbf{P}$ and $\mathbf{Q}$ are generated in degree s, which implies that M is a d-Koszul module if and only if K and N are both d-Koszul modules. ∎

Corollary 3.3. *Let M be a finite direct sum of finitely generated graded A-modules and A be a d-Koszul algebra. That is, $M = \bigoplus_{i=1}^{n} M_i$. Then M is a d-Koszul module if and only if all M_i are d-Koszul modules.*

Proof. It is immediate from Lemma 3.2. ∎

Lemma 3.4. [9] *Let $M = \bigoplus_{i \geq 0} M_i$ be a weakly d-Koszul module with $M_0 \neq 0$. Set $K_M = \langle M_0 \rangle$. Then*
(1) *K_M is a d-Koszul module;*
(2) *$K_M \cap J^kM = J^kK_M$ for each $k \geq 0$;*
(3) *M/K_M is a weakly d-Koszul module.*

Lemma 3.5. [9] *Let $0 \to K \to M \to N \to 0$ be an exact sequence in $gr(A)$ and A be a d-Koszul module. Then we have the following statements:*
(1) *If K and M are weakly d-Koszul modules with $J^kK = K \cap J^kM$ for all $k \geq 0$, then N is a weakly d-Koszul module.*
(2) *If K and N are weakly d-Koszul modules with $JK = K \cap JM$, then M is a weakly d-Koszul module.*

Lemma 3.6. [9] *Let $0 \to K \to M \to N \to 0$ be an exact sequence in $gr(A)$.*

Then the following statements are equivalent:
(1) $J^kK = K \cap J^kM$ *for all* $k \geq 0$*;*
(2) $A/J^k \otimes_A K \to A/J^k \otimes_A M$ *is a monomorphism for all* $k \geq 0$*;*
(3) $0 \to J^kK \to J^kM \to J^kN \to 0$ *is exact for all* $k \geq 0$*;*
(4) $0 \to J^kK/J^{k+1}K \to J^kM/J^{k+1}M \to J^kN/J^{k+1}N \to 0$ *is exact for all* $k \geq 0$*;*
(5) $0 \to J^kK/J^mK \to J^kM/J^mM \to J^kN/J^mN \to 0$ *is exact for all* $m > k$*.*

Theorem 3.7. *Let* A *be a graded* $\mathbb{F}$*-algebra and* $M = M_{k_0} \oplus M_{k_1} \oplus M_{k_2} \oplus \cdots$ *be a finitely generated* A*-module with* $M_{k_0} \neq 0$*. Let* $K = \langle M_{k_0} \rangle$ *be the graded submodule of* M *generated by* M_{k_0}*. Then we have a split exact sequence in* $gr(G(A)) = gr(A)$

$$0 \to G(K) \to G(M) \to G(M/K) \to 0.$$

Proof. Set $M/K = N$ for simplicity. By Lemma 3.4(2), we get a short exact sequence

$$0 \to K \to M \to N \to 0$$

with $J^kK = K \cap J^kM$ for all $k \geq 0$. By Lemma 3.6, we have the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & J^{k+1}K & \longrightarrow & J^{k+1}M & \longrightarrow & J^{k+1}N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & J^kK & \longrightarrow & J^kM & \longrightarrow & J^kN & \longrightarrow & 0
\end{array}$$

where the vertical arrows are natural embeddings. By the "Snake Lemma", we can get the following exact sequence

$$0 \to J^kK/J^{k+1}K \to J^kM/J^{k+1}M \to J^kN/J^{k+1}N \to 0$$

for all $k \geq 0$. Applying the exact functor "$\bigoplus$" to the above exact sequence, we have

$$0 \to \bigoplus J^kK/J^{k+1}K \to \bigoplus J^kM/J^{k+1}M \to \bigoplus J^kN/J^{k+1}N \to 0.$$

That is, we have the exact sequence

$$0 \to G(K) \to G(M) \to G(M/K) \to 0.$$

Now we claim that the above exact sequence splits. Since M is finitely generated, it is no harm to assume that the generators lie in degree $k_0 < k_1 < \cdots < k_p$ parts and $k_0 = 0$. For each j, let S_{k_j} denote a A_0 complement in M_{k_j} of the degree k_j part of the submodule of M generated by the degree k_0, k_1, $\cdots$, k_{j-1} parts. Let $S = S_{k_1} \oplus \cdots \oplus S_{k_p}$. Then it is easy to see that $M/JM = M_0 \oplus S$, $G(M) = G(K) + \langle S \rangle$ and $\langle S \rangle = G(N)$, and at the degree 0 part, we have $G(M)_0 = M/JM = M_0 \oplus S$. Now we only need to show that $G(M) = G(K) \oplus \langle S \rangle$. Indeed, let $\bar{x} \in G(K) \cap \langle S \rangle$ be a homogeneous element of degree i, then $\bar{x} = \sum \bar{a}\bar{y}$ where $\bar{a} = a + J^{i+1} \in G(A)_i$ and $\bar{y} = y + JK \in G(K)_0$ since $\bar{x} \in G(K)$. On the other hand, since $\bar{x} \in \langle S \rangle$, we can write $\bar{x}$ in the form

$$\bar{x} = \sum \bar{\alpha}\bar{\mu} + \sum \bar{\beta}\bar{\nu} + \cdots ,$$

where $\bar{\alpha}$, $\bar{\beta}$, $\cdots$ are in $G(A)_i$ and $\bar{\mu} = \mu + JM$ with $\mu \in M_{k_1}$, $\bar{\nu} = \nu + JM$ with $\nu \in M_{k_2}$, $\cdots$. Hence in M we have $\sum ay - (\sum \alpha\mu + \sum \beta\nu + \cdots) \in J^{i+1}M$, since the degree of $\sum ay$ is i and that of $\sum \alpha\mu$ is $i + k_1$, $\cdots$, which implies that $\bar{x} = 0$. Therefore the exact sequence

$$0 \to G(K) \to G(M) \to G(M/K) \to 0$$

splits. ■

Now we can investigate the relations between weakly d-Koszul modules and d-Koszul modules, the following theorem also provides a criteria theorem for a finitely generated graded module to be a weakly d-Koszul module in terms of the associated graded module of it and the Koszul dual of M.

Theorem 3.8. *Let A be a d-Koszul algebra and $M \in gr(A)$. Then the following are equivalent,*

(1) *M is a weakly d-Koszul module,*

(2) *$G(M)$ is a d-Koszul module,*

(3) *The Koszul dual of $G(M)$, $\mathcal{E}(G(M)) = \bigoplus_{n\geq 0} \mathrm{Ext}\,_A^n(G(M), A_0)$ is generated in degree 0 as a graded $E(A)$-module.*

Proof. We only need to prove the equivalence between assertion (1) and assertion (2), since the equivalence between assertion (2) and assertion (3) is obvious from [6]. Since M is finitely generated, assume that M is generated by a minimal set of homogeneous elements lying in degrees $k_0 < k_1 < \cdots < k_p$. Set $K = \langle M_{k_0} \rangle$. By Theorem 3.7, we get a split exact sequence

$$0 \to G(K) \to G(M) \to G(N) \to 0.$$

Now suppose assertion (1) holds, we prove (2) by induction on p. If $p = 0$, M is a pure weakly d-Koszul module, by Proposition 2.4 and Proposition 3.1, we get that M is a d-Koszul module and $M \cong G(M)$ as a graded A-module. Hence $G(M)$ is a d-Koszul module. Now we assume that the statement holds for less than p. By Lemma 3.4, K is a d-Koszul module, by Proposition 2.4, K is a weakly d-Koszul module. Consider the exact sequence $0 \to K \to M \to N \to 0$, by Lemmas 3.4 and 3.5, we get that N is a weakly d-Koszul module. Since the number of generators of N is less than p, by the induction assumption, $G(N)$ is a d-Koszul module. Since $G(K)$ is obviously a d-Koszul module, by Proposition 3.2, we get that $G(M)$ is a d-Koszul module.

Conversely, assume that $G(M)$ is a d-Koszul module, by Proposition 3.2, we get that $G(K)$ and $G(N)$ are d-Koszul modules. By the induction assumption, K and N are weakly d-Koszul modules. By Lemma 3.5 and Lemma 3.4, we get that M is a weakly d-Koszul module. ■

Proposition 3.9. *Let A be a d-Koszul algebra and M be a d-Koszul module. Then for all integers $k \geq 1$, we have $\mathbf{E_k}(M) = \oplus_{n\geq 0} Ext\ _A^{2kn}(M, A_0)$ is a Koszul module.*

Proof. We claim that $\mathbf{E_k}(M)$ is generated in degree 0 as a graded $\mathbf{E_k}(A)$-module. In fact, ${\mathbf{E_k}}^n(M) = \mathrm{Ext}\ _A^{2kn}(M, A_0) = \mathrm{Ext}\ _A^{2kn}(A_0, A_0)\cdot\mathrm{Hom}_A(M, A_0) = {\mathbf{E_k}}^n(A)\cdot \mathrm{Hom}_A(M, A_0) = {\mathbf{E_k}}^n(A) \cdot {\mathbf{E_k}}^0(M)$.

Similar to the proof of Theorem 6.1 in [6], we have the following exact sequences for all $n, k \in \mathbb{N}$:

$$0 \to \mathrm{Ext}\ _A^{2kn-1}(JM, A_0) \to \mathrm{Ext}\ _A^{2kn}(M/JM, A_0) \to \mathrm{Ext}\ _A^{2kn}(M, A_0) \to 0$$

such that all the modules in the above exact sequences are concentrated in degree $\delta(2nk,\, 0)$ in the shift-grading.

We have the following exact sequences since

$$\mathrm{Ext}\ _A^{2kn-1}(JM, A_0) = \mathrm{Ext}\ _A^{2k(n-1)}(\Omega^{2k-1}(JM), A_0),$$
$$0 \to \mathrm{Ext}\ _A^{2k(n-1)}(\Omega^{2k-1}(JM), A_0) \to \mathrm{Ext}\ _A^{2kn}(M/JM, A_0) \to \mathrm{Ext}\ _A^{2kn}(M, A_0) \to 0.$$

By taking the direct sums of the above exact sequences, we have

$$\begin{aligned} 0 &\to \oplus_{n\geq 0}\mathrm{Ext}\ _A^{2k(n-1)}(\Omega^{2k-1}(JM), A_0) \\ &\to \oplus_{n\geq 0}\mathrm{Ext}\ _A^{2kn}(M/JM, A_0) \to \oplus_{n\geq 0}\mathrm{Ext}\ _A^{2kn}(M, A_0) \to 0. \end{aligned}$$

Now we claim that $\mathbf{E_k}(M/JM)$ is a projective cover of $\mathbf{E_k}(M)$ and it is generated in degree 0. In fact, $\mathbf{E_k}(M/JM)$ is a $\mathbf{E_k}(A)$-projective module since M/JM is semi-simple. M/JM is a d-Koszul module since A is a d-Koszul algebra. We have proved that if M is a d-Koszul module, then $\mathbf{E_k}(M)$ is generated in degree 0 as a graded $\mathbf{E_k}(A)$-module. Hence $\mathbf{E_k}(M/JM)$ is generated in degree 0 as a graded $\mathbf{E_k}(A)$-module and it is the graded projective cover of $\mathbf{E_k}(M)$.

Therefore the first syzygy is $\oplus_{n\geq 0}\mathrm{Ext}\ _A^{2k(n-1)}(\Omega^{2k-1}(JM), A_0)$, from [6], we have that $\Omega^{2k-1}(JM)$ is generated in degree $\delta(2k,\, 0)$ and clearly $\Omega^{2k-1}(JM)$ is again a d-Koszul module. To complete the proof of this proposition, we only need to show that $\oplus_{n\geq 0}\mathrm{Ext}\ _A^{2k(n-1)}(\Omega^{2k-1}(JM), A_0)$ is generated in degree 1. It is obvious that $\mathbf{E_k}(\Omega^{2k-1}(JM)[-kd])$ is generated in degree 0. In the shift-grading, $\oplus_{n\geq 0}\mathrm{Ext}\ _A^{2k(n-1)}(\Omega^{2k-1}(JM), A_0)$ is generated in degree $\delta(2k,\, 0) = kd$. By the definition of $\mathbf{E_k}(\Omega^{2k-1}(JM))$, we have that $\mathbf{E_k^1}(\Omega^{2k-1}(JM)) = \mathrm{Ext}\ _A^{2k}(\Omega^{2k-1}(JM), A_0) = \mathrm{Ext}\ _A^{2k}(\Omega^{2k-1}(JM), A_0)_{kd}$, it follows that $\oplus_{n\geq 0}$ $\mathrm{Ext}\ _A^{2k(n-1)}(\Omega^{2k-1}(JM), A_0)$ is generated in degree 1. By an induction, we finish the proof. ∎

As some applications of Theorem 3.8, we can discuss the relations among weakly d-Koszul modules, d-Koszul modules and Koszul modules.

Corollary 3.10. *Let M be a weakly d-Koszul module. Then*
(1) *All the $2n^{th}$ syzygies of $G(M)$ denoted by $\Omega^{2n}(G(M))$ are d-Koszul modules,*

(2) *For all $n \geq 0$, all the Koszul duals of $\Omega^{2n}(G(M))$, $\mathcal{E}(\Omega^{2n}(G(M))$, are generated in degree 0 as a graded $E(A)$-module.*

From a given weakly d-Koszul module, we can construct a lot of Koszul modules. Therefore weakly d-Koszul modules have a close relation to Koszul modules in this view.

Proposition 3.11. *Let M be a weakly d-Koszul module. Then*
(1) *$\overline{M} = \bigoplus_{n\geq 0} Ext\,_A^{2kn}(G(M), A_0)$ are Koszul modules for all integers $k \geq 1$,*
(2) *$G\overline{(M)} = \bigoplus_{n\geq 0} Ext\,_A^{2kn}(\Omega^{2m}G(M), A_0)$ are Koszul modules for all integers $k \geq 1$ and $m \geq 0$.*

Proof. If we note that $G(M)$ is a d-Koszul module, where M is a weakly d-Koszul module, then the proof will be clear by Proposition 3.9 and Corollary 3.10.

■

4. The Finite Generation of $\mathcal{E}(M)$

In this section, let M be a weakly d-Koszul module and $\mathcal{E}(M)$ be the corresponding Koszul dual of M. We will show that $\mathcal{E}(M)$ is finitely generated as a graded $E(A)$-module.

From [3], we can get the following useful result and we omit the proof since it is evident.

Lemma 4.1. *Let A be a d-Koszul algebra and M be a d-Koszul module. Then the Koszul dual of M, $\mathcal{E}(M)$, is finitely generated as a graded $E(A)$-module.*

Lemma 4.2. *Let*

$$0 \to K \xrightarrow{f} M \xrightarrow{g} N \to 0$$

be an exact sequence in $Gr(A)$ and A be a graded algebra. If K and N are finitely generated, then M is finitely generated.

Proof. Let $\{x_1, x_2, \cdots, x_n\}$ and $\{y_1, y_2, \cdots, y_m\}$ be the generators of K and N respectively. We claim that $\{f(x_1), f(x_2), \cdots, f(x_n), g^{-1}(y_1), g^{-1}(y_2), \cdots, g^{-1}(y_m)\}$ is the set of generators of M. For the simplicity, let $g^{-1}(y_i) = z_i$ for all $0 \leq i \leq m$. Let $x \in M$ be a homogeneous element, it is trivial that $g(x) = \sum a_i y_i$, where $a_i \in A$. In M, we consider the element, $\sum a_i z_i - x$. Since $g(\sum a_i z_i - x) = 0$, we have $\sum a_i z_i - x \in \ker g = \operatorname{im} f$, there exists $w = \sum b_i x_i \in K$, such that $f(w) = \sum a_i z_i - x$. Hence we have $x = \sum a_i z_i - \sum b_i f(x_i)$. Therefore, M is generated by $\{f(x_1), f(x_2), \cdots, f(x_n), g^{-1}(y_1), g^{-1}(y_2), \cdots, g^{-1}(y_m)\}$ and of course finitely generated. ■

Now we can state and prove the main result in this section.

Theorem 4.3. *Let A be a d-Koszul algebra and $M \in gr(A)$ be a weakly d-Koszul module. Then the Koszul dual of M, $\mathcal{E}(M)$ is finitely generated as a graded $E(A)$-module.*

Proof. Suppose that the generators of M lie in the degree $k_0 < k_1 < \cdots < k_p$ part. we will prove the theorem by induction. If $p = 0$, then M is pure, by Proposition 2.4, M is a d-Koszul module. Then by Lemma 4.3, $\mathcal{E}(M)$ is finitely generated as a graded $E(A)$-module. Assume that the statement holds for less than p. Since M is a weakly d-Koszul module, by Lemma 3.4, M admits a chain of submodules

$$0 \subset U_0 \subset U_1 \subset \cdots \subset U_p = M,$$

such that all U_i/U_{i-1} are d-Koszul modules. Consider the following exact sequence

$$0 \to U_0 \to M \to M/U_0 \to 0.$$

From the proof of Lemma 2.5 [9], we get the following exact sequence for all $n \geq 0$

$$0 \to \Omega^n(U_0) \to \Omega^n(M) \to \Omega^n(M/U_0) \to 0,$$

which implies an exact sequence for all $n \geq 0$

$$0 \to \text{Hom}_A(\Omega^n(U_0), A_0) \to \text{Hom}_A(\Omega^n(M), A_0) \to \text{Hom}_A(\Omega^n(M/U_0), A_0) \to 0,$$

that is to say we have the following exact sequence for all $n \geq 0$

$$0 \to \text{Ext}_A^n(M/U_0, A_0) \to \text{Ext}_A^n(M, A_0) \to \text{Ext}_A^n(U_0, A_0) \to 0.$$

Applying the exact functor "$\bigoplus$" to the exact sequence above, we get

$$0 \to \bigoplus_{n\geq 0} \text{Ext}_A^n(M/U_0, A_0) \to \bigoplus_{n\geq 0} \text{Ext}_A^n(M, A_0) \to \bigoplus_{n\geq 0} \text{Ext}_A^n(U_0, A_0) \to 0.$$

That is, we have the following exact sequence in $Gr(A)$

$$0 \to \mathcal{E}(M/U_0) \to \mathcal{E}(M) \to \mathcal{E}(U_0) \to 0.$$

It is evident that $\mathcal{E}(U_0)$ is a finitely generated graded $E(A)$-module and the number of the generating spaces of M/U_0 is less than p, by induction assumption, we have that $\mathcal{E}(M/U_0)$ is a finitely generated graded $E(A)$-module. Now by Lemma 4.2, we have that $\mathcal{E}(M)$ is a finitely generated graded $E(A)$-module. ∎

References

1. A. Beilinson, V. Ginszburg, and W. Soergel, Koszul duality patterns in representation theory, *J. Amer. Math. Soc.* **9** (1996) 473–525.

2. R. Berger, Koszulity for nonquadratic algebras, *J. Alg.* **239** (2001) 705–734.
3. E. L. Green and E. N. Marcos, δ-Koszul algebras, *Comm. Alg.* **33** (2005) 1753–1764.
4. E. L. Green and R. Martinez-Villa, *Koszul and Yoneda Algebras,* Representation theory of algebras (Cocoyoc, 1994), CMS Conference Proceedings, Vol. 18, AMS, Providence, RI, 1996, pp. 247–297.
5. E. L. Green and R. Martinez-Villa, *Koszul and Yoneda algebras II Algebras and modules, II* (Geiranger, 1996), CMS Conference Proceedings, Vol. 24, AMS, Providence, RI, 1998, pp. 227–244.
6. E. L. Green, E. N. Marcos, R. Martinez-Villa, Pu Zhang, D-Koszul algebras, *J. pure and Appl. Algebra* **193** (2004) 141–162.
7. E. L. Green, R. Martinez-Villa, I. Reiten, ϕ. Solberg, and D. Zacharia, On modules with linear presentations, *J. Alg.* **205** (1998) 578–604.
8. J.-W. He, D.-M. Lu, Higher Koszul Algebras and A-infinity Algebras, *J. Alg.* **293** (2005) 335–362.
9. J.-F. Lu, J.-W. He, and D.-M. Lu, On modules with d-Koszul towers, *Chi. J. Cont. Math.* (2006) (submitted).
10. C. Lofwall, *On the subalgebra generated by the one-dimensional elements in the Yoneda Ext-algebra*, Algebra, Algebraic Topology and Their Interactions (Stockholm, 1983), 291-338, LNM, Vol. 1183, Springer–Verlag, Berlin-New York, 1986.
11. R. Martinez-Villa and D. Zacharia, Approximations with modules having linear resolutions, *J. Alg.* **266** (2003) 671–697.
12. S. Priddy, Koszul resolutions, *Trans. Amer. Math. Soc.* **152** (1970) 39–60.
13. C. A. Weible, *An Introduction to Homological Algebra*, Cambridge Studies in Avanced Mathematics, Vol. 38, Cambridge Univ. Press, 1995.