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The Embedding of Haagerup L^p Spaces

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Abstract. The aim of this paper is to give a proof for a theorem due to S. Goldstein that: If there is a σ- weakly continuous faithful projection of norm one from a von Neumann algebra M onto its von Neumann subalgebra N, then $L^p(N)$ can be canonically embeded into $L^p(M)$. Here $L^p(A)$ [6] denotes the Haagerup L^p space over the von Neumann algebra A.

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Let M be a von Neumann algebra acting in a Hilbert space H and ψ a normal faithful semifinite weight on M. Let $\{\sigma_t^\psi\}_{t\in\mathbb{R}}$ denote the modular automorphism group on M associated with ψ. The crossed product $\mathbb{M} = M \rtimes_{\sigma_t} \mathbb{R}$ is a von Neumann algebra acting on $\overline{H} = L^2(\mathbb{R}, H)$ generated by

$$\begin{aligned}(\pi_M(a)\xi)(t) &= \sigma_{-t}^\psi(a)\xi(t),\\ (\lambda_M(s)\xi)(t) &= \xi(t-s) \quad \xi \in \overline{H}, t \in \mathbb{R}.\end{aligned} \tag{1}$$

Theorem. *Let N be a von Neumann subalgebra of M. Suppose that $\psi|N$ is semifinite and $\sigma_t^\psi|N = \sigma_t^{\psi|N}$ for each $t \in \mathbb{R}$. Then $\mathbb{N}$, the crossed product of N, is canonically embeded into $\mathbb{M}$ and for each $p \in [1, \infty]$ the space $L^p(N)$ can be canonically embeded into $L^p(M)$, so that for any $k \in L^p(N)$*

$$\|k\|_p^N = \|k\|_p^M,$$

where $\|.\|_p^N$ and $\|.\|_p^M$ denote the norms of $L^p(N)$ and $L^p(M)$ respectively.

Proof. The condition $\sigma_t^{\psi}|_N = \sigma_t^{\psi|N}$ means that $\forall b \in N$, $\sigma_t^{\psi}(b) = \sigma_t^{\psi|N}(b) \in N$, i.e. σ_t^{ϕ} leaves N invariant; Together with the condition that $\psi|N$ is semifinite, it implies, by a theorem of Takesaki [5], that there is a σ-weakly continuous projection E of norm one of M onto N such that $\psi = (\psi|_N) \circ E$. It is not hard to show that $E \circ \sigma^{\psi} = \sigma^{\psi} \circ E$ (see for example, [4, Proposition 3.2]).

Let $\mathbb{N} = N \rtimes_{\sigma_t^{\psi|N}} \mathbb{R}$, it is a von Neumann algebra acting on $L^2(\mathbb{R}, H) = \overline{H}$, generated by operators $\pi_N(b), b \in \mathbb{N}$ and $\lambda_N(s), s \in \mathbb{R}$; defined by

$$\begin{aligned}&(\pi(b)\xi(t) = \sigma_{-t}^{\psi|N}(b)\xi(t)),\\ &(\lambda(s)\xi(t) = \xi(t-s)) \quad \xi \in \overline{H}, t \in \mathbb{R}.\end{aligned} \tag{2}$$

Sine $\sigma_{-t}^{\psi|N}(b) = (\sigma_{-t}^{\psi}|N)(b)$ for $b \in N$; (1) and (2) imply

$$\begin{aligned}\pi_M|N &= \pi_N,\\ \lambda_M &= \lambda_N,\end{aligned} \tag{3}$$

and $\mathbb{M}, \mathbb{N}$ act on the same Hilbert space $\overline{H}$.

Let $\mathbb{M}_0$ be the * algebra generated algebraically by operators $\pi_M(a)$, $a \in M$ and $\lambda_M(s), s \in \mathbb{R}$. Then $\mathbb{M}$ is the σ-weak closure of $\mathbb{M}_0$ and any element $x_0 \in \mathbb{M}_0$ can be represented as

$$x_0 = \sum_{k=1}^{n} \lambda_M(s_k)\pi_M(a_k) \text{ for some } \{s_k\}_1^n \subset \mathbb{R}; \{a_k\}_1^n \subset M.$$

We define $\mathbb{N}_0$ in the same way. Thus $\forall y_0 \in \mathbb{N}_0$,

$$y_0 = \sum_{k=1}^{m} \lambda_N(s_k)\pi_N(b_k) = \sum_{k=1}^{m} \lambda_M(s_k)(\pi_M|N)(b_k) \in \mathbb{M}_0$$

for some $\{s_k\}_1^m \subset \mathbb{R}; \{b_k\}_1^m \subset N$. The σ-weak closure of $\mathbb{N}_0$ is $\mathbb{N}$. Then we have $\mathbb{N}_0 \subset \mathbb{M}_0$ and their σ-weak closures verify $\mathbb{N} \subset \mathbb{M}$. It is clear that $\forall x \in \mathbb{N} \subset \mathbb{M}; ||x||^{(N)} = ||x||^{(M)}$.

Consider now the dual action θ_s of $\mathbb{R}$ in $\mathbb{M}$, characterized by

$$\begin{aligned}\theta_s(\pi_M(a)) &= \pi_M(a), \ \forall a \in M,\\ \theta_s(\lambda_M(t)) &= e^{-ist}\lambda_M(t), \ \forall t, s \in \mathbb{R}.\end{aligned} \tag{4}$$

By (3), we have

$$\begin{aligned}\theta_s(\pi_N(a)) &= \pi_N(a), \ \forall a \in N,\\ \theta_s(\lambda_N(t)) &= e^{-ist}\lambda_N(t), \quad \forall t, s \in \mathbb{R}.\end{aligned}$$

Thus $\theta_s(y_0) \in \mathbb{N}_0$ for $y_0 \in \mathbb{N}_0$, $\forall s \in \mathbb{R}$. So that $\theta_s(\mathbb{N}_0) \subset \mathbb{N}_0 \subset \mathbb{N}$. Since θ_s is σ-weakly continuous on $\mathbb{M}$; for all $s \in \mathbb{R}$ we have

$$\theta_s(\mathbb{N}) \subset \mathbb{N}.$$

The continuity of θ_s in measure implies also

$$\theta_s(\widetilde{\mathbb{N}}) \subset \widetilde{\mathbb{N}}$$

and

$$\theta_s^{\mathbb{M}}|_{\mathbb{N}} = \theta_s^{\mathbb{N}}, \quad \forall s \in \mathbb{R},$$

where $\theta_s^{\mathbb{M}}$ and $\theta_s^{\mathbb{N}}$ denote the dual action θ_s of $\mathbb{R}$ on $\mathbb{M}$ and on $\mathbb{N}$ respectively.

By definition of $L^p(N)$ and $L^p(M)$ and the above results, it follows that

$$\begin{aligned} L^p(N) &= \{k \in \widetilde{\mathbb{N}} | \forall s \in \mathbb{R} : \theta_s^{\mathbb{N}} k = e^{-\frac{s}{p}} k\} \\ &= \{k \in \widetilde{\mathbb{N}} \subset \widetilde{\mathbb{M}} | \forall s \in \mathbb{R} : \theta_s^{\mathbb{M}} k = e^{-\frac{s}{p}} k\} \subset L^p(M). \end{aligned} \tag{5}$$

Then we have $L^p(N) \subset L^p(M)$.

It remains now to show that

$$\|k\|_p^M = \|k\|_p^N \text{ for any } k \in L^p(N) \subset L^p(M).$$

It suffices to demonstrate it for the case $p = 1$. Note that $L^1(M) \simeq M_*; L^1(N) \simeq N_*$ and for any $\phi \in N_*; \phi \circ E \in M_*$. In [1, 2] the author has proved that E can be extended canonically to $\hat{E} : M_+^\wedge \to N_+^\wedge; \widetilde{E} : \widetilde{\mathbb{M}} \to \widetilde{\mathbb{N}}$ and $E_1 : L^1(M) \to L^1(N)$, given by $h_{(\phi)} \to h_{\phi\circ E}$. It is extended also to $E_p : L^p(M) \to L^p(N)$; and for any $\phi \in N_*$

$$\overline{\phi} = (\phi \circ E)^-|_{\mathbb{N}}.$$

Let us calculate the norm of $h_\phi^N = h_{\phi\circ E}^M$. Note that $||h_\phi^N||_1^{(N)} = ||\phi||^{(N)}$ and $||h_{\phi\circ E}^M||_1^{(M)} = ||\phi \circ E||^{(M)}$ for any $\phi \in N_*$. We have

$$\begin{aligned} \|\phi\|^{(N)} &= \sup_{b\in N, \|b\|\le 1} |\phi(b)| \ge \sup_{a \in M, \|a\| \le 1} |(\phi \circ E)(a)| = \|\phi \circ E\|^{(M)} \\ &\ge \sup_{b\in N, \|b\|\le 1} |(\phi \circ E)(b)| = \sup_{b\in N, \|b\|\le 1} |\phi(b)| = \|\phi\|^{(N)}. \end{aligned} \tag{6}$$

This implies $\|\phi\|^{(N)} = \|\phi \circ E\|^{(M)}$; i.e. $\|h_\phi^N\|_1^{(N)} = \|h_{\phi\circ E}^M\|_1^{(M)}$, which shows that for any $k \in L^1(N) \subset L^1(M)$, one has $\|k\|_1^{(N)} = \|k\|_1^{(M)}$. It is now obvious that, for each $p \in [1, \infty]$,

$$\|k\|_p^{(N)} = \|k\|_p^{(M)}, \quad \forall k \in L^p(N) \subset L^p(M).$$

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