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Robust Stability of Metzler Operator and Delay Equation in $L^p([-h, 0]; X)$

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Abstract. In this paper we study how the spectral bound of Mezler operator changes under multi-perturbations. Characterizations of the stability radius of Metzler operators with respect to this type of disturbances are established. We will then apply the obtained results to study the stability radius of delays equation in $L^p([-1,0],X)$.

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1. Introduction

In the last two decades, a considerable attention has been paid to problems of robust stability of dynamic systems in infinite-dimensional spaces. The interested readers are referred to [3, 5, 6, 9, 15] and the biography therein for further references. One of the most important problems in the study of robust stability is the calculation of the stability radius of a dynanmic system subjected to various classes of parameter perturbations. In [5, 15] explicit formulas for the complex stability radius of a given (uniformly) exponentially stable linear system $\dot{x}(t) = Ax(t)$ under structured perturbations of the form

$$A \to A + D\Delta E \tag{1}$$

(where A is a closed unbounded operator in a Banach space X, $D \in \mathcal{L}(U,X), E \in \mathcal{L}(X,Y)$ are given linear bounded operators and $\Delta \in \mathcal{L}(Y,U)$ is unknown perturbation) have been established, extending the classical results in finite-dimensional case obtained by Hinrichsen and Pritchard in [8]. The case of time-varying systems has been considered in [9] and [3] where various formulas and estimates of complex stability radius have been obtained for evolution operators. In [6] it was shown that, for the case of structured perturbation (1), if the operator A is a Metzler operator (i.e. the resolvent $R(\lambda; A) = (\lambda I - A)^{-1}$ is positive operator), then the real stability radius coincide with the complex stability radius and can be calculated by a simple formula.

The main purpose of paper is to extend the main result of [6] to the case where the system operator A is subjected to affine multi-perturbations of the form

$$A \to A + \sum_{i=1}^{N} D_i \Delta_i E_i. \tag{2}$$

The result is then applied to study the stability radii of delay equations in the Banach space $L^p([-h,0];X)$. To simplify the presentation, we shall make use of the notation used in [6].

2. Main Result

Let X be a complex Banach space. For a closed linear operator A, let $\sigma(A)$ denote the spectrum of A, $\rho(A) = C \backslash \sigma(A)$ the resolvent set of A, and $R(\lambda; A) = (\lambda I - A)^{-1} \in \mathcal{L}(X)$ the resolvent of A defined on $\rho(A)$. The spectral radius $r(A)$ and the spectral bound $s(A)$ of A are defined by

$$r(A) = \sup\big\{|\lambda| : \lambda \in \sigma(A)\big\}, \qquad s(A) = \sup\big\{\Re\lambda : \lambda \in \sigma(A)\big\}.$$

Denote the open complex left half-plane by $C_- = \{\lambda \in C : \Re\lambda < 0\}$.A closed operator A on X is said to be *Hurwitz stable* if $\sigma(A) \subset C_-$ and *strictly Hurwitz stable* if $s(A) < 0$. Clearly, every strictly Hurwitz stable operator is Hurwitz stable. Let X, Y be complex Banach lattices and X^+, Y^+ denote positive cones of X and Y respectively; and $\mathcal{L}^R(X,Y)$ ($\mathcal{L}^+(X,Y)$) are the set of all the real (the positive) linear operators from X to Y, respectively. If $Y = X$ then we use $\mathcal{L}^R(X)$, $\mathcal{L}^+(X)$ to denote the above spaces. A closed operator A is said to be a *Metzler operator* if there exists $\omega \in R$ such that $(\omega, \infty) \subset \rho(A)$ and $R(t; A)$ is positive for $t \in (\omega, \infty)$).It is clear that if $A \in \mathcal{L}^+(X)$ then A is a Metzler operator.

We recall some results of [5] and [6] which will be used in the sequel.

Theorem 2.1. *Suppose $T \in \mathcal{L}^+(X)$. Then*
i) $r(T) \in \sigma(T)$.
ii) $R(\lambda; T) \geqslant 0$ *if and only if* $\lambda \in R$ *and* $\lambda > r(T)$.

Theorem 2.2. *Let A be a Metzler operator on X. Then*
i) $s(A) \in \sigma(A)$

ii) *the function* $R(\cdot;A)$ *is positive and decreasing for* $t > s(A)$

$$s(A) < t_1 \leqslant t_2 \Longrightarrow 0 \leqslant R(t_2;A) \leqslant R(t_2;A).$$

Lemma 2.3. *Let* A *be a Metzler operator on* X *and* $E \in \mathcal{L}^+(X,Y)$. *Then*

$$|ER(\lambda;A)x| \leqslant ER(\Re\lambda;A)|x|, \quad \Re\lambda > s(A), \quad x \in X.$$

(Remind that for x in a complex Banach lattice X, $|x|$ denotes the *modulus* of x : $|x| = \sup\{x, -x\}$).

Now we assume that A is a Hurwitz stable closed operator on a complex Banach lattice X and that A is subjected to under multi-perturbations of the form

$$A \to A_\Delta = A + \sum_{i=1}^{N} D_i \Delta_i E_i \tag{3}$$

where $D_i \in \mathcal{L}(U_i,X), E_i \in \mathcal{L}(X,Y_i), i \in \overline{N} = \{1,...,N\}$ are given linear bounded operators determining the structure of perturbations and $\Delta_i \in \mathcal{L}(Y_i,U_i), i \in \overline{N}$ are unknown disturbance operators.

The transfer function $G_{ij} : \rho(A) \to \mathcal{L}(U_j,Y_i)$ associated with the triplet (A,E_i,D_j) is defined by

$$G_{ij}(\lambda) = E_i R(\lambda;A) D_j, \ \lambda \in \rho(A), \quad i,j \in \overline{N}.$$

It is clear that each $G_{ij}(\cdot)$ is analytic on $\rho(A)$. We have the following result.

Proposition 2.1. *Let* $\lambda \in \rho(A)$ *and* $\Delta_i \in L(Y_i,U_i), i \in \overline{N}$. *If*

$$\sum_{i=1}^{N} ||\Delta_i|| < \frac{1}{\max\limits_{i,j\in\overline{N}} ||G_{ij}(\lambda)||}, \tag{4}$$

then A_Δ *is closed and* $\lambda \in \rho(A_\Delta)$.

Proof. Let us consider the Banach spaces $U = \prod_{i=1}^{N} U_i, \quad Y = \prod_{i=1}^{N} Y_i$ provided with the norm

$$\|u\| = \sum_{i=1}^{N} \|u_i\|, u = (u_1,\dots,u_N) \in U, \quad u_i \in U_i, \quad i \in \overline{N}, \tag{5}$$

$$\|y\| = \sum_{i=1}^{N} \|y_i\|, u = (y_1,\dots,y_N) \in Y, \quad y_i \in Y_i, \quad i \in \overline{N}. \tag{6}$$

Let us define the linear operators $E \in \mathcal{L}(X,Y), \quad D \in \mathcal{L}(U,X)$ by setting

$$Ex = (E_1x,\cdots,E_Nx), \ Du = \sum_{i=1}^{N} D_i u_i, \ \text{ for } x \in X, \ \ u = (u_1,\cdots,u_N) \in U. \tag{7}$$

For any $\Delta_i \in \mathcal{L}(Y_i, U_i), i \in \overline{N}$ we define the "block-diagonal" operator $\Delta : Y \longrightarrow U$ by setting

$$\Delta y = (\Delta_1 y_1, \cdots, \Delta_N y_N), \quad y = (y_1, \cdots, y_N) \in Y, \tag{8}$$

It is clear that $\Delta \in \mathcal{L}(Y, U)$. Assume $\lambda \in \rho(A)$, then, by definition, we have, for each $u = (u_1, \cdots, u_N) \in U$,

$$\Delta E R(\lambda; A) D u = (\sum_{j=1}^{N} \Delta_1 G_{1j}(\lambda) u_j, \cdots, \sum_{j=1}^{N} \Delta_N G_{Nj}(\lambda) u_j).$$

Therefore,

$$\|\Delta E R(\lambda; A) D u\| = \sum_{i=1}^{N} \|\Delta_i \sum_{j=1}^{N} G_{ij}(\lambda) u_j\| \leqslant \max_{i,j \in \overline{N}} ||G_{ij}(\lambda)|| \sum_{i=1}^{N} ||\Delta_i|| \|u\|,$$

and hence, by (4), $\|\Delta E R(\lambda; A) D\| < 1$. It follows that the operator $[I - \Delta E R(\lambda; A) D]$ is invertible and $[I - \Delta E R(\lambda; A) D]^{-1} \in \mathcal{L}(U)$. Therefore, $[I - D\Delta E R(\lambda; A)]$ is invertible and $[I - D\Delta E R(\lambda; A)]^{-1} \in \mathcal{L}(X)$. Since, obviously,

$$[I - D\Delta E R(\lambda; A)](\lambda I - A) = \lambda I - A - D\Delta E = \lambda I - A_\Delta, \tag{9}$$

it follows that $\lambda I - A_\Delta$ is a closed operator on X and $\lambda I - A_\Delta : \mathcal{D}(A) \to X$ is invertible. Moreover, by (9),

$$(\lambda I - A_\Delta)^{-1} = R(\lambda; A)[I - D\Delta E R(\lambda; A)]^{-1} \in \mathcal{L}(X),$$

which implies that $\lambda \in \rho(A_\Delta) = \rho(A + \sum_{i=1}^{N} D_i \Delta_i E_i)$, completing the proof. ■

Defition 2.4. *Let A be Hurwitz stable. The complex, the real and the positive Hurwitz stability radii of A with respect to the multi-perturbations of the form* (2) *is defined, respectively, by*

$$r_C = \inf \Big\{ \sum_{i=1}^{N} ||\Delta_i|| : \Delta_i \in \mathcal{L}(Y_i, U_i), i \in \overline{N}, \quad \sigma(A_\Delta) \not\subset C_- \Big\},$$

$$r_R = \inf \Big\{ \sum_{i=1}^{N} ||\Delta_i|| : \Delta_i \in \mathcal{L}^R(Y_i, U_i), i \in \overline{N}, \quad \sigma(A_\Delta) \not\subset C_- \Big\},$$

$$r_+ = \inf \Big\{ \sum_{i=1}^{N} ||\Delta_i|| : \Delta_i \in \mathcal{L}^+(Y_i, U_i), i \in \overline{N}, \quad \sigma(A_\Delta) \not\subset C_- \Big\},$$

where we set $\inf \emptyset = \infty$.

Note that the first two stability radii are well defined without the assumption that the underlying spaces are Banach lattices. Moreover, by definition,

$$r_C \leqslant r_R \leqslant r_+.$$

The following theorem gives a formula for calculation of the complex stability radius with respect to multi-perturbations.

Theorem 2.5. *Let A be Hurwitz stable. Then*

$$\frac{1}{\max\limits_{i,j\in\overline{N}}\sup\limits_{\Re s\geqslant 0}||G_{ij}(s)||}\leqslant r_C\leqslant\frac{1}{\max\limits_{i\in\overline{N}}\sup\limits_{\Re s\geqslant 0}||G_{ii}(s)||}. \tag{10}$$

In particular, if $D_i=D_j$ or $E_i=E_j$ for all $i,j\in\overline{N}$, then

$$r_C=\frac{1}{\max\limits_{i\in\overline{N}}\sup\limits_{\Re s\geqslant 0}||G_{ii}(s)||}. \tag{11}$$

Proof. Assume to the contrary that the first inequality in (10) is not true, that is

$$r_C<\frac{1}{\max\limits_{i,j\in\overline{N}}\sup\limits_{\Re s\geqslant 0}||G_{ij}(s)||}=:\gamma. \tag{13}$$

Then, by the definition of r_C, there exist $\lambda_0, \Re\lambda_0\geqslant 0$ and $\Delta^0=(\Delta^0_1,...,\Delta^0_N), \Delta^0_i\in\mathcal{L}(Y_i,U_i), i\in\overline{N}$ such that $\lambda_0\in\sigma(A_{\Delta^0})$ and

$$\sum_{i=1}^{N}\|\Delta^0_i\|<\gamma\leqslant\frac{1}{\max\limits_{i,j\in\overline{N}}\|G_{ij}(\lambda_0)\|}. \tag{13}$$

On the other hand, since A is Hurwitz stable, $\lambda_0\in\rho(A)$ and hence, by Proposition 2.1, it follows from (13) that $\lambda_0\in\rho(A_{\Delta^0})$, a contradiction. Thus we have

$$r_C\geqslant\frac{1}{\max\limits_{i,j\in\overline{N}}\sup\limits_{\Re s\geqslant 0}\|G_{ij}(s)\|} \tag{14}$$

We now prove that

$$r_C\leqslant\frac{1}{\max\limits_{i\in\overline{N}}\sup\limits_{\Re s\geqslant 0}\|G_{ii}(s)\|}. \tag{15}$$

Let us fix $\lambda\in C$ with $\Re\lambda\geqslant 0, i\in\overline{N}$ and $\varepsilon>0$. Then, there exists $\hat{u}_i\in U_i, \|\hat{u}_i\|=1$ satisfying $\|G_{ii}(\lambda)\|\geqslant\|G_{ii}(\lambda)\hat{u}\|\geqslant\|G_{ii}(\lambda)\|-\varepsilon$. By Hahn-Banach theorem there exists $\hat{y}^*_i\in Y^*_i$ such that$\|\hat{y}^*_i\|=1, \hat{y}^*_i(G_{ii}(\lambda)\hat{u}_i)=\|G_{ii}(\lambda)\hat{u}_i\|$. We define $\overline{\Delta}_i: Y_i\to U_i$ by setting

$$\overline{\Delta}_i y_i=\frac{1}{\||G_{ii}(\lambda)\hat{u}\|}\hat{y}^*_i(y_i)\hat{u}_i, \quad \forall y_i\in Y_i.$$

Then, it is clear that $\overline{\Delta}_i\in\mathcal{L}(Y_i,U_i)$ and

$$\|\overline{\Delta}_i\|\leqslant\frac{1}{\|G_{ii}(\lambda)\hat{u}\|}\leqslant\frac{1}{\|G_{ii}(\lambda)\|-\varepsilon}$$

Now we define the disturbance $\Delta=(\Delta_1,...,\Delta_N)$ by setting, for $j\in\overline{N}$,

$$\Delta_j = \begin{cases} \overline{\Delta}_i & \text{if } \; j = i, \\ 0 & \text{if } \; j \neq i. \end{cases} \tag{16}$$

Then $\sum_{j=1}^{N} ||\Delta_j|| = ||\overline{\Delta}_i||$ and, taking $\hat{x} = R(\lambda; A)D\hat{u} \in \mathcal{D}(A)$ we can easily verify that $\hat{x} \neq 0$ and $(A + \sum_{j=1}^{N} D_j\Delta_j E_j)\hat{x} = A_\Delta \hat{x} = \lambda\hat{x}$. This implies $\lambda \in \sigma(A_\Delta)$ and so $\sigma(A_\Delta) \not\subset C_-$. Consequently, by the definition of r_C

$$r_C \leqslant \sum_{j=1}^{N} \|\Delta_j\| = \|\overline{\Delta}_i\| \leqslant \frac{1}{\|G_{ii}(\lambda)\| - \varepsilon}. \tag{17}$$

Since the above inequality has been established for arbitrary $\lambda \in C$ with $\Re\lambda \geqslant 0, i \in \overline{N}$ and $\varepsilon > 0$, the inequality (15) follows. Furthermore, if $D_i = D_j, \forall i, j \in \overline{N}$ (resp. $E_i = E_j, \forall i, j \in \overline{N}$) then, by the definition $G_{ii}(s) = G_{ij}(s), \forall i, j \in \overline{N}$ (resp. $G_{jj}(s) = G_{ij}(s), \forall i, j \in \overline{N}$). Thus, in this case, (10) implies (11). The proof is complete. ∎

Remark that for the Hurwitz stable operator A, the functions $G_{ij}(\cdot)$ are analytic in the complex right-half plane, therefore, by the maximum modulus principle, one has

$$\sup_{\Re s \geqslant 0} ||G_{ij}(s)|| = \sup_{s \in R} ||G_{ij}(\imath s)||$$

and hence the formulas (10) and (11) can be rewriten accordingly with the supremum is taken over the real line.

Theorem 2.6. *Let A be a Hurwitz Metzler stable operator and all operator $D_i, E_i, i \in \overline{N}$ are positive. If $D_i = D_j$ (or) $E_i = E_j$ for all $i, j \in \overline{N}$ then*

$$r_C = r_R = r_+ = \frac{1}{\max\limits_{i \in \overline{N}} \|G_{ii}(0)\|}.$$

Proof. Since $s(A) < 0$ and $E_i, D_i, \forall i \in \overline{N}$, are positive operators, it follows from Theorem 2.2 that all $G_{ii}(t)$ are decreasing for $t \geqslant 0$:

$$0 \leqslant t_1 \leqslant t_2 \;\Rightarrow\; 0 \leqslant G_{ii}(t_2) \leqslant G_{ii}(t_1) \;\text{ and }\; \|G_{ii}(t_2)\| \leqslant \|G_{ii}(t_1)\|, \quad \forall i \in \overline{N}. \tag{18}$$

Applying Lemma 2.3 and the lattice norm property, from (18), we get, for all $\lambda = t + \imath\omega \in C$ with $t = \Re\lambda \geqslant 0$,

$$\|G_{ii}(\lambda)\| \leqslant \|G_{ii}(t)\| \leqslant \|G_{ii}(0)\|, \quad \forall i \in \overline{N}.$$

Therefore, by formula (11),

$$r_C = \frac{1}{\max\limits_{i \in \overline{N}} \|G_{ii}(0)\|}.$$

To show that $r_+ \leqslant r_C$, let us fix $i \in \overline{N}$ and an arbitrary $\varepsilon > 0$. As in [6], using the Krein-Rutman Theorem, one can construct an one-rank positive destabilizing perturbation $\Delta = (\Delta_1, \cdots, \Delta_N), \Delta_j \in \mathcal{L}^+(Y_j, U_j), \forall j \in \overline{N}$ such that $\|\Delta\| = \|\Delta_i\| < \|G_{ii}(0)\|^{-1} + \varepsilon$. This implies

$$r_+ \leqslant \|\Delta\| < \frac{1}{\|G_{ii}(0)\|} + \varepsilon = r_C + \varepsilon,$$

concluding the proof. ∎

3. Stability Radii of Delay Equation in $L^p([-h,0];X)$

In this section, we apply the results of the above section to study robust stability for linear delay equations in Banach spaces.

Assume that A_0 is a generator of a *uniformly continuous* C_0-semigroup $(T(t))_{t\geqslant 0}$ on a complex Banach space X. We also fix $p \in [1,\infty)$ and non-negative real numbers $0 \leqslant h_1 < h_2 < ... < h_n =: h$. Given bounded linear operators $A_1, ..., A_n$ on X, we will study the delay equation

$$\begin{cases} \dot{u}(t) = A_0 u(t) + \sum\limits_{i=1}^{n} A_i u(t-h_i), \quad t \geqslant 0 \\ u(0) = x, \\ u(t) = f(t), \quad t \in [-h,0). \end{cases} \tag{19}$$

Here, $x \in X$ is the initial value and $f \in L^p([-h,0];X)$ is the 'history' function. A mild solution of (19) is the function $u(\cdot) \in L^p_{loc}([-h,\infty);X)$ satisfying $u(t) = f(t), t \in [-h,0)$ and

$$u(t) = T(t)x + \int_0^t T(t-s)\sum_{i=1}^{n} A_i u(s-h_i)ds, \ t \geqslant 0. \tag{20}$$

The delay equation (19) is called exponentially stable is there exist $M > 0$ and $\omega > 0$ such that the solution $u(t)$ of (19) satisfies

$$\|u(t)\| \leqslant Me^{-\omega t}(\|x\|^p + \|f\|^p_{L^p([-h,0];X)}), \quad t \geqslant 0.$$

In order to study the asymptotic behavior of solutions of (19) by semigroup methods, we introduce the product space

$$\mathcal{X} := X \times L^p([-h,0];X)$$

(endowed with the norm $\|(x,f)\|^p = \|x\|^p + \|f\|^p_{L^p([-h,0];X)}$) and the operator $\mathcal{A}$ on $\mathcal{X}$ define by

$$\mathcal{A}(x,f) = (A_0 x + \sum_{i=1}^{n} A_i f(-h_i), f'),$$

with the domain

$$D(\mathcal{A}) = \{(x,f) \in \mathcal{X} : f \in W^{1,p}([-h,0];X), f(0) = x \in D(A_0)\}$$

(here $W^{1,p}([-h,0];X)$ denotes the space of absolutely contiuous with X-valued functions f on $[-h,0]$ that are strongly differentiable a.e. with derivatives

$f'(t) \in L^p([-h,0];X))$. Then, as it has been shown in [2], $\mathcal{A}$ generates a C_0-semigroup $(\mathcal{T}(t))_{t\geqslant 0}$ on $\mathcal{X}$ which is defined by

$$(\mathcal{T}(t))(x,f) = (u(t), u_t), \quad t \geqslant 0,$$

where $u(t)$ is a mild solution of (19) and $u_t(s) := u(t+s), \quad s \in [-h,0]$. Moreover, the delay equation (19) is exponentially stable if and only if $\omega_0(\mathcal{T}) < 0$. Note that $s(\mathcal{A}) = \omega_0(\mathcal{T})$ because $\mathcal{T}(t)$ is uniformly continuous semigroup for $t > h$ (see [5], p.94).

In what follows we assume X a complex Banach lattice and we consider $L^p([-h,0];X)$ as the Banach lattice with respect to the pointwise order relation. Then the product space $\mathcal{X}$ becomes a Banach lattice as well. The following result follows directy from the definition.

Proposition 3.1. *If A_0 generates a positive C_0-semigroup and $A_i \in \mathcal{L}^+(X)$, for all $i = 1,...,n$, then $\mathcal{T}$ is a positive C_0-semigroup.*

Now we define an operator quasi-polynomial $P(\lambda) = A_0 + \sum\limits_{i=1}^{n} e^{-\lambda h_i} A_i$. The spectral set, the resolvent set, and the spectral bound of $P(\cdot)$ are defined respectively by $\sigma(P(\cdot)) = \{\lambda \in C : \lambda \in \sigma(P(\lambda))\}, \quad \rho(P(\cdot)) = C\backslash\sigma(P(\cdot)), \quad s(P(\cdot)) = \sup\{\Re\lambda : \lambda \in \sigma(P(\cdot))\}$. Then, by definition, it is easy to show

Remark 3.1. We have $\sigma(P(\cdot)) = \sigma(\mathcal{A})$ and $s(P(\cdot)) = s(\mathcal{A})$.

The following result will address the properties about the monotonicity and the positivity of the resolvent $R(\cdot, P(\cdot))$.

Lemma 3.2. *Suppose that A_0 generates a positive C_0-semigroup and $A_i \in \mathcal{L}^+(X)$, for all $i = 1,...,n$. Then the resolvent $R(\cdot; P(\cdot))$ is positive and decreasing for $t > s(P(\cdot)) = s(\mathcal{A})$:*

$$s(\mathcal{A}) = s(P(\cdot)) < t_1 \leqslant t_2 \Longrightarrow 0 \leqslant R(t_2; P(t_2)) \leqslant R(t_1; P(t_1)).$$

Proof. By Proposition 3.1, $\mathcal{A}$ is a generator of a positive C_0-semigroup. This implies that $\mathcal{A}$ is Metzler operator. By Theorem 2.2, we have that the resolvent $R(\cdot, \mathcal{A})$ is positive and decreasing for $t > s(\mathcal{A}) = s(P(\cdot))$. The assertion now follows from the formula about relationship between $R(\cdot, \mathcal{A})$ and $R(\cdot; P(\cdot))$ (see [5, Proposition 3.2]).

Proposition 3.2. *Suppose that A generates a positive C_0-semigroup and $A_i \in \mathcal{L}^+(X)$ for all $i = 1,...,n$. For $E \in \mathcal{L}^+(X,Y), x \in X$ we have*

$$|ER(\lambda; P(\lambda))x| \leqslant ER(\Re\lambda; P(\Re\lambda))|x|, \quad \Re\lambda > s(A) = s(P(\cdot)).$$

Proof. We set the operator $\widetilde{E} : \mathcal{X} \to Y$ defined by $\widetilde{E}(x,f) = Ex$ and we choose the vector $(x,0) \in \mathcal{X}$. Applying Lemma 2.5 we have

$$|\widetilde{E}R(\lambda;\mathcal{A})(x,0)| \leqslant \widetilde{E}R(\Re\lambda;\mathcal{A})(x,0)|, \quad \Re\lambda > s(A) = s(P(\cdot)).$$

or equivalent

$$|ER(\lambda;P(\lambda))x| \leqslant ER(\Re\lambda;P(\Re\lambda))|x|, \quad \Re\lambda > s(A) = s(P(\cdot)).$$

The proof is complete. ∎

Now we return to study the stability radii of the delay equation (19). Suppose that the equation (19) is exponentially stable, or the operator $\mathcal{A}$ generates the exponentially stable C_0-semigroup. This is equivalent to $\omega_0(\mathcal{T}) = s(\mathcal{A}) = s(P(\cdot)) < 0$. Suppose that the operators $A_i, i = 0,1,...,n$ are subjected to perturbations of the form

$$A_i \to A_i + D_i\Delta_i E_i, i = 0,1,...,n, \tag{21}$$

where $D_i \in \mathcal{L}(U_i, X), E_i \in \mathcal{L}(X, Y_i), i = 0,1,...,n$ are given operators determining structure of perturbation and $\Delta_i \in \mathcal{L}(Y_i, U_i), i = 0,1,...,n$ are unknown operators. Then, the perturbed equation has the form

$$\begin{cases} \dot{u}(t) = (A_0u(t) + D_0\Delta_0E_0) + \sum\limits_{i=1}^{n}(A_i + D_i\Delta_iE_i)u(t-h_i), \quad t \geqslant 0 \\ u(0) = x, \\ u(t) = f(t), \quad t \in [-h,0). \end{cases} \tag{22}$$

We also set

$$\mathcal{A}_\Delta(x,f) = ((A_0 + D_0\Delta_0E_0)x + \sum_{i=1}^{n}(A_i + D_i\Delta_iE_i)f(-h_i), f'),$$

and

$$P_\Delta(\lambda) = (A_0 + D_0\Delta_0E_0) + \sum_{i=1}^{n} e^{-\lambda h_i}(A_i + D_i\Delta_iE_i).$$

Definition 3.3. *Let equation* (19) *be exponentially stable. The complex, the real and the positive stability radii of* (19) *under perturbations of the form* (21) *are defined respectively by*

$$r_C^{(DE)} = \inf\Big\{\sum_{i=0}^{n}\|\Delta_i\| : \Delta_i \in \mathcal{L}(Y_i,U_i), i = 0,1,...,n \text{ and } (22) \text{ is not exponentially stable}\Big\}$$

$$r_R^{(DE)} = \inf\Big\{\sum_{i=0}^{n}\|\Delta_i\| : \Delta_i \in \mathcal{L}^R(Y_i,U_i), i = 0,1,...,n \text{ and } (22) \text{ is not exponentially stable}\Big\}$$

$$r_+^{(DE)} = \inf\Big\{\sum_{i=0}^{n}\|\Delta_i\| : \Delta_i \in \mathcal{L}^+(Y_i,U_i), i = 0,1,...,n \text{ and } (22) \text{ is not exponentially stable}\Big\}$$

(*we set* $\inf\emptyset = \infty$*).*

Since A_0 is the generator of a C_0-semigroup which is uniformly continuous for $t > 0$ and D_i, Δ_i, E_i are all bounded operators, it can be easily verified, by using Lemma 2.4 in [5] that the operator $\mathcal{A}_\Delta$ generates a uniformly continuous C_0-semigroup. This follows that the equation (22) is not exponentially if and only if $s(\mathcal{A}_\Delta) = s(P_\Delta(\cdot)) \geqslant 0$. Thus we can rewrite the definition of the stability radii of (19) as follows

$$r_C^{(DE)} = \inf\Big\{ \sum_{i=0}^{n} ||\Delta_i|| : \Delta_i \in \mathcal{L}(Y_i, U_i), i = \overline{0, n}, s(P_\Delta(\cdot)) \geqslant 0 \Big\}$$
$$r_R^{(DE)} = \inf\Big\{ \sum_{i=0}^{n} ||\Delta_i|| : \Delta_i \in \mathcal{L}^R(Y_i, U_i), i = \overline{0, n}, s(P_\Delta(\cdot)) \geqslant 0 \Big\}$$
$$r_+^{(DE)} = \inf\Big\{ \sum_{i=0}^{n} ||\Delta_i|| : \Delta_i \in \mathcal{L}^+(Y_i, U_i), i = \overline{0, n}, s(P_\Delta(\cdot)) \geqslant 0 \Big\}$$

Assume that the equation (19) is exponentially stable. For $\lambda \in \rho(P(\cdot))$ we introduce the transfer function associated with the triplet $(P(\lambda), D_j, E_i)$

$$G_{ij}(\lambda) = E_i R(\lambda; P(\lambda)) D_j, i, j \in \{0, 1, ..., n\}.$$

Noticing that for $\lambda \in C$ with $\Re\lambda \geqslant 0$, $\|e^{-\lambda h_i}\Delta_i G_{ij}(\lambda)\| \leqslant \max_{i,j} \|G_{ij}(\lambda)\|, \forall i, j \in \{0, 1, ..., n\}$ we can prove the following proposion similarly as it was done for Proposition 2.1.

Propostion 3.3. *Let $\lambda \in \rho(P(\cdot)), \Re\lambda \geqslant 0$ and $\Delta_i \in \mathcal{L}(Y_i, U_i)$, for all $i = 0, 1, ..., n$. If*

$$\sum_{i=1}^{n} \|\Delta_i\| < \frac{1}{\max\limits_{i,j\in\{0,1,\dots,n\}} \|G_{ij}(\lambda)\|}, \tag{23}$$

then $\lambda \in \rho(P_\Delta(\lambda))$.

Using the above proposition we get the following theorem. The proof is similar to that of Theorem 2.5 and is therefore omitted.

Theorem 3.4. *Let the equation (19) be exponentially stable. Then*

$$\frac{1}{\max\limits_{i,j\in\{0,1,\dots,n\}} \sup\limits_{s\in R} \|G_{ij}(is)\|} \leqslant r_C^{(DE)} \leqslant \frac{1}{\max\limits_{i\in\{0,1,\dots,n\}} \sup\limits_{s\in R} \|G_{ii}(is)\|}. \tag{24}$$

In particular, if $D_i = D_j$ or $E_i = E_j$ for all $i, j \in \{0, 1, ..., n\}$, then

$$r_C^{(DE)} = \frac{1}{\max\limits_{i\in\{0,1,\dots,n\}} \sup\limits_{s\in R} \|G_{ii}(is)\|}. \tag{25}$$

In general, the three stability radii are distinct. However, for the case of positive delay equations, they coinside and, moreover, can be computed easily, as shown by the following

Theorem 3.5. *Suppose that equation* (19) *is exponentially stable, A_0 generates a positive C_0-semigroup and the equation is subjected to affine perturbations of the form* (21) *with $A_i \in \mathcal{L}^+(X), D_i \in \mathcal{L}^+(U_i, X), E_i \in \mathcal{L}^+(X, Y_i), \quad \forall i = 1, ..., n$. If $D_i = D_j$ or $E_i = E_j$ for all $i, j \in \{0, 1, ..., n\}$, then*

$$
\begin{aligned}
r_C^{(DE)} = r_R^{(DE)} = r_+^{(DE)} &= \frac{1}{\max\limits_{i \in \{0,1,\ldots,n\}} \|G_{ii}(0)\|} \\
&= \frac{1}{\max\limits_{i \in \{0,1,\ldots,n\}} \|E_i(-A_0 - A_1 - \ldots - A_n)^{-1} D_i\|}.
\end{aligned}
\tag{26}
$$

The proof similar to that of Theorem 2.6 and is omitted.

We note that the results obtained in Theorems 2.5 and 2.6 generalize results in [6] to multi-perturbations, while Theorems 3.4 and 3.5 extend results due to [18] to the case of delay systems in Banach spaces. Similarly, we can also consider the case of delay systems in Banach spaces where operators $A_i, i \in \{0, 1, ..., n\}$ are subjected to multi-perturbations of the form

$$
A_i \longrightarrow A_i + \sum_{j=0}^{k} D_{ij} \Delta_{ij} E_{ij}, i \in \{0, 1, ..., n\}.
$$

It is a natural open problem to extend the results regarding stability radii of functional differential equations obtained recently in [19, 13] to Banach spaces.

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