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Survey

Interpolation Conditions and Polynomial Projectors Preserving Homogeneous Partial Differential Equations

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Abstract. We give a brief survey on a new approach in study of polynomial projectors that preserve homogeneous partial differential equations or homogeneous differential relations, and their interpolation properties in terms of space of interpolation conditions. Some well-known interpolation projectors as, Abel-Gontcharoff, Birkhoff and Kergin interpolation projectors are considered in details.

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1. Introduction

1.1. We begin with some preliminary notions. Let us denote by $H(\mathbb{C}^n)$ the space of entire functions on $\mathbb{C}^n$ equipped with its usual compact convergence topology, and $\mathcal{P}_d(\mathbb{C}^n)$ the space of polynomials on $\mathbb{C}^n$ of total degree at most d. A *polynomial projector of degree* d is defined as a continuous linear map Π from

$H(\mathbb{C}^n)$ into $\mathcal{P}_d(\mathbb{C}^n)$ for which

$$\Pi(p) = p, \quad \forall p \in \mathcal{P}_d(\mathbb{C}^n).$$

Let $H'(\mathbb{C}^n)$ denote the space of linear continuous functionals on $H(\mathbb{C}^n)$ whose elements are usually called analytic functionals. We define the space $\mathcal{I}(\Pi) \subset H'(\mathbb{C}^n)$ as follows : an element $\varphi \in H'(\mathbb{C}^n)$ belongs to $\mathcal{I}(\Pi)$ if and only if for any $f \in H(\mathbb{C}^n)$ we have

$$\varphi(f) = \varphi(\Pi(f)).$$

This space is called *space of interpolation conditions* for Π.

Let $\{p_\alpha : |\alpha| \le d\}$ be a basis of $\mathcal{P}_d(\mathbb{C}^n)$ whose elements are enumerated by the multi-indexes $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{Z}^n_+$ with length $|\alpha| := \alpha_1 + \cdots + \alpha_n$ not greater than d. Then there exists a unique sequence of elements $\{a_\alpha : \ |\alpha| \le d\}$ in $H'(\mathbb{C}^n)$ such that Π is represented as

$$\Pi(f) = \sum_{|\alpha| \le d} a_\alpha(f) p_\alpha, \ f \in H(\mathbb{C}^n), \tag{1}$$

and $\mathcal{I}(\Pi)$ is given by

$$\mathcal{I}(\Pi) = \langle a_\alpha, |\alpha| \le d \rangle$$

where $\langle \cdots \rangle$ denotes the linear hull of the inside set. In particular, we may take in (1)

$$p_\alpha(z) = u_\alpha(z) := z^\alpha / \alpha!,$$

where $z^\alpha := \prod_{j=1}^n z_j^{\alpha_j}, \ \alpha! : = \ \prod_{j=1}^n \alpha_j!$.

Notice that as sequences of elements in $H(\mathbb{C}^n)$ and $H'(\mathbb{C}^n)$ respectively, $\{p_\alpha : |\alpha| \le d\}$ and $\{a_\alpha : \ |\alpha| \le d\}$ are a biorthogonal system, i.e.,

$$a_\alpha(p_\beta) = \delta_{\alpha\beta}.$$

Moreover, $\mathcal{I}(\Pi)$ is nothing but the range of the adjoint of Π and the restriction of $\mathcal{I}(\Pi)$ to $\wp_d(\mathbb{C}^n)$ is the dual space $\wp_d^*(\mathbb{C}^n)$. Clearly, we have for the dimension of $\mathcal{I}(\Pi)$

$$N_d(n) := \ \dim \mathcal{I}(\Pi) = \ \dim \mathcal{P}_d(\mathbb{C}^n) = \ \binom{n+d}{n}.$$

Conversely, if $\mathbf{I}$ is a subspace of $H'(\mathbb{C}^n)$ of dimension $N_d(n)$ such that the restriction of its element to $\wp_d(\mathbb{C}^n)$ spans $\wp_d^*(\mathbb{C}^n)$, then there exists a unique polynomial projector $\mathcal{P}(\mathbf{I})$ such that $\mathbf{I} = \mathcal{I}(\mathcal{P}(\mathbf{I}))$. In that case we say that $\mathbf{I}$ *is an interpolation space for* $\mathcal{P}_d(\mathbb{C}^n)$ and, for $p \in \mathcal{P}_d(\mathbb{C}^n)$, we have

$$\wp(\mathbf{I})(f) = p \Leftrightarrow \varphi(p) = \varphi(f), \ \forall \varphi \in \mathbf{I}.$$

Obviously, for every projector Π we have $\wp(\mathcal{I}(\Pi)) = \Pi$.

Thus, polynomial projector Π of degree d can be completely described by its space of interpolation conditions $\mathcal{I}(\Pi)$. It is useful to notice that one can in one hand, study interpolation properties of known polynomial projectors, and in the other hand, define new polynomial projectors via their space of interpolation conditions.

1.2. A polynomial projector Π of degree d is said to *preserve homogeneous partial differential equations (HPDE) of degree* k if for every $f \in H(\mathbb{C}^n)$ and every homogeneous polynomial of degree k,

$$q(z) = \sum_{|\alpha|=k} a_\alpha z^\alpha,$$

we have

$$q(D)f = 0 \Rightarrow q(D)\Pi(f) = 0,$$

where

$$q(D) := \sum_{|\alpha|=k} a_\alpha D^\alpha$$

and $D^\alpha = \partial^{|\alpha|}/\partial z_1^{\alpha_1} \dots \partial z_n^{\alpha_n}$.

If a polynomial projector preserves HPDE of degree k for all $k \geq 0$ of degree d, it is said to *preserve homogeneous differential relations (HDR)* .

It should be emphasised that this definition does not make sense in the univariate case as every univariate polynomial projector preserves HDR.

1.3. Preservation of HDR or HPDE is a quite natural and substantial property specific only to multivariate interpolation. Thus, well-known examples of polynomial projectors preserving HDR, are the Taylor projectors T_a^d of degree d (at the point $a \in \mathbb{C}^n$) that are defined by

$$T_a^d(f)(z) := \sum_{|\alpha| \leq d} D^\alpha(f)(a) u_\alpha(z-a).$$

Abel-Gontcharoff, Kergin, Hakopian and mean-value interpolation projectors provide other interesting examples of polynomial projectors preserving HDR.

1.4. In the present paper, we shall discuss a new approach in study of polynomial projectors that preserve HPDE or HDR, and their interpolation properties in terms of space of interpolation conditions. Some interpolation projectors as Abel-Gontcharoff, Birkhoff, Kergin, Hakopian and mean-value interpolation projectors are considered in details. In particular, we shall be concerned with recent papers [5, 11] and [12] investigating these problems.

In [5] Calvi and Filipsson gave a precise description of the polynomial projectors preserving HDR in terms of space of interpolation conditions of D-Taylor projectors. In particular, they showed that a polynomial projector preserves HDR if and only if it preserves HPDE of degree 1 or equivalently, preserves ridge functions.

Polynomial projectors that preserve HPDE where investigated by Dinh Dũng, Calvi and Trung [11, 12]. There naturally arises the question of the existence of polynomial projectors preserving HPDE of degree $k > 1$ without preserving HPDE of smaller degree. In [12] the authors proved that such projectors do indeed exist and a polynomial projector Π preserves HPDE of degree k, $1 \leq k \leq d$, if and only if there are analytic functionals $\mu_k, \mu_{k+1}, \dots, \mu_d \in H'(\mathbb{C}^n)$ with $\mu_i(1) \neq 0,\ i = k, \dots, d$, such that Π is represented in the following form

$$\Pi(f) = \sum_{|\alpha|<k} a_\alpha(f) u_\alpha + \sum_{k \le |\alpha| \le d} D^\alpha \mu_{|\alpha|} u_\alpha,$$

with some $a_\alpha's \in H'(\mathbb{C}^n),\ |\alpha| < k$. Moreover, a polynomial projector which preserves HPDE of degree k necessarily preserves HPDE of every degree not smaller than k.

The results on polynomial projectors preserving HDR lead to a new characterization of well-known interpolation projectors as Abel-Gontcharoff, Kergin, Hakopian and mean-value interpolation projectors *et cetera*. Thus, Calvi and Filipsson [5] have used their results to give a new characterization of Kergin interpolation. They have shown that a polynomial projector of degree d preserving HDR, interpolates at most at $d+1$ points taking multiplicity into account, and only the Kergin interpolation projectors interpolate at maximal $d+1$ points. Dinh Dũng, Calvi and Trung [11, 12] have established a characterization of Abel-Gontcharoff interpolation projectors as the only Birkhoff interpolation projectors that preserve HDR.

Many questions treated in this paper originally come from real interpolation. However, we prefer to discuss the complex version, i.e., we will work in $\mathbb{C}^n$. In the last section we will explain how to transfer our results to the real version. The complex variables setting simplifies rather than complicates the study. Techniques of proofs of results employed in [5, 12] are "almost elementary". Apart from very basic facts on holomorphic functions of several complex variables, the authors only used the Laplace transform $\hat{\varphi}$ of an analytic functional $\varphi \in H'(\mathbb{C}^n)$. The mapping $\varphi \to \hat{\varphi}$ is an isomorphism between the analytic functionals and the space of entire functions of exponential type. (Recall that an entire function f is of exponential type if there exists a constant τ such that $|f(z)| = O(\exp \tau |z|)$ as $|z| \to \infty$.) This allowed them to transform the statement of results into the space of entire functions of exponential type which is more convenient for processing the proof.

2. D-Taylor Projectors and Preservation of HDR

2.1. Let us discuss different characterizations of polynomial projectors that preserve HDR. Calvi introduced in [4] a general class of interpolation spaces characterizing the polynomial projectors preserving HDR. The following assertion proven in [5], gives a possibility to describe the polynomial projectors that preserve HDR via their space of interpolation conditions.

Let k be a positive integer and $\mu_0, \mu_1, \dots, \mu_d$ be $d+1$ not necessarily distinct analytic functionals on $H(\mathbb{C}^n)$ such that $\mu_i(1) \neq 0$ for $i = 0, \dots, d$. Then

$$\mathbf{I} := \langle D^\alpha \mu_{|\alpha|},\ |\alpha| \le k \rangle \tag{2}$$

is an interpolation space for $\mathcal{P}_d(\mathbb{C}^n)$. Recall that for the analytic functional $\varphi \in H'(\mathbb{C}^n)$ and multi-index α the derivative $D^\alpha \varphi$ is defined by

$$D^\alpha \varphi(f) := \varphi(D^\alpha f),$$

for all $f \in H(\mathbb{C}^n)$.

The projectors $\mathcal{P}(\mathbf{I})$ corresponding to spaces $\mathbf{I}$ as in (2) is called *decentered-Taylor projectors of degree k* or, for short, D-Taylor projectors [5]. It is not difficult to see that every univariate projector is a D-Taylor projector.

For $a \in \mathbb{C}^n$, the analytic functional $[a]$ is defined by taking the value of $f \in H(\mathbb{C}^n)$ at the point a, i.e.,

$$[a](f) = f(a).$$

For $\alpha \in \mathbb{Z}_+^n$ and $a \in \mathbb{C}^n$, we have

$$D^\alpha[a](f) = [a] \circ D^\alpha(f) = D^\alpha f(a), \ f \in H(\mathbb{C}^n).$$

An analytic functional of the form $[a]$ or $D^\alpha[a]$ is called *a discrete functional*.

Let $a_0, \ldots, a_d \in \mathbb{C}^n$ be not necessary distinct points. A typical D-Taylor projector is the *Abel-Gontcharoff interpolation projector* $G_{[a_0,\ldots,a_d]}$ for which the space of interpolation condition is defined by

$$\mathcal{I}(G_{[a_0,\ldots,a_d]}) := \langle D^\alpha[a_{|\alpha|}], \, |\alpha| \le k \rangle.$$

2.3. Let us consider polynomial projectors preserving HPDE of degree 1, the simplest case. An entire function f is a solution of the equations

$$b_1 \frac{\partial f}{\partial z_1} + \cdots + b_n \frac{\partial f}{\partial z_n} = 0$$

for every b with $a.b = 0$ if and only if it is of the form

$$f(z) = h(a.z)$$

with $h \in H(\mathbb{C})$, where

$$y.z := \sum_{i=1}^n y_i z_i, \ y, z \in \mathbb{C}^n.$$

These functions f composed of a univariate function with a linear form are called *ridge functions*. Let Π be a polynomial projector preserving HPDE of degree 1. From the definition we can easily see that Π also preserves ridge functions, that is, if $f(z) = h(a.z)$, then there exists a univariate polynomial p such that

$$\Pi(h(a.\cdot))(z) = p(a.z).$$

This formula defines a univariate polynomial projector which is denoted by Π_a, satisfying the following property

$$\Pi_a(h)(a.z) = \Pi(h(a.\cdot))(z).$$

As shown below the converse is true. More precisely, Π preserves ridge functions if it preserves HPDE of degree 1.

2.4. Calvi and Filipsson [5] recently have proven the following theorem giving different characterizations of the polynomial projectors that preserve HDR.

Theorem 1. *Let Π be a polynomial projector of degree d in $H(\mathbb{C}^n)$. Then the following four conditions are equivalent.*

(1) Π *preserves HDR.*
(2) Π *preserves ridge functions.*
(3) Π *is a D-Taylor projector.*
(4) *There are analytic functionals* $\mu_0, \mu_1, \dots, \mu_d \in H'(\mathbb{C}^n)$ *with* $\mu_i(1) \neq 0,\ i = 0, 1, \dots, d$, *such that* Π *is represented in the following form*

$$\Pi(f) = \sum_{|\alpha| \le d} D^\alpha \mu_{|\alpha|}(f) u_\alpha.$$

This theorem shows that a polynomial projector Π preserving HPDE of degree 1 also preserves HDR.

Let Π be a D-Taylor projector of degree d on $H(\mathbb{C}^n)$ and $\varphi \in H'(\mathbb{C}^n)$. If α is a multi-index such that $D^\alpha \varphi \in \mathcal{I}(\Pi)$ then $D^\beta \varphi \in \mathcal{I}(\Pi)$ for every β with $|\beta| = |\alpha|$. Furthermore, if $\varphi(1) = 1$ then there exists a representing sequence μ for Π such that $\mu_{|\alpha|} = \varphi$ (see [5]).

2.5. Kergin [17, 18] introduced in a natural way a real multivariate interpolation projector which is a generalization of Lagrange interpolation projector. Let us give a complex version of Kergin interpolation polynomial projector $K_{[a_0,\dots,a_d]}$, associated with the points $a_0, \dots, a_d \in \mathbb{C}^n$. (For a full complex treatment see [1].) This is done by requiring the polynomial $K_{[a_0,\dots,a_d]}(f)$ to interpolate f not only at $a_0, \dots, a_d$, but also derivatives of f of order k somewhere in the convex hull of any $k+1$ of the points. More precisely, he proved the following

Theorem 2. *Let be given not necessarily distinct points* $a_0, \dots, a_d \in \mathbb{C}^n$. *Then there exists a unique linear map* $K_{[a_0,\dots,a_d]}$ *from* $H(\mathbb{C}^n)$ *into* $\mathcal{P}_d(\mathbb{C}^n)$, *such that for every* $f \in H(\mathbb{C}^n)$, *every* k, $1 \le k \le d$, *every homogeneous polynomial* q *of degree* k, *and every set* $J \subset \{0, 1, ..., d\}$ *with* $|J| = k+1$, *there exists a point* b *in the convex hull of* $\{a_j : j \in J\}$ *such that*

$$q(D)(f, b) = q(D)(K_{[a_0,\dots,a_d]}(f), b).$$

Moreover, $K_{[a_0,\dots,a_d]}$ *is a polynomial projector of degree* d, *and preserves HDR.*

An explicit description of the space of interpolation conditions of Kergin interpolation projectors is given by Michelli and Milman [21] in terms of simplex functionals. More precisely, they proved the following

Theorem 3. *Let be given not necessarily distinct points* $a_0, \dots, a_d \in \mathbb{C}^n$. *Then the Kergin interpolation projector* $K_{[a_0,\dots,a_d]}$ *of degree* d *is a D-Taylor projector and*

$$\mathcal{I}(K_{[a_0,\dots,a_d]}) = \langle D^\alpha \mu_{|\alpha|}, |\alpha| \le k \rangle,$$

where μ_i *is a simplex functional, i.e.,*

$$\mu_i(f) = i! \int_{S_i} f(s_0 a_0 + s_1 a_1 + \dots + s_i a_i) dm(s) \quad (0 \le i \le k), \tag{3}$$

the simplex S_i *is defined by*

$$S_i := \{(s_0, s_1, \ldots, s_i) \in [0,1]^{i+1} : \sum_{j=0}^{i} s_j = 1\},$$

and dm is the Lebesgue measure on S_i.

3. Derivatives of D-Taylor Projector

3.1. If Π is a D-Taylor projector and $\mu := (\mu_0, \ldots, \mu_d)$ a sequence such that

$$\mathcal{I}(\Pi) = \langle D^\alpha \mu_{|\alpha|}, |\alpha| \leq k \rangle,$$

then clearly, μ is not unique, even when we normalize the functionals by $\mu_i(1) = 1,\ i = 0, 1, \ldots, d$. For example, using the fact that a Kergin interpolation operator is invariant under any permutation of the points, we may take for $\Pi = K[a_0, \ldots, a_d]$ the functionals

$$\mu_i^\sigma(f) = i! \int_{S_i} f(s_0 a_{\sigma(0)} + s_1 a_{\sigma(1)} + \cdots + s_d a_{\sigma(i)}) dm(s) \quad (0 \leq i \leq d)$$

where σ is any permutation of $\{0, 1, 2, \ldots, d\}$. Let us discuss this question in details. Given a sequence of functionals of length $d+1$ $\mu = (\mu_0, \ldots, \mu_d)$ with $\mu_i \in H'(\mathbb{C}^n)$, we set

$$\Pi^\mu := \wp(\langle D^\alpha \mu_{|\alpha|}, |\alpha| \leq d \rangle).$$

When $\Pi = \Pi^\mu$, we say that μ is a *representing sequence* for the D-Taylor projector Π (or that μ represents Π), and if in addition, $\mu_i(1) = 1$, $i = 0, 1, \ldots, d$, a *normalized representing sequence.* As already noticed a normalized representing sequence is not unique. However the sequences representing the same D-Taylor projector are in an equivalence relation determined by the following assertion.

Let $\mu := (\mu_0, \ldots, \mu_d)$ and $\mu' := (\mu'_0, \ldots, \mu'_d)$ be two normalized sequences. In order that both sequences represent the same D-Taylor projector, i.e. $\Pi^\mu = \Pi^{\mu'}$, it is necessary and sufficient that there exist complex coefficients $c_l,\ l \in \{1, ..., n\}^j,\ 0 \leqslant j \leqslant d$, such that

$$\mu'_i = \mu_i + \sum_{j=1}^{d-i} \sum_{l \in \{1,\ldots,n\}^j} c_l D^l \mu_{|\beta|+j}, \quad 0 \leq i \leq d. \tag{4}$$

The relation (4) between μ and μ' is clearly an equivalence relation. We shall write $\mu \sim \mu'$. Note that the last normalized functional is always unique, i.e. $\mu \sim \mu' \implies \mu_d = \mu'_d$.

3.2. Let us now define the k-th derivative of a D-Taylor projector of degree d for $1 \leq k \leq d$ introduced in [5]. Given a normalized sequence $\mu = (\mu_0, \ldots, \mu_d)$ of length $d+1$, we define a normalized sequence μ^k of the length $d - k + 1$ by setting $\mu^k := (\mu_k, \ldots, \mu_d)$. In view of (4), if $\mu_1 \sim \mu_2$ then $\mu_1^k \sim \mu_2^k$ and this shows that the following definition is consistent. Let Π be a D-Taylor projector of degree d. We define $\Pi^{(k)}$ as Π^{μ^k} where μ is any representing sequence for Π. This is a D-Taylor projector of degree $d-k$. We shall call it *the k-th derivative* of Π. This notion is motivated by the following argument.

Let Π be a D-Taylor projector of degree d and let $1 \leq k \leq d$. Then for every homogeneous polynomial q of degree k we have

$$q(D)\Pi(f) = \Pi^{(k)}(q(D)f) \quad (f \in H(\mathbb{C}^n)).$$

The derivatives of an Abel-Gontcharoff interpolation projector are again Abel-Gontcharoff interpolation projectors, namely

$$G^{(k)}_{[a_0,a_1,\dots,a_d]} = G_{[a_k,a_{k+1},\dots,a_d]}$$

and, for the more particular case of Taylor interpolation projectors, we have

$$(T^d_a)^{(k)} = T^{d-k}_a.$$

The concept of derivative of D-Taylor projector provides an interesting new approach to some well-known projectors.

3.3. Let $A = \{a_0, \dots, a_{d+n-1}\}$ be $n+d$ (pairwise) distinct points in $\mathbb{C}^n$ which are in general position, that is, every subset $B = \{a_{i_1}, \dots, a_{i_n}\}$ of cardinality n of A defines a proper simplex of $\mathbb{C}^n$. For every $B = \{a_{i_1}, \dots, a_{i_n}\}$, we define μ_B as the simplex functional corresponding to the points of B:

$$\mu_B(f) = \int_{S_{n-1}} f(t_1 a_{i_1} + t_2 a_{i_2} + \cdots + t_n a_{i_n})dt.$$

Hakopian [16] has shown that given numbers c_B, there exists a unique polynomial $p \in \mathcal{P}_d$ such that $\mu_B(p) = c_B$ for every B. When $c_B = \mu_B(f)$ the map $f \mapsto p = H_A(f)$ is called the Hakopian interpolation projector with respect to A. Notice that the polynomial projector H_A is actually defined for functions merely continuous on the convex hull of the points of A. In fact, using properties of the simplex functional, this projector can be seen as the extension of a projector naturally defined on analytic functions and much related to Kergin interpolation (for Hakopian interpolation projectors we refer to [16] or [2] and the references therein). More precisely, the polynomial projector $p = H_A(f)$ is determined by the space of interpolation conditions generated by the analytic functionals

$$\int\limits_{S_l} D^\alpha f(t_0 a_0 + t_1 a_1 + \cdots + t_l a_l)dt,$$

with $|\alpha| = l - n + 1,\ n-1 \leq l \leq n+d-1$. Whereas the Kergin interpolation projector corresponding to the set of nodes $\{a_0, \dots, a_d\}$, is characterized by (3). Hence we can see that

$$K^{(n-1)}_{[a_0,a_1,\dots,a_{d+n-1}]} = H_{[a_0,\dots,a_{d+n-1}]}.$$

3.4. The Kergin interpolation is also related to the so called mean value interpolation which appears in [10] and [15], (see also [8]). The mean value interpolation projector is the lifted multivariate version of the following univariate operator.

Let Ω be a simply connected domain in $\mathbb{C}$ and $A = \{a_0, \dots, a_d\}$ $d+1$ not necessarily distinct points in Ω. For $f \in H(\Omega)$ we define $f^{(-m)}$ to be any m-th integral of f, that is, $(f^{(-m)})^{(m)} = f$. Since Ω is simply connected, $f^{(-m)}$ exists in $H(\Omega)$ but, of course, is not unique. Now, using $L_A(u)$ to denote the Lagrange-Hermite interpolation polynomial of the function u corresponding to the points of A, the univariate mean value polynomial projector $L_A^{(m)}$ is defined for $0 \le m \le d$ by the relation

$$L_A^{(m)}(f) = [L_A(f^{(-m)})]^{(m)}.$$

It turns out that the definition does not depend on the choice of integral and is therefore correct. Now, let A be a subset of $d+1$ non necessarily distinct points in the convex set Ω in $\mathbb{C}^n$. Then it can be proven that there exists a (unique) continuous polynomial projector of degree d on $H(\Omega)$, denoted by $\mathcal{L}_A^{(m)}$, which lifts the univariate projector $L_{l(A)}^{(m)}$, that is,

$$\mathcal{L}_A^{(m)}(f) = L_{l(A)}^{(m)}(h) \circ l$$

for every ridge function $f = h \circ l$ where h is a univariate function and l a linear form on $\mathbb{C}^n$. This polynomial projector $\mathcal{L}_A^{(m)}$ is called the m-th mean value interpolation operator corresponding to A. The interpolation conditions of the projector can be expressed in terms of the simplex functionals. For details the reader can consult [5, 13] for the complex case, [15] for the real case.

The derivatives of a Kergin interpolation projector are nothing else than the mean value interpolation projectors. More precisely, we have from [13] and [5]

$$K_{[a_0,a_1,\dots,a_d]}^{(m)} = \mathcal{L}_{\{a_0,\dots,a_d\}}^{m}.$$

4. Interpolation Properties

4.1. Let Π be a polynomial projector on $H(\mathbb{C}^n)$. We say that Π interpolates at a with the multiplicity $m = m(a) \ge 1$ if there exists a sequence $\alpha(i)$, $i = 0, \dots, m-1$ with $|\alpha(i)| = i$ and $D^{\alpha(i)}[a] \in \mathcal{I}(\Pi)$, i.e.,

$$D^{\alpha(i)}(\Pi(f))(a) = D^{\alpha(i)} f(a), \ \forall f \in H(\mathbb{C}^n).$$

In the contrary case, we set $m(a) = 0$. Note that we always have $m(a) \le d+1$ where d is the degree of Π.

We shall say that Π interpolates at k points taking multiplicity into account if $\sum_{a\in\mathbb{C}^n} m(a) = k$.

From the remark in Subs. 2.4 we can see that if Π is a polynomial projector of degree d preserving HDR and $D^\alpha[a] \in \mathcal{I}(\Pi)$ for some multi-index α and $a \in \mathbb{C}^n$, then $D^\beta[a] \in \mathcal{I}(\Pi)$ for every β with $|\beta| = |\alpha|$. Moreover, there exists a representing sequence μ for Π such that $\mu_{|\alpha|} = [a]$. Thus, we arrive at the following interpolation properties of polynomial projectors preserving HDR.

Let Π be a polynomial projector of degree d preserving HDR, and $a \in \mathbb{C}^n$. Then the following conditions are equivalent.

(i). Π interpolates at a with the multiplicity m.

(ii). There is a representing sequence μ of Π such that $\mu_i = [a]$ for $0 \le i \le m-1$.

(iii). $\Pi^{(k)}$ interpolates at a with the multiplicity $m-k$ for $k=0,\dots,m-1$.

The next theorem shows that the simplex functionals (and behind them the Kergin interpolation projectors) are involved in every polynomial projector that preserves HDR and interpolates at sufficiently many points. It would be possible, more generally, to prove a similar theorem in which the game played by Kergin interpolation would be taken by some lifted Birkhoff interpolant constructed in [8].

Theorem 4. *A polynomial projector* Π *of degree* d*, preserving HDR, interpolates at most at* $d+1$ *points taking multiplicity into account. Moreover, a polynomial projector* Π *of degree* d *is Kergin interpolation projector if and only if it preserves HDR and interpolates at a maximal number of* $d+1$ *points.*

Theorem 3 describes a new characterization of the Kergin interpolation projectors of degree d as the polynomial projectors Π of degree d that preserve HDR and interpolate at $d+1$ points taking multiplicity into account.

4.2. The Abel-Gontcharoff projectors can be characterized as the Birkhoff interpolation projectors preserving HDR. Let us first define Birkhoff interpolation projectors. Denote by $S = S_d$ the set of n-indices of length $\le d$ and $Z = \{z_1,\dots,z_m\}$ a set of m pairwise distinct points in $\mathbb{C}^n$. A *Birkhoff interpolation matrix* is a matrix E with entries $e_{i,\alpha}$, $i \in \{1,\dots,m\}$ and $\alpha \in S$ such that $e_{i,\alpha} = 0$ or 1 and $\sum_{i,\alpha} e_{i,\alpha} = |S|$ where $|\cdot|$ denote the cardinality. Thus the number of nonzero entries of E (which is also the number of 1-entries of E) is equal to the dimension of the space P_d of polynomials of n variables of degree at most d. Notice that E is a $m \times |S|$ matrix. Then, given numbers $c_{i,\alpha}$, the (E,Z)*-Birkhoff interpolation problem* consists in finding a polynomial $p \in P_d$ such that

$$D^\alpha p(z_i) = c_{i,\alpha} \text{ for every } (i,\alpha) \text{ such that } e_{i,\alpha} = 1. \tag{5}$$

When the problem is solvable for every choice of the numbers $c_{i,\alpha}$ (and, therefore, in this case, uniquely solvable), one says that the Birkhoff interpolation problem (E,Z) is poised. If (E,Z) is poised and the values $c_{i,\alpha}$ are given by $D^\alpha[z_i](f)$, then there is a unique polynomial $p_{(E,Z)}(f)$ solving equations (5) which is called the (E,Z)-*a Birkhoff interpolation polynomial of* f. The map $f \mapsto p_{(E,Z)}(f)$ is then a polynomial projector of degree d and denoted by $B_{(E,Z)}$ which is called *a Birkhoff interpolation projector.* Its space of interpolation conditions

$$\mathcal{I}(B_{(E,Z)}) = \langle D^\alpha[z_i],\ e_{i,\alpha} = 1\rangle$$

is easily described from (5). Thus, a Birkhoff interpolation projector can be defined as a polynomial projector Π for which $\mathcal{I}(\Pi)$ is generated by discrete functionals. A basic problem in Birkhoff interpolation theory is to give conditions on the matrix E in order that the problem (E,Z) be poised for almost every choice of Z. A general reference for multivariate Birkhoff interpolation is [18] (see also [19]) in which the authors characterize all the matrices E for which (E,Z) is poised for every Z.

The Abel-Gontcharoff interpolation projectors are a very particular case of poised Birkhoff interpolation problems. They are obtained in taking $m = d+1$ and $e_{i,\alpha} = 1$ if and only if $|\alpha| = i-1$. In that case the problem (E, Z) is easily shown to be poised for every Z. A treatment of multivariate Gontcharoff interpolation emphasizing its relation with its univariate counterpart with the use of ridge functions can be found in [8].

The following theorem proven in [12], characterises the Abel-Gontcharoff interpolation projectors as Birkhoff interpolation projectors preserving HDR.

Theorem 5. *Let $n \geq 2$. Then a polynomial projector Π is a Birkhoff interpolation projector of degree d, preserving HDR if and only if it is an Abel-Gontcharoff interpolation projector, that is, there are $a_0, \ldots, a_d \in \mathbb{C}^n$ not necessarily distinct such that*

$$\mathcal{I}(\Pi) = \langle D^\alpha[a_s],\ |\alpha| = s,\ \ s = 0, \ldots, d\rangle.$$

It is worth noting that this result is typical of the higher dimension. It is indeed not true in dimension 1 in which the concept of projector preserving HDR reduces to a triviality.

5. Polynomial Projectors Preserving HPDE

5.1. As mentioned in Sec. 2, if a polynomial projector of degree d preserves HPDE of degree 1, then it preserves HDR. If $1 < k \leq d$, there arises a natural question: does exist a polynomial projector of degree d which preserves HPDE of degree k but not HPDE of all degree smaller than k, and how to characterize the polynomial projectors preserving HPDE of degree k. The following theorem proven in [12], and its consequences give an answer to this question.

Theorem 6. *A polynomial projector Π of degree d preserves HPDE of degree k, $1 \leq k \leq d$ if and only if there are analytic functionals $\mu_k, \mu_{k+1}, \ldots, \mu_d \in H'(\mathbb{C}^n)$ with $\mu_i(1) \neq 0,\ i = k, \ldots, d$, such that Π is represented in the following form*

$$\Pi(f) = \sum_{|\alpha|<k} a_\alpha(f) u_\alpha + \sum_{k \leq |\alpha| \leq d} D^\alpha \mu_{|\alpha|}(f) u_\alpha, \tag{6}$$

with some $a_\alpha{}'s \in H'(\mathbb{C}^n),\ |\alpha| < k$.

5.2. From Theorem 1 we can derive some interesting properties of polynomial projectors preserving HPDE of degree k (see [12] for details).

(i). If the polynomial projector Π of degree d preserves HPDE of degree k, $1 \leq k \leq d$, then Π preserves also HPDE of all degree greater than k.

(ii). If $1 < k \leq d$, there is a polynomial projector of degree d which preserves HPDE of degree k but not HPDE of all degree smaller than k.

Such a polynomial projector can be constructed as follows. Since the set $\{u_\alpha : |\alpha| \leq d\}$ is linearly independent, there exist distinct $\mu_1, \mu_2 \in H'(\mathbb{C}^n)$ such that $\mu_j(1) = 1,\ j = 1,2$ and $\mu_j(u_\alpha) = 0,\ 1 \le |\alpha| \le d,\ j = 1,2$. Fix two multi-indices α^1, α^2 with $|\alpha^1| = |\alpha^2| = k-1$. We have

$$D^{\alpha^j}\mu_j(u_\beta) = \delta_{\alpha^j\beta},\ j = 1,2.$$

Then the polynomial projector Π of degree d defined by

$$\Pi(f) = \sum_{j=1}^{2} D^{\alpha^j}\mu_j(f)u_{\alpha^j} \ + \sum_{|\alpha|\le d,\ \alpha\ne\alpha^1,\alpha^2} D^\alpha[0](f)u_\alpha,$$

preserves HPDE of degree k but not HPDE of any degree smaller than k.

Finally, the following corollary can be considered as a generalization of the formula (4) on representing sequences for D-Taylor projectors.

(iii). Let Π be a polynomial projector of degree d preserving HPDE of degree $k,\ 1 \le k \le d$. Then there are functionals $\mu_k, \mu_{k+1}, \dots, \mu_d$ with $\mu_i(1) = 1$ $(k \le i \le d)$ such that the set

$$\langle D^\alpha\mu_s,\ |\alpha| = s,\ s = k, \dots, d\rangle$$

is a proper subset of $\mathcal{I}(\Pi)$. Moreover, if Π is represented as in (6) with $\mu_i(1) = 1,\ i = k, \dots, d$, and if for some β with $|\beta| \ge k$, we have $D^\beta\nu \in \mathcal{I}(\Pi)$, then there exists a relation

$$\nu = \mu_{|\beta|} + \sum_{j=1}^{d-|\beta|} \sum_{l\in\{1,\dots,n\}^j} c_l D^l \mu_{|\beta|+j}.$$

6. Some Final Remarks

6.1. Runge Domain

Recall that Ω is a Runge domain if entire functions are dense in $H(\Omega)$. We do not lose generality on studying projectors on $H(\mathbb{C}^n)$ rather than $H(\Omega)$ where Ω is any Runge domain of $\mathbb{C}^n$. In particular, all formulated above results for $H(\mathbb{C}^n)$, can be extended to $H(\Omega)$. Indeed, if Π is a polynomial projector on $H(\Omega)$, we may apply these results to the restriction of Π to $H(\mathbb{C}^n)$ which in that case completely characterizes the global projector Π.

6.2. Real Version

A real version of the discussed aproach can be applied to study the polynomial projectors on $C^\infty(\mathbb{R}^n)$ whose coefficients are distributions with compact support. As noticed in Introduction, the Laplace transform played a central role in the proof of main results discussed in the present paper. The same methods employed in [5, 12] will work if we use the Fourier transform instead of the Laplace transform together with a (multivariate) Paley-Wiener Theorem to play the game of the isomorphism between analytic functionals and entire functions of exponential type.

6.3. Open Problems

(i). How many are points at which can interpolate a polynomial projectors preserving HPDE of degree $k > 1$, and how to describe the polynomial projectors preserving HPDE of degree $k > 1$ and interpolating at a maximal number points? (In the case $k = 1$ the answer is given in Theorem 4.)

(ii). Characterize the Birkhoff projectors preserving HPDE of degree $k > 1$? (In the case $k = 1$ the answer is given in Theorem 5.)

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