Vietnam Journal of Mathematics **34**:3 (2006) 255–264

Vietnam Journal
of
MATHEMATICS

Weighted Composition Operators Between Different Weighted Bergman Spaces in Polydiscs

Li Songxiao*

Department of Math., Shantou University,515063, Shantou, Guangdong, China and Department of Mathematics, Jiaying University, 514015, Meizhou, Guangdong, China

Received April 04, 2004
Revised September 18, 2005

Abstract. Let D^n be the unit polydiscs of $\mathbb{C}^n$, $\varphi(z)=(\varphi_1(z),\ldots,\varphi_n(z))$ be a holomorphic self-map of D^n and $\psi(z)$ a holomorphic function on D^n. Necessary and sufficient conditions are established for the weighted composition operator ψC_φ induced by $\varphi(z)$ and $\psi(z)$ to be bounded or compact between different weighted Bergman spaces in polydiscs.

2000 Mathematics Subject Classification: 47B38, 32A36.
Keywords: Bergman space, polydiscs, weighted composition operator.

1. Introduction

We adopt the notation described in [4-6]. Denote by D^n the unit polydisc in $\mathbb{C}^n$, by T^n the distinguished boundary of D^n, by $A^p_\alpha(D^n)$ the weighted Bergman spaces of order p with weights $\prod_{i=1}^n(1-|z_i|^2)^\alpha$, $\alpha > -1$. We use m_n to denote the n-dimensional Lebesgue area measure on T^n, normalized so that $m_n(T^n) = 1$. By σ_n we shall denote the volume measure on $\overline{D^n}$ given by $\sigma_n(D^n) = 1$, and by $\sigma_{n,\alpha}$ we shall denote the weighted measure on $\overline{D^n}$ given by $\sigma_{n,\alpha} = \prod_{i=1}^n(1 - |z_i|^2)^\alpha\sigma_n$. If R is a rectangle on T^n, then $S(R)$ denote the corona associated to

*The author is partially supported by NNSF(10371051) and ZNSF(102025).

R. In particular, if $R = I_1 \times I_2 \times \cdots \times I_n \subset T^n$, with I_i being the intervals on T^n of length δ_i and centered at $e^{i(\theta_i^0+\delta_i/2)}$ for $i = 1, \cdots, n$, then $S(R)$ is given by $S(R) = S(I_1) \times S(I_2) \times \cdots \times S(I_n)$, where

$$S(I_i) = \{re^{i\theta} \in D : 1 - \delta_i < r < 1, \theta_0^i < \theta < \theta_i^0 + \delta_i\}.$$

For $\alpha > -1, 0 < p < \infty$, recall that the weighted Bergman space $A_\alpha^p(D^n)$ consists of all holomorphic functions on the polydisc satisfying the condition

$$\|f\|_{A_\alpha^p}^p = \int_{D^n} |f(z)|^p \prod_{i=1}^n (1 - |z_i|^2)^\alpha d\sigma_{n,\alpha} < +\infty.$$

Denoted by $H(D^n)$ the class of all holomorphic functions with domain D^n. Let φ be a holomorphic self-map of D^n, the composition operator C_φ induced by φ is defined by $(C_\varphi f)(z) = f(\varphi(z))$ for z in D^n and $f \in H(D^n)$. If, in addition, ψ is a holomorphic function defined on D^n, the weighted composition operators ψC_φ induced by ψ and φ is defined by

$$(\psi C_\varphi f)(z) = \psi(z) f(\varphi(z))$$

for z in D^n and $f \in H(D^n)$.

It is interesting to characterize the composition operator on various analytic function spaces. The book [2] contains plenty of information. It is well known that composition operator is bounded on the Hardy space and the Bergman space in the unit disc. This result does not carry over to the case of several complex variables. Singh and Sharma has showed in [7] that not every holomorphic map from D^n to D^n induces a composition operator on $H^p(D^n)$. For example, $\varphi(z_1, z_2) = (z_1, z_2)$ does not induce a bounded composition operator on $H^2(D^n)$.

Jafari studied the composition operator on Bergman spaces $A_\alpha^p(D^n)$ in [6]. His results can be stated as follows.

Theorem A. *Let $1 < p < \infty, \alpha > -1$ and let φ be a holomorphic self-map of D^n. Define μ to be $\mu(E) = \sigma_{n,\alpha}(\varphi^{-1}(E))(E \subset \overline{D^n})$. Then C_φ is bounded (compact) on $A_\alpha^p(D^n)$ if and only if μ is an (compact) α Carleson measure.*

Theorem B. *Let $1 < p < \infty$ and $\alpha > -1$ and let φ be a holomorphic self-map of D^n. Then*

(i) *C_φ is a bounded composition operator on $A_\alpha^p(D^n)$ if and only if*

$$\sup_{z_0 \in D^n} \int_{D^n} \prod_{i=1}^n \Big[\frac{1 - |(z_0)_i|^2}{|1 - (\overline{z_0})_i \varphi_i|^2}\Big]^{2+\alpha} d\sigma_{n,\alpha} \le M < \infty.$$

(ii) *C_φ is a compact composition operator on $A_\alpha^p(D^n)$ if and only if*

$$\limsup_{z_0 \in D^n} \int_{D^n} \prod_{i=1}^n \Big[\frac{1 - |(z_0)_i|^2}{|1 - (\overline{z_0})_i \varphi_i|^2}\Big]^{2+\alpha} d\sigma_{n,\alpha} = 0$$

as $\|z_0\| \to 1$.

In this paper, we study the weighted composition operators between different weighted Bergman spaces in polydiscs. Some measure characterizations and function theoretic characterizations are given for the boundedness and compactness of the weighted composition operators.

Throughout the remainder of this paper C will denote a positive constant, the exact value of which may vary from one appearance to the next.

2. Measure Characterization of Weighted Composition Operators

In this section, we give the measure characterization of weighted composition operators between different weighted Bergman spaces. For this purpose, we should need some lemmas which will be stated as follows.

Definition 1. *A finite, nonnegative, Borel measure μ on $\overline{D^n}$ is said to be a $\eta-\alpha$ Carleson measure if*

$$\mu^{\frac{1}{\eta}}(S(R)) \le C\prod_{i=1}^{n}\delta_i^{2+\alpha}$$

for all $R \subset T^n$.

μ is said to be a compact $\eta-\alpha$ Carleson measure if

$$\lim_{\delta_i\to 0}\sup_{\theta\in T^n}\frac{\mu^{\frac{1}{\eta}}(S(R))}{\prod_{i=1}^{n}\delta_i^{2+\alpha}} = 0.$$

Remark. When $\eta = 1$, the definition of Carleson measures for polydiscs is due to Chang(see [1]).

Modifying the proof of Theorem 2.5 in [5], we get the following lemma.

Lemma 1. *Suppose that $1 < p < \infty$, $\alpha > -1, \eta \ge 1$. Let I be the identity operator from $A^p_\alpha(D^n)$ into $L^{\eta p}(D^n,\mu)$. Then I is a bounded operator if and only if μ is an $\eta-\alpha$ Carleson measure.*

Proof. We prove that if $f \in A^p_\alpha(D^n)$ then

$$\Big\{\int_{\overline{D^n}}|f|^{\eta p}d\mu\Big\}^{\frac{1}{\eta}} \le C\int_{\overline{D^n}}|f|^p d\sigma_{n,\alpha} \qquad (1)$$

if and only if

$$\mu^{\frac{1}{\eta}}(S(R)) \le C\prod_{j=1}^{n}\delta_j^{\alpha+2}. \qquad (2)$$

For this purpose, suppose that (1) holds for all $f \in A^p_\alpha(D^n)$. Define

$$f(z) = \prod_{j=1}^{n}(1-\overline{\alpha_j}z_j)^{-(\alpha+2)/p},$$

where $\alpha_j = (1-\delta_j)e^{i(\theta_j^0+\delta_j/2)}$. It is easy to see that $f(z) \in A^p_\alpha(D^n)$. In addition, since on $S(R)$,

$$|f(z)|^{\eta p} > 2^{-\eta(\alpha+2)} \prod_{j=1}^{n} \delta_j^{-\eta(\alpha+2)},$$

we have

$$\Big\{ \int_{\overline{D^n}} |f(z)|^{\eta p} d\mu \Big\}^{\frac{1}{\eta}} \geq \Big\{ \int_{S(R)} |f(z)|^{\eta p} d\mu \Big\}^{\frac{1}{\eta}} \geq 2^{-(\alpha+2)} \prod_{j=1}^{n} \delta_j^{-(\alpha+2)} \mu^{\frac{1}{\eta}}(S(R)). \tag{3}$$

Then the result follows from (1) and (3).

Conversely, suppose that (2) holds for all rectangles in T^n. Fix $z \in D^n$ and let $1 - |z_j|^2 = \delta_j$, consider a polydisc W_z centered at z and of radius $\delta_j/2$ in the z_j coordinate. If $R = I_1 \times \ldots \times I_n$ is the rectangle on T^n with I_j centered at $z_j/|z_j|$ and $|I_j| = 2\delta_j$, then $W_z \subset S(R)$ (see [5]). Therefore for any $f \in A^p_\alpha(D^n)$, by the sub mean value property for $|f|$, we get (see [5])

$$|f(z)| \leq \frac{C}{\prod_{j=1}^{n} \delta_j^{\alpha+2}} \int_{S(R)} |f| d\sigma_{n,\alpha}.$$

Since $\sigma_{n,\alpha}(S(R)) = C \prod_{j=1}^{n} \delta_j^{\alpha+2}$, then

$$|f(z)| \leq \frac{C}{\sigma_{n,\alpha}(S(R))} \int_{S(R)} |f| d\sigma_{n,\alpha}. \tag{4}$$

Now define

$$M(f) = \sup_R \frac{1}{\sigma_{n,\alpha}(S(R))} \int_{S(R)} |f| d\sigma_{n,\alpha}.$$

We get

$$|f(z)| \leq CM(f)(z). \tag{5}$$

We will show that there exists a constant C independent of s such that

$$\mu\{M(f) > s\} \leq C(s^{-1}\|f\|_{1,\alpha})^{\eta}. \tag{6}$$

Given (6), since M is a sublinear operator of type (∞, ∞), it is obvious that $\|M(f)\|_\infty \leq \|f\|_\infty$. If we define $\frac{1}{p} = \theta, \frac{1}{q} = \frac{\theta}{\eta}$, $0 < \theta < 1$, i.e. $q = p\eta$, by the Marcinkiewicz interpolation theorem we obtain that

$$\Big\{ \int_{\overline{D^n}} |M(f)|^{\eta p} d\mu \Big\}^{\frac{1}{\eta}} \leq C \int_{\overline{D^n}} |f|^p d\sigma_{n,\alpha}. \tag{7}$$

Combining (5) and (7) we get

$$\Big\{ \int_{\overline{D^n}} |f(z)|^{\eta p} d\mu \Big\}^{\frac{1}{\eta}} \leq C \Big\{ \int_{\overline{D^n}} |M(f)(z)|^{\eta p} d\mu \Big\}^{\frac{1}{\eta}} \leq C \int_{\overline{D^n}} |f(z)|^p d\sigma_{n,\alpha}.$$

This prove (1).

To complete the proof we need to show that if $\mu^{\frac{1}{\eta}}(S(R)) \leq C\sigma_{n,\alpha}(S(R))$, then (6) holds. Let $R_z = I_1 \times \ldots \times I_n$ denote rectangle on T^n with I_j denoting intervals centered at $z_j/|z_j|$ and of radius $(1-|I_j|)/2$. Let S_z denote the corona associated with R_z. Note that $z \in S_z$, define

$$A_s^{\epsilon} = \Big\{ z \in D^n : \int_{S_z} |f| d\sigma_{n,\alpha} > s(\epsilon + \sigma_{n,\alpha}(S_z)) \Big\}. \tag{8}$$

It is easily to check that the following equality holds

$$\Lambda = \{z \in D^n : M(f) > s\} = \bigcup_{\epsilon > 0} A_s^{\epsilon},$$

i.e. $\mu(\Lambda) = \lim_{\epsilon \to 0} \mu(A_s^{\epsilon})$. Furthermore, if $z \in A_s^{\epsilon}$ and S_z are disjoint for the different $z \in \Lambda$, then by (8) we have

$$s \sum_{z \in \Lambda} (\epsilon + \sigma_{n,\alpha}(S_z)) < \sum_{z \in \Lambda} \int_{S_z} |f| d\sigma_{n,\alpha} \leq \|f\|_{1,\alpha}.$$

Hence

$$s \sum_{z \in \Lambda} (\epsilon + \sigma_{n,\alpha}(S_z)) \leq \|f\|_{1,\alpha}. \tag{9}$$

Consider the last inequality (9), it shows that there are only finitely many $z \in A_s^{\epsilon}$ so that their corresponding S_z are disjoint. From these extract the points, $z_1, \ldots, z_l$, that in addition have the property that if their associated S_z radius are multiplied by five in each coordinate then the resulting sets cover A_s^{ϵ}. This follows from covering lemma. Write the S_z associated with these points as $S_1, S_2, \ldots, S_l$. Since $A_s^{\epsilon} \subset \bigcup_{k=1}^{l} 5S_k$, S_k are pairwise disjoint,

$$\mu(A_s^{\epsilon}) \leq 5^n \sum_{k=1}^{l} \mu(S_k), \tag{10}$$

(see [5]). Also, by hypothesis

$$\mu^{\frac{1}{\eta}}(S_k) \leq C\sigma_{n,\alpha}(S_k), \tag{11}$$

combining (9), (10) and (11), we get

$$\mu(A_s^{\epsilon}) \leq 5^n \sum_{k=1}^{l} \mu(S_k) \leq C \sum_{k=1}^{l} (\epsilon + \sigma_{n,\alpha}(S_k))^{\eta} \leq C(s^{-1}\|f\|_{1,\alpha})^{\eta}.$$

Letting ϵ tend to zero we obtain

$$\mu\{z \in D^n : M(f) > s\} \leq C(s^{-1}\|f\|_{1,\alpha})^{\eta}.$$

This completes the proof. ■

Using Lemma 1, we give a characterization of the boundedness of the weighted composition operator $\psi C_{\varphi} : A_{\alpha}^p(D^n) \to A_{\beta}^{\eta p}(D^n)$.

Theorem 1. *Suppose that $1 < p < \infty$, $\beta, \alpha > -1$, $\eta \geq 1$. Let φ be a holomorphic self-map of D^n and ψ be a holomorphic function on D^n, $d\nu = |\psi|^{\eta p} d\sigma_{n,\beta}$,*

$\mu(E)=\nu(\varphi^{-1}(E))(E\subset\overline{D^n})$. Then $\psi C_\varphi: A^p_\alpha(D^n)\to A^{\eta p}_\beta(D^n)$ is bounded if and only if μ is a $\eta-\alpha$ Carleson measure.

Proof. If $\psi C_\varphi: A^p_\alpha(D^n)\to A^{\eta p}_\beta(D^n)$ is bounded, then there exists a constant C such that

$$\|\psi f\circ\varphi\|_{A^{\eta p}_\beta}\le C\|f\|_{A^p_\alpha}$$

for all $f\in A^p_\alpha(D^n)$, i.e.

$$\Big\{\int_{D^n}|\psi f\circ\varphi|^{\eta p}d\sigma_{n,\beta}(z)\Big\}^{\frac{1}{\eta}}\le C\int_{D^n}|f|^p d\sigma_{n,\alpha}(z).$$

By the definition of μ, we have (see [3, p. 163])

$$\int_{D^n}|\psi f\circ\varphi|^{\eta p}d\sigma_{n,\beta}=\int_{D^n}|f|^{\eta p}d\mu.$$

Hence

$$\Big\{\int_{D^n}|f|^{\eta p}d\mu\Big\}^{\frac{1}{\eta}}\le C\int_{D^n}|f|^p d\sigma_{n,\alpha}(z).$$

The assertion follows from Lemma 1.

Conversely, suppose that μ is an $\eta-\alpha$ Carleson measure. By Lemma 1, there exists a constant C such that

$$\Big\{\int_{D^n}|f|^{\eta p}d\mu\Big\}^{\frac{1}{\eta}}\le C\int_{D^n}|f|^p d\sigma_{n,\alpha}(z)$$

for all $f\in A^p_\alpha(D^n)$. By the definition of μ, $\|\psi f\circ\varphi\|_{A^{\eta p}_\beta}\le C\|f\|_{A^p_\alpha}$. Hence $\psi C_\varphi: A^p_\alpha(D^n)\to A^{\eta p}_\beta(D^n)$ is bounded. We are done. ∎

To characterize the compactness of weighted composition operators between different weighted Bergman spaces, we will need the following lemma, whose proof is an easy modification of that Proposition 3.11 in [2], we omit the proof.

Lemma 2. *Suppose that $1<p<\infty$, $\beta,\alpha>-1$, $\eta\ge 1$. Let φ be a holomorphic self-map of D^n and ψ be a holomorphic function on D^n. Then the weighted composition operator $\psi C_\varphi: A^p_\alpha(D^n)\to A^{\eta p}_\beta(D^n)$ is compact if and only if for any bounded sequence $\{f_k\}$ converging to zero in $A^p_\alpha(D^n)$, $\|\psi f_k\circ\varphi\|^{\eta p}_{\eta p,\beta}\to 0$ as $k\to\infty$.*

Theorem 2. *Suppose that $1<p<\infty$, $\beta,\alpha>-1$, $\eta\ge 1$. Let φ be a holomorphic self-map of D^n and ψ be a holomorphic function on D^n, $d\nu=|\psi|^{\eta p}d\sigma_{n,\beta}$, $\mu(E)=\nu(\varphi^{-1}(E))(E\subset\overline{D^n})$. Then $\psi C_\varphi: A^p_\alpha(D^n)\to A^{\eta p}_\beta(D^n)$ is compact if and only if μ is a compact $\eta-\alpha$ Carleson measure.*

Proof. Suppose $\psi C_\varphi: A^p_\alpha(D^n)\to A^{\eta p}_\beta(D^n)$ is compact. Let

$$f_\delta(z_1,z_2,\dots,z_n)=\prod_{i=1}^n\frac{\delta_i^{\beta-(\alpha+2)/p}}{(1-\overline{\eta_i}z_i)^\beta},$$

where $0 < \delta_j < 1$, $\beta > (\alpha+2)/p$ and $\eta_j = (1-\delta_j)e^{i(\theta_j^0+\delta_j/2)}$. These functions are bounded in $A_\alpha^p(D^n)$, and tend to zero weakly as $\delta_i \to 0$. Since on regions $S(R)$, $|1-\overline{\eta_i}z_i| < 2\delta_i$, we have

$$|f_\delta(z_1, z_2, \ldots, z_n)|^p > \prod_{i=1}^{n} \frac{\delta_i^{(\beta-(\alpha+2)/p)p}}{(2^\beta \delta_i^\beta)^p} = \prod_{i=1}^{n} \frac{1}{\delta_i^{\alpha+2} 2^{p\beta}},$$

then we get

$$\begin{aligned}\epsilon(\delta) = \|\psi f_\delta \circ \varphi\|_{\eta p,\beta}^{\eta p} &= \int_{D^n} |\psi f_\delta \circ \varphi|^{\eta p} d\sigma_{n,\beta} \\ &\geq \int_{\overline{D^n}} |f_\delta|^{\eta p} d\mu \geq \frac{\mu(S(R))}{\prod_{i=1}^n \delta_i^{\eta(\alpha+2)} 2^{\eta p\beta}},\end{aligned}$$

where $\epsilon(\delta) \to 0$ as $\delta_i \to 0$ for some i. Hence $\mu^{\frac{1}{\eta}}(S) \leq \epsilon(\delta) 2^{n\beta p} \prod_{i=1}^n \delta_i^{\alpha+2}$, i.e.

$$\lim_{\delta_i \to 0} \sup_{\theta \in T^n} \frac{\mu^{\frac{1}{\eta}}(S(R))}{\prod_{i=1}^n \delta_i^{\alpha+2}} = 0.$$

Therefore μ is a compact $\eta-\alpha$ Carleson measure.

Conversely, suppose that μ is a compact $\eta-\alpha$ Carleson measure, then for every $\epsilon > 0$, there is δ such that

$$\sup_{\theta \in T^n} \frac{\mu^{\frac{1}{\eta}}(S(R))}{\prod_{i=1}^n \delta_i^{\alpha+2}} \leq \epsilon$$

for all $\delta_i < \delta$. Let $f_k \subset A_\alpha^p(D^n)$ converge uniformly to 0 on each compact subsets of D^n. It just only need to show that $\|\psi f_k \circ \varphi\|_{\eta p,\beta} \to 0$. We have

$$\begin{aligned}\|\psi f_k \circ \varphi\|_{\eta p,\beta}^{\eta p} &= \int_{D^n} |\psi f_k \circ \varphi|^{\eta p} d\sigma_{n,\beta} = \int_{\overline{D^n}} |f_k|^{\eta p} d\mu \\ &= \int_{\overline{D^n}\setminus(1-\delta)\overline{D^n}} |f_k|^{\eta p} d\mu + \int_{(1-\delta)\overline{D^n}} |f_k|^{\eta p} d\mu \\ &= I_1 + I_2.\end{aligned}$$

Write $\mu = \mu_1 + \mu_2$, where μ_1 is the restriction of μ to $(1-\delta)\overline{D^n}$ and μ_2 lies on the complement of this set in $\overline{D^n}$. Then, since $\mu_2 \leq \mu$, we get

$$\sup \frac{\mu_2(S(R))}{t^{n(\alpha+2)}} \leq \sup \frac{\mu(S(R))}{t^{n(\alpha+2)}},$$

where the supremums are extended over all $\theta \in T^n$ and for all $0 < t < \delta$. Then it is clearly that μ_2 is a compact $\eta-\alpha$ Carleson measure. Hence

$$I_1 = \int_{\overline{D^n}\setminus(1-\delta)\overline{D^n}} |f_k|^{\eta p} d\mu \leq \sup \frac{\mu_2(S(R))}{t^{\eta n(\alpha+2)}} \|f_k\|_{A_\alpha^p}^{\eta p} \leq C\epsilon \|f_k\|_{A_\alpha^p}^{\eta p}.$$

Because $\{f_k\}$ converges uniformly to 0 on $(1-\delta)\overline{D^n}$, I_2 can be made arbitrarily small by choosing large k. Since ϵ is arbitrary, we have $\psi C_\varphi f_k \to 0$ in $A_\beta^{\eta p}(D^n)$, that is $\psi C_\varphi : A_\alpha^p(D^n) \to A_\beta^{\eta p}(D^n)$ is compact. This completes the proof. ∎

3. Function Theoretic Characterization of Weighted Composition Operators

In this section, we give some function theoretic characterizations of weighted composition operators. For this purpose, we should first modify the Proposition 8 and Proposition 12 of [6] and give the following lemma.

Lemma 3. *Let μ be a nonnegative, Borel measure on $\overline{D^n}$. Then*
(i) *μ is an $\eta-\alpha$ Carleson measure if and only if*

$$\sup_{z_0\in D^n}\int_{D^n}\prod_{i=1}^{n}\Big[\frac{1-|(z_0)_i|^2}{|1-(\overline{z_0}_i z_i)|^2}\Big]^{(2+\alpha)\eta}d\mu\le C<\infty. \tag{12}$$

(ii) *μ is a compact $\eta-\alpha$ Carleson measure if and only if*

$$\lim\sup_{z_0\in D^n}\int_{D^n}\prod_{i=1}^{n}\Big[\frac{1-|(z_0)_i|^2}{|1-(\overline{z_0}_i z_i)|^2}\Big]^{(2+\alpha)\eta}d\mu=0 \tag{13}$$

as $\|z_0\|\to 1$.

Proof. Suppose that (12) holds, we show that μ is an $\eta-\alpha$ Carleson measure. Note that

$$R=\{(e^{i\theta_1},\cdots,e^{i\theta_n})\in T^n:|\theta_i-(\theta_0)_i|<\delta_i\}$$

and

$$S=S(R)=\{(r_1e^{i\theta_1},\cdots,r_ne^{i\theta_n})\in D^n:1-\delta_i<r_i<1,|\theta_i-(\theta_0)_i|<\delta_i\}.$$

Hence if $z_0=0$, (12) implies that $\mu(\overline{D^n})\le M<\infty$. Therefore we can assume that $\delta_i<\frac{1}{4}$ for all i. Take$(z_0)_i=\big(1-\frac{\delta_i}{2}e^{i(\theta_0)_i}\big)$, then for all $z\in S$, we have

$$\prod_{i=1}^{n}\frac{C}{(1-|(z_0)_i|^2)^{2+\alpha}}\le\prod_{i=1}^{n}\Big(\frac{1-|(z_0)_i|^2}{|1-(\overline{z_0})_i z_i|^2}\Big)^{2+\alpha}.$$

So

$$\begin{aligned}\mu^{\frac{1}{\eta}}(S)&=\Big\{\int_S d\mu\Big\}^{\frac{1}{\eta}}\\&=C\prod_{i=1}^{n}(1-|(z_0)_i|^2)^{2+\alpha}\Big\{\int_S\prod_{i=1}^{n}\Big[\frac{1-|(z_0)_i|^2}{|1-(\overline{z_0}_i z_i)|^2}\Big]^{(2+\alpha)\eta}d\mu\Big\}^{\frac{1}{\eta}}\\&\le CM^{\frac{1}{\eta}}\prod_{i=1}^{n}\delta_i^{2+\alpha}.\end{aligned}$$

Hence μ is an $\eta-\alpha$ Carleson measure.

Conversely, suppose that μ is an $\eta-\alpha$ Carleson measure. Let $z_0\in D^n$, if $||z_0||\le 1$, it is obviously that (12) holds, since the integrand can be bounded uniformly. Also, if $|(z_0)_i|<\frac{3}{4}$, the term corresponding to this i in the integrand in (12) can be bounded. So let us suppose $|(z_0)_i|>\frac{3}{4}$ for all i, and let

$$E_k = \Big\{ z \in D^n : \max_i \frac{|z_i - \frac{(z_0)_i}{|(z_0)_i|}|}{1 - |(z_0)_i|} < 2^k \Big\}.$$

Note that if $z \in E_1$, then

$$\prod_{i=1}^n \frac{C}{(1 - |(z_0)_i|^2)^{2+\alpha}} \geq \prod_{i=1}^n \Big(\frac{1 - |(z_0)_i|^2}{|1 - (\overline{z_0})_i z_i|^2} \Big)^{2+\alpha}$$

and for $k \geq 2$ if $z \in E_k - E_{k-1}$, then

$$\prod_{i=1}^n \frac{C}{(1 - |(z_0)_i|^2)^{2+\alpha}} \geq \prod_{i=1}^n \Big(\frac{1 - |(z_0)_i|^2}{|1 - (\overline{z_0})_i z_i|^2} \Big)^{2+\alpha}.$$

Since μ is an $\eta - \alpha$ Carleson measure, we have

$$\begin{aligned} &\int_{D^n} \prod_{i=1}^n \Big[\frac{1 - |(z_0)_i|^2}{|1 - (\overline{z_0}_i z_i)|^2} \Big]^{(2+\alpha)\eta} d\mu \\ &\leq \int_{E_1} + \sum_{k=2}^\infty \int_{E_k - E_{k-1}} \prod_{i=1}^n \Big[\frac{1 - |(z_0)_i|^2}{|1 - (\overline{z_0}_i z_i)|^2} \Big]^{(2+\alpha)\eta} d\mu \\ &\leq C \sum_{k=2}^\infty \frac{\mu(E_k - E_{k-1})}{\prod_{i=1}^n (1 - |(z_0)_i|)^{(2+\alpha)\eta}} \\ &\leq C \sum_{k=2}^\infty \frac{\mu(E_k)}{\prod_{i=1}^n \delta_i^{(2+\alpha)\eta}} \leq C < \infty. \end{aligned}$$

This completes the proof of (i). With the same manner of (i) and the proof of Proposition 8 (ii) in [6] we can give the proof of (ii), we omit the details. ∎

Theorem 3. *Suppose that $1 < p < \infty, \beta, \alpha > -1$, $\eta \geq 1$. Let φ be a holomorphic self-map of D^n and ψ be a holomorphic function on D^n. Then*
(i) *$\psi C_\varphi : A^p_\alpha(D^n) \to A^{\eta p}_\beta(D^n)$ is bounded if and only if*

$$\sup_{z_0 \in D^n} \int_{D^n} |\psi(z)|^{\eta p} \prod_{i=1}^n \Big[\frac{1 - |(z_0)_i|^2}{|1 - (\overline{z_0}_i \varphi_i)|^2} \Big]^{(2+\alpha)\eta} d\sigma_{n,\beta} < \infty.$$

(ii) *$\psi C_\varphi : A^p_\alpha(D^n) \to A^{\eta p}_\beta(D^n)$ is compact if and only if*

$$\lim \sup_{z_0 \in D^n} \int_{D^n} |\psi(z)|^{\eta p} \prod_{i=1}^n \Big[\frac{1 - |(z_0)_i|^2}{|1 - (\overline{z_0}_i \varphi_i)|^2} \Big]^{(2+\alpha)\eta} d\sigma_{n,\beta} = 0$$

as $\|z_0\| \to 1$.

Proof. By Lemma 2, we know that ψC_φ is bounded or compact if and only if μ is a bounded or compact $\eta - \alpha$ Carleson measure. Then by Lemma 3 we get the desired results. ∎

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