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Simply Presented Inseparable $V(RG)$ Without R Being Weakly Perfect or Countable

Peter Danchev

13 General Kutuzov Str., block 7, floor 2, flat 4, 4003 Plovdiv, Bulgaria

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Abstract. It is constructed a special commutative unitary ring R of characteristic 2, which is not necessarily weakly perfect (hence not perfect) or countable, and it is selected a multiplicative abelian 2-group G that is a direct sum of countable groups such that $V(RG)$, the group of all normed 2-units in the group ring RG, is a direct sum of countable groups. So, this is the first result of the present type, which prompts that the conditions for perfection or countability on R can be, probably, removed in general.

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Let RG be a group ring where G is a p-primary abelian multiplicative group and R is a commutative ring with identity of prime characteristic p. Let $V(RG)$ denote the normalized p-torsion component of the group of all units in RG. For a subgroup D of G, we shall designate by $I(RG; D)$ the relative augmentation ideal of RG with respect to D, that is the ideal of RG generated by elements 1-d whenever $d \in D$.

Warren May first proved in [11] that $V(RG)$ is a direct sum of countable groups if and only if G is, provided R is perfect and G is of countable length. More precisely, he has argued that if G is an arbitrary direct sum of countable groups and R is perfect, $V(RG)/G$ and $V(RG)$ are both direct sums of countable groups (for their generalizations see [12] and [2, 8] as well).

At this stage, even if the group G is reduced, there is no results of this kind which are established without additional restrictions on the ring R. These restrictions are: *perfection* ($R = R^p$), *weakly perfection* ($R^{p^i} = R^{p^{i+1}}$ for some $i \in \mathbb{N}$) and *countability* ($|R| \leqslant \aleph_0$). In that aspect see [2-4] and [6-8], too.

That is why, we have a question in [1] that whether $V(RG)$ simply presented does imply that R is weakly perfect. When both R and G are of countable powers, it is self-evident that $V(RG)$ is countable whence it is simply presented. Moreover, if G is a direct sum of cyclic groups then, by using [5], the same property holds true for $V(RG)$. Thus, the problem in [1] should be interpreted for uncountable and inseparable simply presented abelian p-groups $V(RG)$. In this case, it was obtained in [6] a negative answer to the query, assuming R is without nilpotents. Here, we shall characterize the general situation for a ring with nilpotent elements.

The motivation of the current paper is to show that the condition on R being perfect, weakly perfect or countable may be dropped off in some instances. We do this via the following original ring construction.

Definition. *The commutative ring R with 1 and of prime characteristic p is called heightly-additive if $R = \cup_{n<\omega} R_n$, $R_n \subseteq R_{n+1}$ and for every natural number n, for every two elements $r \in R_n$ and $f \in R_n$ such that $r \notin R^{p^n}$ and $f \notin R^{p^n}$, we have $r+f \notin R^{p^n}$ or $r+f \in R^{p^\omega} = \cap_{n<\omega} R^{p^n}$. Inductively, either $r+f+\ldots+e \notin R^{p^n}$ or $r+f+\ldots+e \in R^{p^\omega}$ whenever $r \in R_n \setminus R^{p^n}, f \in R_n \setminus R^{p^n}, \ldots, e \in R_n \setminus R^{p^n}$.*

It is worthwhile noticing that the countable union of an ascending chain of $(R_n)_{n<\omega}$ is in the set-theoretic sense, i.e. the members R_n of the union are not necessarily subrings of R. Thus, for $r \in R_n$ and $f \in R_n$ it is possible that $r + f \notin R_n$. When all R_n are rings, the union is in the ring-theoretic sense. For the simplicity of the computations, we shall assume further that $r^\varepsilon \in R_n$ whenever $r \in R_n$ and $\varepsilon \geq 1$.

Analyzing the conditions stated in the definition, we plainly check that for each $1 \leqslant n < \omega$ and for each $k \geq n$, if $r \in R_n \setminus R^{p^k}$ and $f \in R_n \setminus R^{p^k}$, then $r + f \notin R^{p^k}$ or $r + f \in R^{p^\omega}$. This is so because $R_n \subseteq R_{n+1}$ for every positive integer n. Moreover, if $r \in R_n \cap (R^{p^j} \setminus R^{p^{j+1}})$ and $f \in R_n \cap (R^{p^j} \setminus R^{p^{j+1}})$ for $j \leqslant n-1$, then $r + f \in R^{p^\omega}$ or otherwise $r + f \in R^{p^j} \setminus R^{p^{j+1}}$ when $j = n-1$, but $r + f \in R^{p^{j+1}}$ is possible when $j < n-1$ although $r \in (R_n \cap R^{p^j}) \setminus R^{p^n}$ and $f \in (R_n \cap R^{p^j}) \setminus R^{p^n}$ imply $r + f \notin R^{p^n}$.

We provide below some examples of classical rings, which are heightly-additive, namely:

(1) For each two $r, f \in R$ so that $r \in R^{p^n} \setminus R^{p^{n+1}}$ and $f \in R^{p^n} \setminus R^{p^{n+1}}$, we have $r + f \in R^{p^n} \setminus R^{p^{n+1}}$ or $r + f \in R^{p^\omega}$.

(2) For any $r \in R$ and $f \in R$, $\text{height}_R(r + f) = \min\,(\text{height}_R(r), \text{height}_R(f))$ holds.

(3) $R = R_0 \times R_1 \times \ldots \times R_n \times \ldots$ such that for any $r_n \in R_n$, we have $r_n \in R^{p^n} \setminus R^{p^{n+1}}, n < \omega$.

(4) $R = \oplus_{n<\omega} R_n$ such that for any $r_n \in R_n$, we have $r_n \in R^{p^n} \setminus R^{p^{n+1}}, n < \omega$.

By definition, R is weakly perfect if and only if there is $m \in \mathbb{N}$ such that

$R^{p^m} = R^{p^{m+1}}$ or, equivalently, $R^{p^m} = R^{p^\omega}$. Thus the finite heights in R are bounded in general at this m.

We observe that for an arbitrary commutative ring R with identity of prime characteristic p, there exists $n \in \mathbb{N}$ such that for any $r, f \in R$ with $r \notin R^{p^n}$ and $f \notin R^{p^n}$, we have $r + f \notin R^{p^n}$ or $r + f \in R^{p^\omega}$. Indeed, for such elements r and f, if $r + f \notin R^{p^\omega}$, then there is $t \geq n$ with $r + f \notin R^{p^t}$. But in this case $r \notin R^{p^t}$ and $f \notin R^{p^t}$, and we are done.

Nevertheless, if R is a commutative unitary ring of prime characteristic p such that for some $n < \omega$ and for every couple $r, f \in R$ with the property that $r \notin R^{p^n}$ and $f \notin R^{p^n}$ yield $r + f \notin R^{p^n}$ or $r + f \in R^{p^\omega}$, then this ring is weakly perfect. Indeed, we can claim that $R^{p^n} = R^{p^{n+1}}$. Suppose the contrary that $R^{p^n} \neq R^{p^{n+1}}$. Choose $r \notin R^{p^n}$ and $s \in R^{p^n} \setminus R^{p^{n+1}}$. Put $f = s - r$. Then $f \notin R^{p^n}$ with $\text{height}_R(r) = \text{height}_R(f)$, but $r + f = s \in R^{p^n} \setminus R^{p^\omega}$. Thus R could not be as defined, which substantiates our claim.

By using the same idea for $s \in (R_n \cap R^{p^n}) \setminus R^{p^{n+1}} = R_n \cap (R^{p^n} \setminus R^{p^{n+1}})$, we see that in the above given definition the series of identities $R_n \cap R^{p^n} = R_n \cap R^{p^{n+1}} = \cdots = R_n \cap R^{p^\omega}$ hold, provided all R_n are subrings of R. However, the heightly-additive rings need not to be neither weakly perfect nor countable. In fact, the latter is true because these rings may have finite heights equal to n for each $n \geq 1$, as it has been demonstrated above, whereas this is not the case for weakly perfect ones.

Without further comments, we will freely use in the sequel the simple but, however, useful fact that

$$\text{height}_{RG}(r_1 g_1 + \cdots + r_t g_t) = \min_{1 \leqslant i \leqslant t} (\text{height}_R(r_i),\ \text{height}_G(g_i)),$$

whenever $r_1 \neq 0, \ldots, r_t \neq 0$ and $g_1 \neq \cdots \neq g_t \neq g_1$, i.e. when $r_1 g_1 + \cdots + r_t g_t$ is written in canonical form.

The above defined sort of heightly-additive commutative unitary rings with characteristic $p = 2$ is interesting for applications to the unit groups of commutative modular group algebras. Combining them with a special chosen class of abelian 2-groups, we establish below criteria for simple presentness in group rings. The next claim deals with such a matter.

We proceed by proving the central attainment, which is on the focus of our examination. Specifically, we prove the following.

Main Theorem. *Suppose R is a heightly-additive commutative ring with unity of characteristic 2 such that $|R^{2^\omega}| = 2$, and suppose G is an abelian 2-group with reduced part G_r that satisfies $|G_r^{2^\omega}| = 2$ and $G_r / G_r^{2^\omega}$ is a direct sum of cyclic groups. Then $V(RG)/G$ is a direct sum of countable groups and so G is a direct factor of $V(RG)$ with a complementary factor which is a direct sum of countable groups.*

Proof. Write down $G = G_d \times G_r$, where G_d is the maximal divisible subgroup of G. Thereby, from [2] or [7], we derive:

$$V(RG)/G \cong V(RG_r)/G_r \times [(1 + I(RG; G_d))/G_d]. \qquad (*)$$

Exploiting [13], we can write $G_r = C \times D$, where C is countable and D is a direct sum of cyclic groups. Clearly G is a direct sum of countable groups. We furthermore obtain $V(RG_r) = V(RC) \times [1 + I(RG_r; D)]$ (see, for example, [2] or [7]) and thus $V(RG_r)/G_r \cong V(RC)/C \times [1+I(RG_r; D)]/D$. By virtue of [5], we deduce that $[1 + I(RG_r; D)]/D$ is a direct sum of cyclic groups.

On the other hand, one may write $C = \cup_{n<\omega} C_n$, where all $C_n \subseteq C_{n+1}$ are finite and $C_n \cap C^{2^n} \subseteq C^{2^\omega}$. Moreover $C^{2^\omega} = G_r^{2^\omega}$, hence C^{2^ω} is of cardinality 2. Under the limitations, $C^{2^\omega} = \{1, g | g^2 = 1\}$ and $R^{2^\omega} = \{0, 1\}$. Then $C^{2^{\omega+1}} = 1$ and $g \in C^{2^\omega} \setminus C^{2^{\omega+1}}$. Besides, we emphasize that $r = -r$ for every $r \in R$ because $\mathrm{char}(R) = 2$.

After this, we construct the groups:

$V_n = \langle 1, g, 1 + f - fg, \sum_{1 \leqslant i \leqslant m_n} r_{in} c_{in} | 1 \neq g \in C^{2^\omega}$ with $g^2 = 1; f \in R_n$ and $f \notin R^{2^n}; r_{in} \in R_n$ so that $\sum_{1 \leqslant i \leqslant m_n} r_{in} = 1$ and $r_{in}^{\varepsilon} \in R_n \setminus R^{2^n}$ or $r_{in}^{\varepsilon} = 0$ for $1 \leqslant \varepsilon \leqslant$ order $\left(\sum_{1 \leqslant i \leqslant m_n} r_{in} c_{in}\right)$ or $r_{in} \in R^{2^\omega}, c_{in} \in C_n;\ 1 \leqslant i \leqslant m_n \in \mathbb{N}\rangle$.

Certainly, V_n are correctly defined generating subgroups of $V(RC)$ such that $V(RC) = \cup_{n<\omega} V_n$ and $V_n \subseteq V_{n+1}$.

What we want to show in the sequel is that all V_n have finite height spectrum of finite heights. In order to do that, we first explore the members of the type $1+f(1-g)$. In fact, $[1+f(1-g)]^2 = 1+f^2(1-g^2) = 1$ and hence $[1+f(1-g)]^{2k} = 1$ together with $[1 + f(1 - g)]^{2k+1} = 1 + f(1 - g)$ for each nonnegative integer k. Clearly, $1+f(1-g) \notin V^{2^n}(RC) = V(R^{2^n} C^{2^n})$. But $[1+f(1-g)][1+r(1-g)] = 1 + f + r - (f + r)g$ and therefore the height condition on R leads us to the fact that the production does not lie in $V^{2^n}(RC)$ when $f + r \notin R^{2^n}$, or otherwise when $f + r \in R^{2^\omega} = \{0, 1\}$ it belongs to C. By induction $[1 + f(1-g)][1 + r(1-g)] \ldots [1 + e(1 - g)] = (1 + f + r + \cdots + e) - (f + r + \cdots + e)g \notin V^{2^n}(RC)$ or $\in C$.

Next, we study the sums $\sum_{1 \leqslant i \leqslant m_n} r_{in} c_{in}$ and also arbitrary degrees of their finite products. It is apparent that $\sum_{1 \leqslant i \leqslant m_n} r_{in} c_{in} \notin V^{2^n}(RC)$ or $\in C^{2^\omega}$. Write $\left(\sum_{1 \leqslant i \leqslant m_n} r_{in} c_{in}\right)\left(\sum_{1 \leqslant j \leqslant t_n} f_{jn} a_{jn}\right) = \sum_{1 \leqslant i \leqslant m_n} \sum_{1 \leqslant j \leqslant t_n} r_{in} f_{jn} c_{in} a_{jn}$. If the canonical record of the double sum contains an element from the group basis of finite height, we are done.

In the remaining case when all members from the support have infinite heights, we demonstrate the following approach. With no harm of generality, we may restrict our attention on $m_n = t_n = 2$ or $m_n = t_n = 3$ (all other possibilities as well as the general procedure are identical). So, we start with

1. $x = ((1+r_1)c_1 - r_1 c_2)((1+f_1)a_1 - f_1 a_2) = (1+r_1)(1+f_1)c_1 a_1 - (1+r_1) f_1 c_1 a_2 - r_1(1 + f_1)\ c_2 a_1 + r_1 f_1 c_2 a_2 = ((1 + r_1)\ (1 + f_1) + r_1 f_1)g - (1 + r_1) f_1 - r_1 (1 + f_1) = (1 + r_1 + f_1)g - (r_1 + f_1)$, where we have assumed that $c_1 a_1 = c_2 a_2 = g$ and $c_1 a_2 = c_2 a_1 = 1$.
2. $x = (r_1 c_1 + r_2 c_2 + r_3 c_3)(f_1 a_1 + f_2 a_2 + f_3 a_3) = r_1 f_1 c_1 a_1 + r_1 f_2 c_1 a_2 + r_1 f_3 c_1 a_3 + r_2 f_1 c_2 a_1 + r_2 f_2 c_2 a_2 + r_2 f_3 c_2 a_3 + r_3 f_1 c_3 a_1 + r_3 f_2 c_3 a_2 + r_3 f_3 c_3 a_3 = (r_1 f_1 + r_2 f_2 + r_3 f_3)g + (r_1 f_2 + r_2 f_1)$, where the following additional relations are fulfilled (all other variants are similar): $c_1 a_1 = c_2 a_2 = c_3 a_3 = g$; $c_1 a_2 = c_2 a_1 = 1$; $c_1 a_3 = c_3 a_1, r_1 f_3 + r_3 f_1 = 0$; $c_2 a_3 = c_3 a_2, r_2 f_3 + r_3 f_2 = 0$. That is why $r_1 = r_1(f_1 + f_2 + f_3) = r_1 f_1 + r_1 f_2 + r_1 f_3 = r_1 f_1 + r_1 f_2 - r_3 f_1$ and

moreover $r_1f_3+r_3f_1+r_2f_3+r_3f_2=0$. Because $f_1+f_2+f_3=r_1+r_2+r_3=1$, it holds that $f_3(1-r_3)+r_3(1-f_3)=0$, i.e. $f_3=r_3$. Thus $r_3+f_2=1+f_1$. But, $x=(r_1-r_1f_2+r_3f_1+r_2f_2+r_3f_3)g+(r_1-r_1f_1+r_3f_1+r_2f_1)=(r_1-r_1f_2+r_2f_2+r_3(1-f_2))g+(r_1+f_1(-r_1+r_2+r_3))=(r_1+r_3-(1-r_2)f_2+r_2f_2)g+(r_1+f_1)=(r_1+r_3+f_2)g+(r_1+f_1)=(1+r_1+f_1)g+(r_1+f_1)$. Consequently, in any event, $x\notin V(R^{2^n}C^{2^n})$ or $x\in V(R^{2^\omega}C^{2^\omega})=C^{2^\omega}$.

For the degrees of these sums, we shall use the standard binomial formula of Newton. In fact, $\big(\sum_{1\leqslant i\leqslant m_n} r_{in}c_{in}\big)^2=\sum_{1\leqslant i\leqslant m_n} r_{in}^2c_{in}^2$, where for every $i\in[1,m_n]$ is valid $c_{in}^2\notin C^{2^n}$, or $c_{in}^2\in C^{2^n}$ hence $c_{in}^2\in C^{2^\omega}$ that is $c_{in}^2=1(\Leftrightarrow c_{in}=1$ or $c_{in}=g)$ or $c_{in}^2=g$. By what we have computed above, we therefore detect that $\big(\sum_{1\leqslant i\leqslant m_n} r_{in}c_{in}\big)^2\notin V^{2^n}(RC)$ or $\big(\sum_{1\leqslant i\leqslant m_n} r_{in}c_{in}\big)^2=1$. By making use of an ordinary induction we have $\big(\sum_{1\leqslant i\leqslant m_n} r_{in}c_{in}\big)^{2k}\notin V^{2^n}(RC)$ or $\big(\sum_{1\leqslant i\leqslant m_n} r_{in}c_{in}\big)^{2k}=1$ for each $k\geq 1$. Moreover, $\big(\sum_{1\leqslant i\leqslant m_n} r_{in}c_{in}\big)^3=\big(\sum_{1\leqslant i\leqslant m_n} r_{in}^2c_{in}^2\big)\big(\sum_{1\leqslant i\leqslant m_n} r_{in}c_{in}\big)$ and so we see that this is precisely the above considered point. Thus, $\big(\sum_{1\leqslant i\leqslant m_n} r_{in}c_{in}\big)^{2k+1}=\big(\sum_{1\leqslant i\leqslant m_n} r_{in}^2c_{in}^2\big)^k\cdot\big(\sum_{1\leqslant i\leqslant m_n} r_{in}c_{in}\big)$ and the method used completes the step.

By the same token, $\big(\sum_{1\leqslant i\leqslant m_n} r_{in}c_{in}\big)(1+f-fg)$ does not lie in $V^{2^n}(RC)$ or belongs to C^{2^ω}, whence via the foregoing described trick $\big(\sum_{1\leqslant i\leqslant m_n} r_{in}c_{in})^{\varepsilon}(1+f-fg)\notin V^{2^n}(RC)$ or $\in C^{2^\omega}$, for each integer $\varepsilon\geq 0$.

And so, in all cases, for an arbitrary element from V_n we infer that $\big(\sum_{1\leqslant i\leqslant m_n} r_{in}c_{in}\big)^{\varepsilon}\ldots\big(\sum_{1\leqslant j\leqslant t_n} f_{jn}a_{jn}\big)^{\tau}\cdot(1+f-fg)\ldots(1+r-rg)\notin V^{2^n}(RC)$ or $\in C^{2^\omega}$; $\tau\geq 0$ is an integer, which guarantees our assertion.

We shall calculate now that $[V_n\cap(CV^{2^n}(RC))]=[V_n\cap(CV(R^{2^n}C^{2^n}))]\subseteq C$, which will substantiate our final claim. And so, given z in the intersection. Then, as above, $z=\big(\sum_k\gamma_k a_{kn}\big)(1+\beta+\beta g)=c\sum_k\alpha_k^{2^n}c_k^{2^n}$, where $a_{kn}\in C_n$ with $\gamma_k\notin R^{2^n}$ when all $a_{kn}\in C^{2^\omega}=\{g,1\}$, or $\gamma_k\in R^{2^\omega}=\{0,1\}$ when there exists some $a_{kn}\notin C^{2^n}$, $\sum_k\gamma_k=1$, $\beta\notin R^{2^n}$ provided $\beta\neq\{0,1\}$ i.e. $\beta\notin R^{2^\omega}=\{0,1\}$; $c\in C,\alpha_k\in R,c_k\in C$. Otherwise, when $\sum_k\gamma_k a_{kn}\in V(R^{2^\omega}C^{2^\omega})=C^{2^\omega}$, we are finished.

Foremost, if each $a_{kn}\in C^{2^\omega}=\{1,g\}$ we deduce as before that $\big(\sum_k\gamma_k a_{kn}\big)(1+\beta+\beta g)=1+\delta+\delta g$ where $\delta\notin R^{2^n}$. So, $z=c\in C$, as desired.

Secondly, in the remaining case when there are some $a_{kn}\notin C^{2^n}$ so that in the support of the canonical record of z there exists a group member, say for instance b_{kn}, such that $b_{kn}\notin C^{2^n}$, we infer that $c\notin C^{2^n}$ since $b_{kn}=cc_k^{2^n}$ for some c_k. That is why, in the canonical form of the left hand-side of z, namely $\big(\sum_k\gamma_k a_{kn}\big)(1+\beta+\beta g)$, all group members do not belong to C^{2^n}. Besides, because $\gamma_k=\{0,1\}$ for every index k, we derive $\gamma_k\beta=0$ or $\gamma_k\beta=\beta\notin R^{2^n}$.

We also only remark that if $\gamma_k\notin R^{2^n}$ and $\beta\notin R^{2^n}$, the relation $\gamma_k\beta\in R^{2^n}$ yields $\gamma_k(1+\beta)=\gamma_k+\gamma_k\beta\notin R^{2^n}$ as well as $\gamma_k\beta\notin R^{2^n}$ implies $\gamma_k(1+\beta)=\gamma_k+\gamma_k\beta\notin R^{2^n}$ or $\gamma_k(1+\beta)=\gamma_k+\gamma_k\beta\in R^{2^\omega}=\{0,1\}$ whenever $\gamma_k\beta\in R_n$.

If now $\beta=0$, we have $z=\big(\sum_k\gamma_k a_{kn}\big)=c\sum_k\alpha_k^{2^n}c_k^{2^n}$. Then we find that $a_{kn}a_{k'n}^{-1}\in C^{2^n}\cap C_n\subseteq C^{2^\omega}=\{1,g\}$ for each different index $k'\neq k$. We therefore obtain that $z\in C$ provided all $\gamma_k\in R^{2^\omega}=\{0,1\}$, as wanted; If there is some

$\gamma_k \notin R^{2^n}$, then we are done.

If now $\beta = 1$, the same method alluded to above works and this finishes the computations to verify the desired ratio.

Finally, observing that $V(RC) = \cup_{n<\omega} V_n$ and $V(RC)/C = \cup_{n<\omega}(V_n C/C)$, by what we have just computed together with the criterion for total projectivity from [10], we conclude that $V(RC)/C$ is a direct sum of countable groups. So, the same holds for $V(RG_r)/G_r$ in fact.

We concentrate now on the second direct factor $(1 + I(RG; G_d))/G_d$ of $(*)$. Consulting with [9], we may write

$$(1 + I(RG; G_d))/G_d \cong [(1 + I(R^{2^\omega} G_d;\ \ G_d))/G_d] \\ \times [(1 + I(RG; G_d))/(1 + I(R^{2^\omega} G_d;\ \ G_d))],$$

where $(1+I(R^{2^\omega} G_d; G_d))/G_d = [(1+I(RG; G_d))/G_d]_d$. It is not hard to see that the maximal (2-)perfect subring of R is R^{2^ω} as well as the maximal (2-)divisible subgroup of G is $G^{2^{\omega+1}} = G_d$.

We show below that $J = (1 + I(RG;\ G_d))/(1 + I(R^{2^\omega} G_d;\ G_d)) = \cup_{n<\omega} I_n$, where $I_n \subseteq I_{n+1}$ and every I_n is with height-finite spectrum.

In fact, an arbitrary element from $[1 + I(RG; G_d)]\backslash[1 + I(R^{2^\omega} G_d;\ G_d)]$ is of the form $1 + \sum_{i,j} r_{ij} g_{ij}(1 - g_d)$ where $R \ni r_{ij} \neq \{0, 1\}$, i.e. $r_{ij} \notin R^{2^\omega}$, or $g_{ij} \in G\backslash G_d$; $g_d \in G_d$. Henceforth, $1 + \sum_{i,j} r_{ij} g_{ij}(1 - g_d) \notin V^{2^{\omega+1}}(RG) = V(R^{2^{\omega+1}} G^{2^{\omega+1}}) = V(R^{2^\omega} G_d) = V(RG)_d$ and so it is a real matter to distribute the heights of such elements at the inequality "$< \omega + 1$" in the following manner by selection of the generating groups

$W_n = \langle w^{(n)} = 1 + \sum_{i,j} r_{ij}^{(n)} g_{ij}^{(n)}(1 - g_d) | r_{ij}^{(n)^\varepsilon} \in R_n\backslash R^{2^n}$ or $r_{ij}^{(n)^\varepsilon} = \{0, 1\}$; all possible finite products of $g_{ij}^{(n)}$'s denoted as $\prod g_{ij}^{(n)^\varepsilon} \in G\backslash G^{2^n}$ or $g_{ij}^{(n)} \in G^{2^\omega}\backslash G^{2^{\omega+1}}$ or $g_{ij}^{(n)} = 1$, but $r_{ij}^{(n)} \neq 1$ if $g_{ij}^{(n)} = 1$ and $g_{ij}^{(n)} \neq 1$ if $r_{ij}^{(n)} = 1$; $1 < \varepsilon <$ order $(w^{(n)})$; $g_d \in G_d\rangle$.

Certainly, $W_n \subseteq W_{n+1}$ are exactly defined subgroups of $1 + I(RG;\ G_d)$ such that $1 + I(RG; G_d) = \cup_{n<\omega} W_n \cup [1 + I(R^{2^\omega} G_d; G_d)]$.

Every element of W_n is of the kind $\prod \left(1 + \sum_{i,j} r_{ij}^{(n)} g_{ij}^{(n)}(1 - g_d)\right)^\varepsilon$, where the product is taken over a finite number of degrees of generators.

As in our preceding tactic, without loss of generality, we may bound our attention to the production of two generating elements both written in canonical type. For example, we can write $\left(\sum_k \alpha_{kn} a_{kn}\right).\left(\sum_k \gamma_{kn} c_{kn}\right) = \sum_{k,m} \alpha_{kn}\gamma_{km} a_{kn} c_{mn}$, where, owing to the construction of the generators, for some indice $k_1, \alpha_{k_1 n} a_{k_1 n} = 1 = \gamma_{k_1 n} c_{k_1 n}$. Moreover, bearing in mind that R is heightly-additive, in the sum $\sum_k \alpha_{kn} a_{kn}$ there exist ring coefficients $\notin R^{2^n}$, or $\in R^{2^\omega} = \{0, 1\}$ and eventually $\in R^{2^n}$; For the other sum $\sum_k \gamma_{kn} c_{kn}$ we do similarly. We also indicate that if $\alpha_{k'n}\gamma_{kn} \in R^{2^n}$ for some index k' and each index k, we obviously obtain $\alpha_{k'n}\left(\sum_k \gamma_{kn}\right) = \alpha_{k'n} \in R^{2^n}$ and that is wrong. Hence there exists an index k' with the property $\alpha_{k'n}\gamma_{k'n} \notin R^{2^n}$. On the other hand, in the canonical record of $\sum_{k,m} \alpha_{kn}\gamma_{km} a_{kn} c_{mn}$, there exists a group member of height $< n$ provided that this double sum contains eventually a group element with finite height. This is fulfilled by adapting the scheme for the proof given in [7, 8].

And so, one can say that the treated sum is not in $V^{2^n}(RG)$ or belongs either to $G^{2^\omega}\setminus G^{2^{\omega+1}}$ or $G^{2^{\omega+1}}$.

We now consider quotients by setting $I_n = W_n[1 + I(R^{2^\omega}G_d;\ G_d)]/[1 + I(R^{2^\omega}G_d;\ G_d)]$. Of course, it is elementary to see that $J = \cup_{n<\omega} I_n$ and $I_n \subseteq I_{n+1}$, as already claimed.

Further, we shall verify that all I_n are height-finite in J with spectrum $\{0, 1, \dots, n-1, \omega\}$ taking into account $J^{2^{\omega+1}} = 1$ and the above demonstrated calculations on W_n. In fact, given an arbitrary element y in $[W_n(1 + I(R^{2^\omega}G_d;\ G_d))] \cap [1 + I(R^{2^n}G^{2^n};\ G_d)]$, by examining only the finite heights, we have $y = w_n.u = v$, where $w_n \in W_n$ is of finite height provided $w_n \neq 1, u \in 1 + I(R^{2^\omega}G_d;\ G_d)$ and $v \in 1 + I^{2^n}(RG; G_d)$. According to our previous conclusions, since w_n has height $< n$ while vu^{-1} possesses height $> n$, we have $w_n = 1$. Thus $y \in 1 + I(R^{2^\omega}G_d; G_d)$, and this assures our assertion.

Furthermore, according to the criterion for total projectivity documented in [10], $[1+I(RG; G_d)]/[1+I(R^{2^\omega}G_d;\ G_d)]$ is totally projective of countable length. So, we see that $(1+I(RG; G_d))/G_d$ should be simply presented which is a direct sum of countable groups.

Finally, we extract via the isomorphism formula (*) that $V(RG)/G$ is a direct sum of countable groups.

The niceness of G in $V(RG)$ (cf. [7]) together with [9] ensure that $V(RG) \cong G \times V(RG)/G$. Henceforth, $V(RG)$ is really a direct sum of countable groups, as promised. The proof is now complete. ■

As an immediate consequence of the theorem, we yield the following.

Corollary 1. *Let R be a commutative ring with* 1 *of characteristic* 2 *which is heightly-additive and so that $|R^{2^\omega}| = 2$, and G a direct sum of countable abelian 2-groups so that $|G^{2^\omega}| = 2$. Then $V(RG)/G$ is a direct sum of cyclic groups, and $V(RG)$ is a direct sum of countable groups.*

Proof. Since G is reduced and G/G^{2^ω} is a direct sum of cyclic groups, we may employ the theorem to get the desired claim, proving our corollary. ■

Remark. In the case when length(G) is countable (eventually limit) and R is perfect, the readers may see [2, 3, 7] and [8].

The next assertion is a direct consequence of the major theorem as well (for the point when $R^p = R^{p^2}$ and p is an arbitrary prime see [4]; it can be realized $R \neq R^p$ provided R is with nilpotents).

Corollary 2. *Let R be a commutative ring with* 1 *of characteristic* 2 *which is heightly-additive and such that $|R^{2^\omega}| = 2$, and G an abelian* 2*-group such that $|G^{2^\omega}| = 2$. Then $V(RG)$ is a direct sum of countable groups if and only if G is.*

Proof. First of all, suppose $V(RG)$ is a direct sum of countable groups. Hence $V(RG)/V^{2^\omega}(RG)$ is a direct sum of cyclic groups and since $G/G^{2^\omega} \cong GV^{2^\omega}(RG)/V^{2^\omega}(RG) \subseteq V(RG)/V^{2^\omega}(RG)$, so does G/G^{2^ω}. Therefore, the theorem is applicable to obtain that $V(RG)/G$ is a direct sum of cyclic groups because it is evidently separable. Thus, because of the well-known fact that G is pure in $V(RG)$, exploiting a classical statement due to Kulikov (e.g. [9]), we infer that $V(RG) \cong G \times V(RG)/G$. Finally, the direct factor G is also a direct sum of countable groups (see, for instance, cf. [9]). This completes the first part.

Another verification of the necessity follows from [13] and the fact that G/G^{2^ω} is a direct sum of cyclic groups along with $|G^{2^\omega}| = 2$.

For the second part, if G is a direct sum of countable groups, then G/G^{2^ω} is a direct sum of cyclic groups. Referring to the theorem we conclude that $V(RG)$ must be a direct sum of countable groups, as expected. ∎

Corollary 3. *The simply presented abelian p-group $V(RG)$ does not imply that R is weakly perfect or countable.*

Proof. It follows automatically from the main theorem by observing that the ring R constructed there needs not to be neither weakly perfect nor countable.

∎

Inspired by the theorem, we now go to the following

Commentary. *If $V(RG)$ is simply presented, then $V(RG)/V^{p^\omega}(RG)$ is a direct sum of cyclic groups. By what we have just described above, G/G^{p^ω} should be a direct sum of cyclic groups whence $G = \cup_{n<\omega} G_n, G_n \subseteq G_{n+1}$ and $G_n \cap G^{p^n} \subseteq G^{p^\omega}$ for each $n < \omega$. Therefore $V(RG) = \cup_{n<\omega} V(RG_n)$. Moreover we can compute $V(RG_n) \cap V^{p^n}(RG) = V(RG_n) \cap V(R^{p^n} G^{p^n}) = V(R^{p^n}(G_n \cap G^{p^n})) \subseteq V(R^{p^n} G^{p^\omega})$. However, as we have established above, $V(R^{p^n} G^{p^\omega})$ needs not to be equal to $V(R^{p^\omega} G^{p^\omega})$, i.e. $R^p \neq R^{p^2}$ or equivalently R is not weakly $(2-)$ perfect.*

As a final note, we mention that the classical Prufer's 2-groups (see [9] for example) satisfy the assumptions listed in the theorem.

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