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Some Results on the Relation Between Pluripolarity of Graphs and Holomorphicity

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Abstract. The aim of this paper is to give some results on the relation between pluripolarity of graphs and holomorphicity of complex-valued functions. In particular, we investigate the relation between the pluripolarity of graphs and the holomorphicity of Hartogs type and of Forelli type.

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1. Introduction

In 2003 Shcherbina proved the following result which solves a problem posed several years ago by Chirka (see [4]): Let D be a domain in $\mathbb{C}^n$ and $f : D \to \mathbb{C}$ be a continuous function. The graph $\Gamma(f)$ of f is pluripolar subset of $\mathbb{C}^{n+1}$ if and only if f is holomorphic (see [10]). Next, Shcherbina extented the above result to the multi-functions. Namely he proved that if $a_1(z), \cdots, a_m(z)$ are continuous on D such that the set:

$$\mathcal{E} = \{(z, w) \in D \times \mathbb{C} : w^m + a_1(z)w^{m-1} + \ldots + a_m(z) = 0\}$$

is a pluripolar set in $\mathbb{C}^{n+1}$ then $a_1(z), \cdots, a_m(z)$ are holomorphic on D (see [11]). In this paper we give some results of type of Shcherbina on the relation between pluriporlarity of graphs and holomorphicity of complex-valued functions defined on a domain in $\mathbb{C}^n$. In particular we establish the relation between the pluripolarity of graphs and the holomorphicity of Hartogs type and of Forelli type (see Theorems 3.2 and 4.1 below). The paper is organized as follows. Beside the introduction, the paper contains three sections. In the second one we recall some well-known facts in the pluripotential theory. The third section is devoted to present holomorphic graphs of Hartogs type. The fourth one deals with holomorphic graphs of Forelli type.

2. Basic Definitions

In this section we recall some notions and fix some notations.

2.1. Let D be a domain in $\mathbb{C}^n$ and $\varphi : D \to [-\infty, +\infty)$. φ is called plurisubharmonic on D if φ is upper semi-continuous and the restriction of φ to the intersection of D with every complex line is subharmonic. The cone of plurisubharmonic functions on D is denoted by $PSH(D)$.

A set E of $\mathbb{C}^n$ is called pluripolar if for every $a \in E$ there exitsts a neighborhood $U = U(a)$ of a, $\varphi \in PSH(U)$, φ is not identity to $-\infty$ such that $\varphi = -\infty$ on $U \cap E$. A basic theorem of Josefson (see [6], Theorem 4.7.4) asserts that if E is pluripolar in a domain D in $\mathbb{C}^n$ then there exists a plurisubharmonic function φ on $\mathbb{C}^n$, φ is not identity to $-\infty$ but $E \subset \{z \in D : \varphi(z) = -\infty\}$.

2.2. We recall the relatively extremal function due to Siciak (see [13]). Let D be a domain in $\mathbb{C}^n$ and $A \subset D$. The relatively extremal function $h_{A,D}$ of A to D is defined by

$$h_{A,D} = \sup\{u : u \in \mathrm{PSH(D)}, u \leq 1, u\big|_A \leq 0\}.$$

By $h^*_{A,D}$ we denote the upper semicontinuous regularization of $h_{A,D}$. A set $A \subset D$ is called locally pluriregular if for any $a \in A$ and any neighborhood $U = U(a)$ of a we have $h^*_{A\cap U,U}(a) = 0$.

2.3. Let $D \subset \mathbb{C}^n$ be a domain and $f : D \to \mathbb{C}$ be a complex-valued function defined on D. The graph of f is the set

$$\Gamma(f) = \{(z, f(z)) : z \in D\} \subset D \times \mathbb{C}.$$

3. Holomorphic Graphs of Hartogs Type

The main aim of this section is to give a result above holomorphicity of a complex-valued function under assumption that it is separately continuous and its graph is pluripolar. As well-known that in the main result of Shcherbina (see [10]), Shcherbina assume that if $f : D \to \mathbb{C}$ is a continuous function and the graph $\Gamma(f)$ of f is a pluripolar set in $\mathbb{C}^{n+1}$ then f is holomorphic on D. Now we prove a such result but under assumption weaker than the theorem of Shcherbina. At the same time, this result itself is a form of Hartogs theorem. However before proving this result we establish some results on the relation between holomorphicity of a valued-complex function and pluripolarity of its graph. Namely we prove the following result.

Theorem 3.1. *Let p be a non-constant holomorphic function on $\mathbb{C}^m \setminus S$, where S is a hypersurface in $\mathbb{C}^m$ and f is a continuous function on a domain $D \subset \mathbb{C}^n$. Assume that*

$$Z = \{(z, w) \in (\mathbb{C}^m \setminus S) \times D : p(z) = f(w)\}$$

is pluripolar in $\mathbb{C}^m \times D$. Then f is holomorphic on D.

Proof. First we consider the case $m = 1$.

Step 1. Given $w_0 \in D$ such that there exists $z_0 \in (\mathbb{C} \setminus S)$ with $p(z_0) = f(w_0)$. Assume that $p'(z_0) \neq 0$. Take two neighborhoods $U \subset (\mathbb{C} \setminus S)$ and V of z_0 and $v_0 = p(z_0)$ respectively such that $p : U \to V$ is biholomorphic. Consider the biholomorphic map: $\theta : U \times f^{-1}(V) \to V \times f^{-1}(V)$ given by $\theta(z, w) = (p(z), w)$. Then we have

$$\theta(Z \cap (U \times f^{-1}(V))) = \{(v, w) \in \mathbb{C} \times f^{-1}(V) : v = f(w)\}.$$

It follows that the graph of $f\,|_{f^{-1}(V)}$ is pluripolar in $f^{-1}(V) \times \mathbb{C}$. By [10] f is holomorphic on $f^{-1}(V)$, a neighborhood of w_0.

Step 2. Let $z_0 \in (\mathbb{C} \setminus S)$, $w_0 \in D$ with $p(z_0) = f(w_0)$, $p'(z_0) = 0$. Since $p \neq$ const then we can choose a neighborhood $U \subset (\mathbb{C} \setminus S)$ of z_0 such that $\forall z \in U \setminus z_0, p^{'}(z) \neq 0$ and $p(U) = V$ is a neighborhood of $v_0 = f(w_0)$. By step 1 f is holomorphic on $f^{-1}(V \setminus v_0) = f^{-1}(V) \setminus f^{-1}(v_0)$. Since f is continuous then the Rado theorem implies that f is holomorphic on $f^{-1}(V)$, a neighborhood of w_0.

Step 3. By Steps 1 and 2 the function f is holomorphic on a neighborhood of $f^{-1}\big(p(\mathbb{C} \setminus S\ )\big)$. Thus again by the Rado theorem without loss of generality we may assume that $\mathbb{C} \setminus p(\mathbb{C} \setminus S)$ is an infinite set. Hence $p(\mathbb{C} \setminus S)$ is hyperbolically embedded into $\mathbb{CT}^1$. By the Kwack theorem (see [7]) p is extended to a holomorphic map $\widehat{p} : \mathbb{C} \to \mathbb{CT}^1$. Since

$$\{(z, w) \in (\mathbb{C} \setminus \widehat{p}^{-1}(\infty)) \times D : \widehat{p}(z) = f(w)\}$$

is pluripolar, Steps 1 and 2 imply that f is holomorphic on a neighborhood of $f^{-1}\big(\widehat{p}(\mathbb{C} \setminus \widehat{p}^{-1}(\infty))\big)$. Since $\widehat{p} \neq$ const, the little Picard theorem implies that $\mathbb{C} \setminus \widehat{p}(\mathbb{C} \setminus \widehat{p}^{-1}(\infty))$ is finite. Again by the Rado theorem f is holomorphic on D.

Now assume that $m > 1$. Since S is a hypersurface we can find a complex line $\ell \subset \mathbb{C}^m$ such that $\ell \cap S$ is discrete and $p\big|_{\ell \cap (\ell \setminus S)} \neq$ const and the set

$$\{(z, w) \in (\ell \setminus \ell \cap S) \times D : p(z) = f(w)\}$$

is pluripolar in $\ell \times D$. By the case $m = 1$ it follows that f is holomorphic on D. ∎

As well-known that in complex analysis the Hartogs theorem says that if D is a domain in $\mathbb{C}^n$ and $f : D \to \mathbb{C}$ is separately holomorphic on D then f is holomorphic on D. Next we give a result of type of Hartogs theorem under the hypothesis f is separately continuous and the graph $\Gamma(f)$ of f is pluripolar in $\mathbb{C}^{n+1}$. Namely we prove the following result.

Theorem 3.2. *Let $D_j \subset \mathbb{C}^{k_j}$ be bounded pseudoconvex domains for $j = 1, 2, \cdots, N$ and $f : D = D_1 \times \cdots \times D_N \to \mathbb{C}$ be a complex - valued function defined on D. Assume that for every $(a_1, \cdots, a_N) \in D_1 \times \cdots \times D_N$ the function $f(a_1, \cdots, a_{j-1}, ., a_{j+1}, \cdots, a_N)$ is continuous on $D_j, j = 1, 2, \cdots, N$. If the graph $\Gamma(f)$ of f:*

$$\Gamma(f) = \{(z_1, \cdots, z_N, w) \in D \times \mathbb{C} : w = f(z_1, \cdots, z_N)\}$$

is pluripolar in $\mathbb{C}^{k_1+\cdots+k_N+1}$ then f is holomorphic on D.

Proof. We prove Theorem 3.2 by induction. For $N = 1$ the result belongs to Shcherbina [10]. We prove if for the case $N = 2$. Let $D_1 \subset \mathbb{C}^n, D_2 \subset \mathbb{C}^m$ be bounded pseudoconvex domains and $f : D_1 \times D_2 \to \mathbb{C}$ be a complex-valued function satisfying the followings:
(i) For every $z_1 \in D_1$ the function $f_{z_1}(z_2) = f(z_1, z_2)$ is continuous on D_2.
(ii) For every $z_2 \in D_2$ the function $f^{z_2}(z_1) = f(z_1, z_2)$ is continuous on D_1.
Assume that the graph $\Gamma(f)$ of f is pluripolar in $\mathbb{C}^{n+m+1}$. From the hypothesis there exists $u \in \mathrm{PSH}(\mathbb{C}^{n+m+1})$, $u \not\equiv -\infty$ such that

$$\Gamma(f) \subset \{(z_1, z_2, w) \in \mathbb{C}^{n+m+1} : u(z_1, z_2, w) = -\infty\}.$$

Assume that $(z_1', z_2', w') \in \mathbb{C}^{n+m+1}$ with $u(z_1', z_2', w') \neq -\infty$. Then the function $u_1(z_1, w) = u(z_1, z_2', w)$ and $u_2(z_2, w) = u(z_1', z_2, w)$ are plurisubharmonic on $D_1 \times \mathbb{C}$ and $D_2 \times \mathbb{C}$ respectively. Setting

$$E_1 = \{z_1 \in D_1 : u_1(z_1, w') = -\infty\},$$
$$E_2 = \{z_2 \in D_2 : u_2(z_2, w') = -\infty\}.$$

Then E_1 and E_2 are pluripolar sets in D_1 and D_2 respectively. Let $A_1 = D_1 \setminus E_1$ and $A_2 = D_2 \setminus E_2$. For each $(z_1^0, z_2^0) \in A_1 \times A_2$ the function $f^{z_2^0} : D_1 \to \mathbb{C}$ is continuous and

$$\begin{aligned}\Gamma(f^{z_2^0}) &= \{(z_1, w) \in D_1 \times \mathbb{C} : w = f^{z_2^0}(z_1)\}\\ &= \{(z_1, w) \in D_1 \times \mathbb{C} : w = f(z_1, z_2^0)\}\\ &\subset \{(z_1, w) \in D_1 \times \mathbb{C} : u(z_1, z_2^0, w) = -\infty\}\end{aligned}$$

with $u(z_1', z_2^0, w) > -\infty$, then Theorem 1 in [10] implies that f^{z_2} is holomorphic on D_1. Similarly, the function $f_{z_1^0} : D_2 \to \mathbb{C}$ is holomorphic on D_2. Hence, f is separately holomorphic on the cross $X = (D_1 \times A_2) \cup (A_1 \times D_2)$. We prove that A_1 and A_2 are pluriregular in D_1 and D_2 respectively. Indeed, let $a_1 \in A_1$ be arbitrary. Take an arbitrary neighborhood U_1 of A_1, $U_1 \subset D_1$. Consider the relatively extremal function $h_{A_1\cap U_1, U_1} = \sup\{u : u \in \mathrm{PSH}(U_1), u \leqslant 1, u|_{A_1\cap U_1} \leqslant 0\}$ and its upper - semicontinuous regularization $h^*_{A_1\cap U_1, U_1}$. Then $h^*_{A_1\cap U_1, U_1}$ is plurisubharmonic on U_1. Put

$$F_1 = \{z_1 \in U_1 : h_{A_1\cap U_1, U_1}(z_1) \neq h^*_{A_1\cap U_1, U_1}(z_1)\}.$$

Theorem 7.1 in [3] implies that F_1 is pluripolar in U_1 and $h^*_{A_1\cap U_1,U_1} = h_{A_1\cap U_1,U_1}$ on $U_1\setminus F_1$. Hence $h^*_{A_1\cap U_1,U_1}(z_1) = 0$ for $z_1 \in A_1\cap U_1\setminus F_1$. However, $A_1\cap U_1\setminus F_1 = U_1 \setminus E_1 \cup F_1$. It follows that $h^*_{A_1\cap U_1,U_1} = 0$ on U_1 and the desired conclusion follows. Similarly, A_2 is pluriregular in D_2. Theorem 2.2.4 in [1] implies that $f : X \to \mathbb{C}$ extends to a holomorphic function $\widehat{f}$ on $\widehat{X} = \{(z_1, z_2) \in D_1 \times D_2 : h^*_{A_1,D_1}(z_1) + h^*_{A_2,D_2}(z_2) < 1\}$. Notice that from the above proof it follows that $h^*_{A_1,D_1} = 0$ on D_1 and $h^*_{A_2,D_2} = 0$ on D_2. Hence $\widehat{X} = D_1 \times D_2$. For each $z_1 \in D_1$ we have that f_{z_1} and $\widehat{f}_{z_1}$ are continuous on D_2 and $f_{z_1}|_{A_2} = \widehat{f}_{z_1}|_{A_2}$. On the other hand, $E_2 = D_2 \setminus A_2$ is pluripolar. Hence $f_{z_1} = \widehat{f}_{z_1}$ on D_2. Thus $f = \widehat{f}$ on $D_1 \times D_2$ and the proof for the case $N = 2$ is completed. Now we assume that the conclusion of the theorem holds for $N - 1 \geq 2$. Put $k = k_1 + \cdots + k_N$. From the hypothesis there exists $u \in PSH(\mathbb{C}^{k+1}), u \not\equiv -\infty$ such that

$$\Gamma(f) \subset \{(z_1, z_2, \cdots , w) \in \mathbb{C}^{k+1} : u(z_1, \cdots , u_N, w) = -\infty\}.$$

Take $(z'_1, z'_2, \cdots , z'_N, w') \in \mathbb{C}^{k+1}$ with $u(z'_1, z'_2, \cdots , z'_N, w') > -\infty$. Then the set

$$F_N = \{z_N \in D_N : u(z'_1, \cdots , z'_{N-1}, z_N, w') = -\infty\}$$

is pluripolar in D_N. For each $z^0_N \in D_N\setminus F_N$ the function $u(z_1, \cdots , z_{N-1}, z^0_N, w) \in PSH(\mathbb{C}^{k+1-k_N})$. Notice that

$$\{(z_1, \cdots , z_{N-1}, w) \in D_1 \times \cdots \times D_{N-1} \times \mathbb{C} : w = f(z_1, \cdots , z_{N-1}, z^0_N)\}$$

$$\subset \{(z_1, \cdots , z_{N-1}, w) \in \mathbb{C}^{k+1-k_N} : u(z_1, \cdots , z_{N-1}, z^0_N, w) = -\infty\}.$$

Hence, the inductive hypothesis implies that the function $f(z_1, \cdots , z_{N-1}, z^0_N)$ is holomorphic on $D_1 \times \cdots \times D_{N-1}$. Since $D' = D_1 \times \cdots \times D_{N-1}$ is bounded pseudoconvex then the arguments in the above proof imply that there exist pluripolar sets $E' \subset D' = D_1 \times \cdots \times D_{N-1}$ and $E_N \subset D_N, E_N \supset F_N$ such that f is separately holomorphic on the cross

$$X = (D \times A_N)\bigcup(A' \times D_N)$$

with $A' = D' \setminus E', A_N = D_N \setminus E_N$. The same arguments as in the above proof complete the proof of Theorem 3.2. ∎

4. Holomorphic Graphs of Forelli Type

In 1978 Forelli proved that if $f : \mathbb{B}^n = \mathbb{B}^n(0, 1) \to \mathbb{C}$ is defined on the unit ball $\mathbb{B}^n(0, 1)$ in $\mathbb{C}^n$ such that restrictions of f on every complex line l through 0 is holomorphic and f is of C^∞-class in a neighborhood of 0 then f is holomorphic on $\mathbb{B}^n$ (see [8]). In this section we establish a result of such type under assumption that f is continuous on every complex line l through 0 and the graph $\Gamma(f)$ is pluripolar in $\mathbb{C}^{n+1}$. Namely we have the following.

Theorem 4.1. *Let $\mathbb{B}^n = \mathbb{B}^n(0, 1)$ be the unit ball in $\mathbb{C}^n$ and $f : \mathbb{B}^n \to \mathbb{C}$ be a function defined on $\mathbb{B}^n$. Suppose that f is continuous on every complex line*

l through $0 \in \mathbb{C}^n$ and on a neighborhood of $0 \in \mathbb{B}^n$. If the graph $\Gamma(f)$ of f is a pluripolar set in $\mathbb{C}^{n+1}$ then there exists a family V of complex lines through $0 \in \mathbb{B}^n$ with $\lambda_{2n}(V) = 0$ and a holomorphic function $\tilde{f}$ on $\mathbb{B}^n$ such that $f = \tilde{f}$ on $\mathbb{B}^n \setminus V$ and a neighborhood of $0 \in \mathbb{B}^n$.
To prove the theorem we need the following.

Lemma 4.2. *Let E be a pluripolar set in $\mathbb{B}^n \setminus \{0\}$ and $V = \bigcup_{z\in E} l_z$, where l_z denotes the complex line through 0 and z. Then $\sigma_1(V \cap \partial\mathbb{B}^n) = 0$ and $\lambda_{2n}(V \cap \mathbb{B}^n) = \lambda_{2n}(V) = 0$, where σ_1 denotes the surface area measure on $\partial\mathbb{B}^n$ and λ_{2n} the Lebesgue measure in $\mathbb{C}^n$.*

Proof. For each $m \geq 1$ we put $E_m = E \setminus \mathbb{B}^n_{\frac{1}{m}}$ and $V_m = \bigcup_{z\in E_{\frac{1}{m}}} l_z$, where $\mathbb{B}^n_{\frac{1}{m}} = \{z \in \mathbb{B}^n : ||z|| < \frac{1}{m}\}$. Then $V_m \subset V_{m+1}$ and $V = \bigcup_{m=1}^{\infty} V_m$. Given $\varepsilon > 0$. By [2], the Hausdorff measure $\Lambda_{2n-1}(E_m) = 0$ and, hence, there exists a family of balls $\{\mathbb{B}^n(z_j, r_j)\}_{j=1}^{\infty}$ such that $r_j < \frac{1}{2m}$ for all $j \geq 1$, $\sum_{j=1}^{\infty} r_j^{2n-1} < \varepsilon$, $E_m \subset \bigcup_{j=1}^{\infty} \mathbb{B}^n(z_j, r_j)$. It follows that

$$\bigcup_{j=1}^{\infty} \mathbb{B}^n(z_j, r_j) \subset \{z \in \mathbb{C}^n : \frac{1}{2m} < ||z|| < 2\}.$$

Let π_2 denote the projection from $\mathbb{C}^n \setminus \{0\}$ to $\partial\mathbb{B}^n$ given by

$$\pi_2(z) = 2\frac{z}{||z||}.$$

Setting, for each $j \geq 1$, $S_j = \pi_2(\mathbb{B}^n(z_j, r_j))$, then we have

$$V_m \cap \partial\mathbb{B}^n_2 \subset \bigcup_{j=1}^{\infty} (\pi_2(B^n(z_j, r_j)).$$

Since $\mathbb{B}^n(z_j, r_j) \subset \{z \in \mathbb{C}^n : \frac{1}{2m} < ||z|| < 2\}$ it follows that there exists $C = C(m, n)$ such that:

$$\sigma_2(S_j) \leq C.r_j^{2n-1} \quad \forall j \geq 1.$$

Hence

$$\sigma_2(V_m \cap \partial\mathbb{B}^n_2) \leq (\bigcup_{j=1}^{\infty} \pi_2(\mathbb{B}^n(z_j, r_j)))$$
$$\leq \sum_{j=1}^{\infty} \sigma_2(S_j) \leq C \sum_{j=1}^{\infty} r_j^{2n-1} \leq C\epsilon.$$

Thus $\sigma_2(V_m \cap \partial\mathbb{B}^n_2) = 0$ and, hence,

$$\sigma_2(V \cap \partial\mathbb{B}^n_2) = \lim_{m\to+\infty} \sigma_2(V_m \cap \partial\mathbb{B}^n_2) = 0.$$

Notice that $\sigma_r(V \cap \mathbb{B}^n_r) = (\frac{r}{2})^{2n-1}\sigma_2(V \cap \mathbb{B}^n_2)$ then we infer that $\sigma_r(V \cap \partial\mathbb{B}^n_r) = 0$ for all $r > 0$. Consequently, $\sigma_1(V \cap \partial\mathbb{B}^n) = 0$.

On the other hand

$$\lambda_{2n}(V) = \int_{\mathbb{R}^{2n}} \chi_V(\xi) d\,\lambda_{2n}(\xi) = 2n \int_0^{+\infty} d\,r \int_{\partial \mathbb{B}^n{}_r} \chi_V(\eta) d\,\sigma_r(\eta) = 0$$

in which χ_V denotes the characteristic function of V. The lemma is proved. ■

Now we are able to prove Theorem 4.1.

Proof of Theorem 4.1. From the hypothesis there exists $u \in PSH(\mathbb{B}^n \times \mathbb{C})$, u is not identity to $-\infty$ such that

$$\Gamma(f) \subset \{(z, w) \in \mathbb{B}^n \times \mathbb{C} : u(z, w) = -\infty\}$$

Assume that $(z^{'}, w^{'}) \in \mathbb{B}^n \times \mathbb{C}$ with $u(z^{'}, w^{'}) \neq -\infty$. Then $v(z) = u(z, w^{'}) \in PSH(\mathbb{B}^n)$. Set $F = \{z \in \mathbb{B}^n : v(z) = -\infty\}$ and $E = F \setminus \{0\}$. Then E is a pluripolar set in $\mathbb{B}^n \setminus \{0\}$. Assume that $z_0 \in \mathbb{B}^n \setminus (E \cup \{0\})$ is arbitrary. Then $f\mid_{\ell_{z_0} \cap \mathbb{B}^n}$ is continuous and $\Gamma(f)\mid_{l_{z_0} \cap \mathbb{B}^n}$ is a pluripolar set. By [10] $f\mid_{\ell_{z_0} \cap \mathbb{B}^n}$ is holomorphic. On the other hand, if we put $V = \bigcup_{z \in E} l_z$ then Lemma 4.2 implies that $\lambda_{2n}(V) = 0$. From the above proof it follows that $f\mid_{\ell \cap \mathbb{B}^n}$ is holomorphic for every complex line $\ell \subset (\mathbb{B}^n \setminus V)$ through $0 \in \mathbb{B}^n$. Moreover, from the hypothesis there exists $0 < r < 1$ such that f is continuous on $\mathbb{B}^n(0, r)$. Again by [10] f is holomorphic on $\mathbb{B}^n(0, r)$. Hence f is of C^∞-class on $\mathbb{B}^n(0, r)$. By shringking r we may assume that f is bounded there. The following proof is presented in [5] but for convenience to the reader we rewrite it here. Putting

$$G_k(z) = \frac{1}{2\pi i} \int_{|\lambda|=1} \frac{f(\lambda z) d\lambda}{\lambda^{k+1}}, z \in \mathbb{B}_r \tag{1}$$

we obtain for each $k \geq 0$ a bounded C^∞-function G_k on $\mathbb{B}_r^n$. Since $\lambda \to f(\lambda z)$ is holomorphic for all $z \in \mathbb{B}^n \setminus V$ we have

$$G_k(\lambda z) = \lambda^k G_k(z) \tag{2}$$

for $z \in \mathbb{B}_r^n \setminus V$ and $|\lambda| \leqslant 1$. Notice that $\mathbb{B}_r^n \setminus V$ is dense in $\mathbb{B}_r^n$ then (2) holds for all $z \in \mathbb{B}_r^n$ and all $|\lambda| \leqslant 1$. From the boundedness of G_k on $\mathbb{B}_r^n$ we deduce that

$$G_k(z) = 0(|z|^k) \quad \text{as } z \to 0. \tag{3}$$

Since $G_k \in C^\infty(\mathbb{B}_r^n)$ and by (3) then the Taylor expansion of G_k at $0 \in \mathbb{B}^n$ has the form

$$G_k(z) = \sum_{\mu+\nu=k} P_{\mu\nu}(z) + |z|^k \gamma(z), \tag{4}$$

where $P_{\mu\nu}$ is a polynomial of degree μ in z and degree ν in $\bar{z}$ and $\gamma(z) \to 0$ as $z \to 0$.

By replacing z by λz into (4) and using (2) we obtain

$$\sum_{\mu+\nu=k} P_{\mu\nu}(z) \lambda^\mu \bar{\lambda}^\nu + |\lambda|^k |z|^k \gamma(\lambda z) = \lambda^k G_k(z)$$

$$= \sum_{\mu+\nu=k} P_{\mu\nu}(z)\lambda^k + \lambda^k |z|^k \gamma(z) \tag{5}$$

for $z \in \mathbb{B}^n_r, |\lambda| \leqslant 1$. If λ is chosen in $(0,1)$ then we deduce that $\gamma(\lambda z) = \gamma(z)$ and let $\lambda \to 0$ we obtain that $\gamma(z) \equiv 0$ for $z \in \mathbb{B}^n_r$ and (5) becomes

$$\sum_{\mu+\nu=k} P_{\mu\nu}(z)\lambda^\mu \bar{\lambda}^\nu = \sum_{\mu+\nu=k} P_{\mu\nu}(z)\lambda^k \tag{6}$$

for $z \in \mathbb{B}^n_r$.

From (6) and using the uniqueness of the expansion it follows that $P_{\mu\nu}(z) = 0$ for all $\nu > 0$ and $z \in \mathbb{C}^n$. Hence from (4) we obtain that

$$G_k(z) = P_{k,0}(z) = \sum_{|\alpha|=k} c_\alpha z^\alpha$$

for $z \in \mathbb{C}^n$ and $G_k(z)$ in fact is a homogeneous holomorphic polynomial of degree k. On the other hand, $\lambda_{2n}(\mathbb{B}^n_r \setminus V) > 0$ and from

$$\sup_k \Big\{ \frac{1}{k} \log |G_k(z)| < \infty \Big\} \quad \text{for} \quad z \in \mathbb{B}^n_r \setminus V$$

by [13] we can find $\varepsilon > 0$ and $C > 0$ such that

$$\sup_k \Big\{ \frac{1}{k} \log |G_k(z)| < \infty : \|z\| < \varepsilon \Big\} < C.$$

This inequality yields that the series $\sum\limits_{k=0}^{\infty} G_k(z)$ is uniformly convergent on $\mathbb{B}^n_{\varepsilon e^{-2C}}$. Thus $f_{\mathbb{B}^n \setminus V}$ is extended to a holomorphic map on $\mathbb{B}^n_\alpha$ where $\alpha = \varepsilon e^{-2C}$. Next for each $1 \leqslant j \leqslant n$ put $\mathbb{B}^n_{*,j} = \{z \in \mathbb{B}^n : z_j \neq 0\}$. Now as in [9] consider the map $\varphi_j : \mathbb{B}^n_{*,j} \longrightarrow \mathbb{C}^n$ given by

$$\varphi_j(z_1, \cdots, z_{j-1}, z_j, z_{j+1}, \cdots, z_n) = \Big(\frac{z_1}{z_j}, \cdots, \frac{z_{j-1}}{z_j}, z_j, \frac{z_{j+1}}{z_j}, \cdots, \frac{z_n}{z_j} \Big)$$

It is clear that φ_j is biholomorphic onto its image. Fix $1 \leqslant j \leqslant n$ which we may assume that $j = n$. By φ we denote φ_n and

$$T := \varphi(\mathbb{B}^n_{*,n}) = \bigcup_{R>0} \mathbb{B}^{n-1}_R \times \triangle^* \Big(0, \sqrt{\frac{1}{1+R^2}}\Big)$$

We shall prove $g := f\varphi^{-1}$ is holomorphic on T. Let

$$A = \{\omega' \in \mathbb{B}^{n-1}_R : \exists \omega_n(\omega') \in \triangle^* \Big(0, \sqrt{\frac{1}{1+R^2}}\Big) \text{ such that } (\omega_n(\omega')\omega', \omega_n(\omega')) \in V\}.$$

Since V is a family of complex lines through 0 then we obtain that $S_{\omega'} \subset V$ and

$$\lambda_2(S_{\omega'}) \geq \pi |\omega_n(\omega')|^2 > 0$$

for all $\omega' \in A$ where

$$S_{\omega'} = \{(\omega_n \omega', \omega_n) : \omega_n \in \mathbb{C}, |\omega_n| \leqslant |\omega_n(\omega')|\}$$

and λ_2 denotes the Lebesgue measure on $\mathbb{C}$. Hence $\bigcup\limits_{\omega' \in A} S_{\omega'} \subset V$ and if λ_{2n} denotes the Lebesgue measure of $\mathbb{C}^n$ then an application of Fubini's theorem gives

$$0 = \int_{\bigcup_{\omega' \in A} S_{\omega'}} d\lambda_n = \int_A d\omega' \int_{S_{\omega'}} d\lambda_1 = \int_A \lambda_2(S_{\omega'}) d\omega'.$$

This yields that $\lambda_{2n-2}(A) = 0$.

Consider $g\big|_{\mathbb{B}_R^{n-1} \times \triangle^*\left(0, \sqrt{\frac{1}{1+R^2}}\right)}$. Since g is holomorphic on $\mathbb{B}_R^{n-1} \times \triangle^*\Big(0, \sqrt{\frac{\alpha^2}{1+R^2}}\Big)$ and we have $g(\omega', \omega_n) = f(\omega_n \omega', \omega_n)$ for $\omega' \in \mathbb{B}_R^{n-1} \backslash A$, $\omega_n \in \triangle^*\Big(0, \sqrt{\frac{1}{1+R^2}}\Big)$ is holomorphic in ω_n, Theorem 3 in [12] implies that g is holomorphic on $\mathbb{B}_R^{n-1} \times \triangle^*\Big(0, \sqrt{\frac{1}{1+R^2}}\Big)$ for all $R > 0$. Therefore g is holomorphic on T. Thus $f_{|\mathbb{B}^n \backslash V}$ is holomorphically extended to $\mathbb{B}^n_{*,j}$ for all $j = 1, 2, \cdots, n$. From the equality

$$\mathbb{B} = \mathbb{B}_\alpha^n \cup \Big(\bigcup_{j=1}^n \mathbb{B}^n_{*,j}\Big)$$

it follows that $f_{|\mathbb{B}^n \backslash V}$ is extended to a holomorphic function $\widetilde{f}$ on $\mathbb{B}^n$ such that $f = \widetilde{f}$ on $\mathbb{B}^n \setminus V$. Since $\lambda_{2n}(V) = 0$ and $f, \widetilde{f}$ are holomorphic on $\mathbb{B}_r^n$, $f\big|_{\mathbb{B}_r^n \backslash V} = \widetilde{f}\big|_{\mathbb{B}_r^n \backslash V}$ it follows that $f = \widetilde{f}$ on $\mathbb{B}_r^n$. Theorem 4.1 is proved. ■

We have the following remark.

Remark 4.3. Under the hypothesis of Theorem 4.1 it could not follow that f is holomorphic in $\mathbb{B}^n$. Indeed, assume that $\mathbb{B}^n = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 < 1\}$ is the unit ball in $\mathbb{C}^2$. Let g be a holomorphic function on $\mathbb{B}^n$. Take a continuous function $h(z)$ on the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ but h is not holomorphic on Δ satisfying condition $h(z) = g(z, 0)$ on $\{|z| < \frac{1}{3}\} \cup \{\frac{2}{3} < |z| < 1\}$. Consider the function on $\mathbb{B}^n$ given by

$$f(z, w) = \begin{cases} g(z, w) & \text{if } w \neq 0 \\ h(z) & \text{if } w = 0. \end{cases}$$

Then f is continuous on every complex line l through $0 \in \mathbb{B}^n$ and $f(z, w) = g(z, w)$ on $\mathbb{B}^n_{\frac{1}{3}}$. Hence $f \in C^\infty(\mathbb{B}^n_{\frac{1}{3}})$. It is easy to see that

$$\Gamma(f) \subset \Gamma(g) \cup \{(z, 0, \eta) \in \mathbb{C}^3\}$$

is a pluripolar set in $\mathbb{C}^3$. However, because h is not holomorphic then f is not holomorphic on $\mathbb{B}^n$.

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