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Solution to an Open Problem on the Integral Sum Graphs*

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Abstract. The concept of the (integral) sum graphs was first introduced by Harary (Congr. Number.72(1990)101; Discrete Math. 124.(1994)99). Let N^* denote the set of positive integers. The (integral) sum graph $G^+(S)$ of a finite subset $S \subset N^*(Z)$ is the graph (S, E) with $uv \in E$ if and only if $u + v \in S$. A graph G is called an (integral) sum graph if it is isomorphic to the (integral) sum graph $G^+(S)$ for some $S \subset N^*(Z)$. In this paper we give a constructive method to show that the odd cycles are regular integral sum graphs, which extends the classes of integral sum graphs and completely solves an open problem posed by Baogen Xu (Discrete Math. 194 (1999) 285-294).

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1. Introduction and Preliminary Results

We follow in general the graph-theoretic notation and terminology of [1] or [2]. Since the concept of the (integral) sum graphs was introduced by Harary, there have been a variety of researches on integral sum graphs, which falls loosely into

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three categories. Firstly, one tries to find some new integral sum graphs (see [1, 3, 4]). Secondly, one tries to construct some new integral sum graph from the old ones (see [5]). Finally people concentrate on the investigation on some combinatorial parameters such as the sum number and the integral sum number of some graphs (see [6, 7, 8]). Indeed, there are many problems in this field remaining unsolved. Here we solve an open problem which was posed by Xu in [4], which extends the area of integer sum graphs.

Let N^* denote the set of positive integers. The *sum graph* $G^+(S)$ of a finite subset S of N^* is the graph (S, E) with vertex set S and edge set E, such that for all $u, v \in S, uv \in E$ if and only if $u + v \in S$. A graph G is called a sum graph if it is isomorphic to the sum graph $G^+(S)$ for some $S \subset N^*$. The *sum number* $\sigma(G)$ of a graph G is defined as the smallest nonnegative integer m for which $G \cup mK_1$ is a sum graph. The *integral sum graph* $G^+(S)$ is defined just as the sum graph, the difference being that $S \subset Z$ instead of $S \subset N^*$, and the *integral sum number* $\zeta(G)$ of G is defined as the smallest nonnegative integer s such that $G \cup sK_1$ is an integral sum graph. For convenience, an integral sum graph is written as $\int \Sigma$-graph. Obviously $\zeta(G) \leq \sigma(G)$ for all graph G by definition, and G is an $\int \Sigma$-graph if and only if $\zeta(G) = 0$. In [4] Xu obtain that if $n \neq 4$ then $\xi(C_n) \leqslant 1$, and he also observed from its proof that C_{2n+1} is an $\int \Sigma$-graph when $1 \leqslant n \leqslant 4$ and $C_{11} \sim G^+(40, 8, -5, 3, 5, -2, 7, -12, 20, -32, 35)$. From these results it is very interesting to answer whether the odd cycle is an $\int \Sigma$-graph or not. In this paper we give a constructive method to show that all the odd cycles are $\int \Sigma$-graphs, which completely solve an open problem posed by Xu in [4].

2. Odd Cycle

In [4], Xu posed an open problem as follows.

Problem 1. *Is it true that any odd cycle is an $\int \Sigma$-graph ?*

In this section, we give an affirmative answer to the above open problem, namely

Theorem 2.1. *If n is an odd number, then the cycle C_n is an $\int \sum$-graph.*

Proof. At first we show that when the odd number $n < 11$, the cycle C_n is an $\int \sum$-graph. In fact,

$$C_3 \cong G^+(0,\ 1,\ -1); \quad C_5 \cong G^+(1, -2, 3, -1, 2);$$
$$C_7 \cong G^+(1, 3, 4, -3, 7, -5, 2); \; C_9 \cong G^+(2, 5, -3, 8, -20, 17, -12, 7, -5).$$

This yields that the odd cycle C_n is an $\int \sum$-graph for $n < 11$.

Next we shall show that, when $n \geq 11$, the odd cycle C_n is also an $\int \sum$-graph. Denote it by $C_n = (a_1, a_2, \ldots, a_n)$, where $n \leqslant 11$. Label its vertices as follows,

$$
\begin{aligned}
&l(a_1) = a, l(a_2) = b, \ldots, l(a_i) = l(a_{i-2}) - l(a_{i-1}) \quad (i = 3, 4, \ldots, n-4),\\
&l(a_{n-3}) = l(a_1) - l(a_{n-4}), l(a_{n-2}) = -l(a_2) - l(a_{n-3}), \qquad (2.1)\\
&l(a_{n-1}) = l(a_1) + l(a_2) = a + b, l(a_n) = -l(a_2) = -b,
\end{aligned}
$$

where $b > 2a > 0$, $b \neq 3a$ and $a, b \in \mathbb{Z}$. Then we get a sequence $S = (l(a_1), l(a_2), \ldots, l(a_n))$ with the following properties.

(1) For $3 \leq i \leq n-4, l(a_i) > 0$ if i is odd; $l(a_i) < 0$ if i is even.

(2) The sequence $(|l(a_1)|, |l(a_3)|, |l(a_2)|, |l(a_{n-1})|, |l(a_4)|, \ldots, |l(a_{n-4})|, |l(a_{n-3})|, |l(a_{n-2})|)$ is strictly increasing.

(3) The sequence $(|l(a_5)| - |l(a_4)|, |l(a_6)| - |l(a_5)|, \ldots, |l(a_{n-4})| - |l(a_{n-5})|)$ is strictly increasing.

The property (1) can be verified easily by (2.1). Then we verify properties (2) and (3).

The proof of property (2)*:* By (2.1), $(|l(a_3)|, |l(a_4)|, \ldots, |l(a_{n-4})|)$ is a strictly increasing sequence. We shall insert $l(a_1), l(a_2), l(a_{n-3}), l(a_{n-2}), l(a_{n-1})$, and $l(a_n)$ to the above sequence such that the resulting sequence is strictly increasing.

For $l(a_{n-3})$, by (2.1) and property (1) we have $l(a_{n-3}) = l(a_1) - l(a_{n-4}) > |l(a_{n-4})|$.

For $l(a_{n-2})$, by (2.1) and property (1) we have $l(a_{n-2}) = -l(a_2) - l(a_{n-3})$, which implies $|l(a_{n-2})| > |l(a_{n-3})|$, therefore $|l(a_{n-2})| > |l(a_{n-3})| > |l(a_{n-4})|$.

For $l(a_{n-1})$, by (2.1) we have $l(a_{n-1}) = l(a_1) + l(a_2) = a + b > b = l(a_2) > a = l(a_1)$, then we get $l(a_{n-1}) > l(a_2) > l(a_1) > 0$.

For $l(a_3)$, by (2.1) we have $|l(a_3)| = |l(a_1) - l(a_2)| = b - a > 2a - a = l(a_1)$. In addition, we get $|l(a_3)| = |l(a_1) - l(a_2)| = b - a < b = l(a_2)$. So $|l(a_2)| > |l(a_3)| > |l(a_1)|$.

For $l(a_4)$, by (2.1) we know $l(a_4) = l(a_2) - l(a_3) = 2b - a = b + b - a > b + a = l(a_{n-1})$. Then we have $|l(a_4)| > |l(a_{n-1})|$.

Therefore, the sequence $(|l(a_1)|, |l(a_3)|, |l(a_2)|, |l(a_{n-1})|, |l(a_4)|, \ldots, |l(a_{n-4})|, |l(a_{n-3})|, |l(a_{n-2})|)$ is strictly increasing, and property(2) holds. ∎

The proof of property (3)*:* By (2.1) and property (2) we have

$$
\begin{aligned}
|l(a_{n-4})| - |l(a_{n-5})| &= |l(a_{n-6})| > |l(a_{n-7})| = |l(a_{n-5})| - |l(a_{n-6})|\\
&> \ldots > |l(a_3)| = |l(a_5)| - |l(a_4)|.
\end{aligned}
$$

Then property (3) holds. ∎

Together with property (2) and $l(a_n) = -l(a_2)$ we obtain the labels in the sequence of $S = (l(a_1), l(a_2), \ldots, l(a_n))$ are all different. Now we verify that the sum of labels of any two adjacent vertices is a label of C_n. In fact, by (2.1),

$$
\begin{aligned}
&l(a_1) + l(a_2) = a + b = l(a_{n-1}).\\
&l(a_i) + l(a_{i+1}) = l(a_{i-1}) \qquad 2 \leq i \leq n-5.\\
&l(a_{n-4}) + l(a_{n-3}) = a = l(a_1).\\
&l(a_{n-3}) + l(a_{n-2}) = -b = l(a_n).
\end{aligned}
$$

$l(a_{n-2}) + l(a_{n-1}) = -b - l(a_{n-3}) + a + b = a - l(a_{n-3}) = l(a_{n-4})$.
$l(a_{n-1}) + l(a_n) = a = l(a_1)$.
$l(a_n) + l(a_1) = a - b = l(a_3)$.

Thus the sum of labels of any two adjacent vertices is a label of C_n. In order to finish our proof, it is sufficient to establish the following lemmas.

Lemma 2.2. *The sum of labels of any two nonadjacent vertices in* $\{a_1, a_2, \dots, a_{n-4}\}$ *are not in* $\{l(a_{n-3}), l(a_{n-2}), l(a_{n-1}), l(a_n)\}$.

Proof. Consider $l(a_i) + l(a_j)$ for $1 \le i,\ j \le n-4$. As a_i and a_j are nonadjacent, we can always suppose that $j > i+1$. By property (2), $|l(a_i)| < |l(a_j)|$.
(i) If $l(a_i) + l(a_j) = l(a_{n-3})$, by properties (1) and (2) we have $l(a_i) + l(a_j) = l(a_{n-3}) > |l(a_{n-4})|$. By $|l(a_i)| < |l(a_j)|$, we get $l(a_j) > 0$. On the other hand, we also have $l(a_i) > 0$, otherwise by property(2) we have $l(a_i) + l(a_j) = l(a_j) - |l(a_i)| \le |l(a_{n-4})| - |l(a_i)| < |l(a_{n-4})|$, a contradiction. So $l(a_i), l(a_j) > 0$. By properties (1) and (2) we have

$$l(a_i) + l(a_j) < l(a_{n-5}) + l(a_{n-7}) < l(a_{n-5}) + |l(a_{n-6})| = |l(a_{n-4})|,$$

which is a contradiction. So $l(a_i) + l(a_j) \ne l(a_{n-3})$.
(ii) If $l(a_i) + l(a_j) = l(a_{n-2})$, by properties (1) and (2) we have $l(a_i) + l(a_j) = l(a_{n-2}) < l(a_{n-4}) < 0$. By $|l(a_i)| < |l(a_j)|$, then we get $l(a_j) < 0$. On the other hand, $l(a_i) < 0$, otherwise by property (2) we have

$$|l(a_i) + l(a_j)| = -l(a_i) - l(a_j) = |l(a_j)| - l(a_i) \le |l(a_{n-4})| - |l(a_i)| < |l(a_{n-4})|.$$

Therefore, $l(a_i) + l(a_j) > l(a_{n-4})$, a contradiction. So $l(a_i), l(a_j) < 0$. Furthermore, we can obtain $j = n-4$, otherwise if $j < n-4$, by properties (1) and (2) we have

$$|l(a_i)+l(a_j)|=|l(a_i)|+|l(a_j)| < |l(a_{n-6})|+|l(a_{n-8})| < |l(a_{n-6})| + l(a_{n-5})=|l(a_{n-4})|,$$

therefore, $l(a_i) + l(a_j) > l(a_{n-4})$, a contradiction. So $l(a_i), l(a_j) < 0$ and $j = n-4$.

For $l(a_{n-2})$, by (2.1) and property(2) we have

$$\begin{aligned}|l(a_{n-2})| &= -l(a_{n-2}) = b + l(a_{n-3}) = (a+b) - l(a_{n-4}) < 3b - 2a - l(a_{n-4}) \\ &= |l(a_5)| + |l(a_{n-4})| = |l(a_5) + l(a_{n-4})|.\end{aligned}$$

On the other hand,

$$|l(a_{n-2})|=(a+b)-l(a_{n-4}) > b-a-l(a_{n-4}) = |l(a_3)|+|l(a_{n-4})|=|l(a_3)+l(a_{n-4})|.$$

But $|l(a_i) + l(a_j)| > |l(a_{n-4}) + l(a_5)|$ or $|l(a_i) + l(a_j)| < |l(a_{n-4}) + l(a_3)|$ for every $i \leqslant n-4$, so $l(a_i) + l(a_j) \ne l(a_{n-2})$.

(iii) If $l(a_i)+l(a_j)=l(a_{n-1})$, by (2.1) we have $l(a_i)+l(a_j)=l(a_{n-1})=a+b>0$, then by $|l(a_i)|<|l(a_j)|$, $l(a_j)>0$. In addition we know $l(a_i)<0$, otherwise by properties (1) and (2) we obtain $l(a_i)+l(a_j)>l(a_1)+l(a_2)=a+b=l(a_{n-1})$, a contradiction. Therefore, $l(a_i)<0,\ l(a_j)>0$. By property(1) we know $i\geq 3$ is odd and $j\geq 6$ is even. Then by property (3) we have

$$\begin{aligned}l(a_i)+l(a_j)&=l(a_j)-|l(a_i)|\geq |l(a_{i+3})|-|l(a_i)|\geq l(a_6)-|l(a_3)|\\&=5b-3a-b+a=4b-2a>a+b=l(a_{n-1}),\end{aligned}$$

which is a contradiction, so $l(a_i)+l(a_j)\neq l(a_{n-1})$.
(iv) If $l(a_i)+l(a_j)=l(a_n)$, by (2.1) we have $l(a_i)+l(a_j)=l(a_n)=-b<0$. Then by $|l(a_i)|<|l(a_j)|$ we obtain $l(a_j)<0$. In addition we know $l(a_i)>0$, otherwise by properties (1) and (2) we get $l(a_i)+l(a_j)<l(a_3)+l(a_5)=3a-4b<-b=l(a_n)$, a contradiction, therefore $l(a_i)>0$ and $l(a_j)<0$.

When $i=1$, we know $j\geq 3$ is odd. By (2.1),

$$\begin{aligned}&l(a_1)+l(a_3)=a+a-b>-b=l(a_n),\quad \text{when } j=3,\\&l(a_1)+l(a_j)\leq l(a_1)+l(a_5)=a+2a-3b=3a-3b<-b=l(a_n),\quad \text{when } j>3,\end{aligned}$$

a contradiction.

When $i\geq 2$, we know $j\geq 5$ is odd and i is even, then we have

$$|l(a_i)+l(a_j)|=|l(a_j)|-l(a_i)\geq |l(a_{i+3})|-|l(a_i)|\geq |l(a_5)|-l(a_2)=3b-2a-b>b,$$

then we obtain $l(a_i)+l(a_j)<-b\neq l(a_n)$, so $l(a_i)+l(a_j)\neq l(a_n)$, which is a contradiction.

Combine (i),(ii),(iii) and (iv), Lemma 2.2 holds. ■

Lemma 2.3. *The sum of labels of two nonadjacent vertices in* $\{a_{n-3},a_{n-2},a_{n-1},a_n\}$ *is not a label.*

Proof. There are three possibilities as the following:
(i) For $l(a_{n-3})+l(a_{n-1})$, by (2.1) we have

$$l(a_{n-3})+l(a_{n-1})=a+b+l(a_{n-3})>b+l(a_{n-3})=|l(a_{n-2})|.$$

So $l(a_{n-3})+l(a_{n-1})$ is not a label by property (2).
(ii) For $l(a_{n-3})+l(a_n)$, by (2.1) and property (1) we have

$$l(a_{n-3})+l(a_n)=a-l(a_{n-4})-b=|l(a_{n-4})|-(b-a)<|l(a_{n-4})|.$$

On the other hand, by (2.1) and properties (1) and (2) we have

$$\begin{aligned}l(a_{n-3})+l(a_n)-|l(a_{n-5})|&=-l(a_{n-4})-(b-a)-l(a_{n-5})=-l(a_{n-6})-(b-a)\\&=|l(a_{n-6})|-(b-a)\geq |l(a_5)|-(b-a)\\&=3b-2a-b+a=2b-a>0.\end{aligned}$$

Hence, $|l(a_{n-5})| < l(a_{n-3})+l(a_{n-1}) < |l(a_{n-4})|$, so $l(a_{n-3})+l(a_n)$ is not a label by property (2).
(iii) For $l(a_{n-2}) + l(a_n)$, by property(1) we have $|l(a_{n-2}) + l(a_n)| > |l(a_{n-2})|$. So $l(a_{n-2}) + l(a_n)$ is not a label by property (2).

Combine (i), (ii) and (iii), Lemma 2.3 holds. ∎

Lemma 2.4. *The sum of labels of any two nonadjacent vertices between* $\{a_{n-3}, a_{n-2}, a_{n-1}, a_n\}$ *and* $\{a_1, a_2, \dots, a_{n-4}\}$ *is not a label of some vertex in* C_n.

Proof. First we shall show that the sum of labels of any two nonadjacent vertices between $\{a_{n-3}, a_{n-2}, a_{n-1}, a_n\}$ and $\{a_1, a_2, a_3\}$ is not a label.

For a_1,
(i) By property (1) $l(a_1)+l(a_{n-3}) > l(a_{n-3})$. As $l(a_{n-3})$ is the largest nonnegative integer in our labelling, $l(a_1) + l(a_{n-3})$ does not exist in our labels.
(ii) By properties (1) and (2), $|l(a_1)+l(a_{n-2})| < |l(a_{n-2})|$. On the other hand, by (2.1) and properties (1) and (2),

$$|l(a_1) + l(a_{n-2})| = -l(a_1) - l(a_{n-2}) = -a + b + l(a_{n-3}) > l(a_{n-3}) = |l(a_{n-3})|.$$

Then $|l(a_{n-3})| < |l(a_1) + l(a_{n-2})| < |l(a_{n-2})|$, so $l(a_1) + l(a_{n-2})$ does not exist in our labels by property (2).
(iii) By (2.1), $l(a_1) + l(a_{n-1}) = 2a + b$. Then we have

$$\begin{aligned} |l(a_{n-1})| < l(a_1) + l(a_{n-1}) < |l(a_4)| \quad &\text{when } b > 3a, \\ |l(a_4)| < l(a_1) + l(a_{n-1}) < |l(a_5)| \quad &\text{when } 2a < b < 3a, \end{aligned}$$

so $l(a_1) + l(a_{n-1})$ does not exist in our labels by property (2).

For a_2,
(i) By (2.1), $l(a_2) + l(a_{n-3}) = b + l(a_{n-3}) = -l(a_{n-2})$, then $l(a_2) + l(a_{n-3})$ is not a label.
(ii) By (2.1), $l(a_2)+l(a_{n-2}) = b-b-l(a_{n-3}) = -l(a_{n-3})$, then $l(a_2)+l(a_{n-2})$ is not a label.
(iii) By (2.1), $l(a_2) + l(a_{n-1}) = a + 2b$, then we have

$$\begin{aligned} |l(a_4)| < |l(a_2) + l(a_{n-1})| < |l(a_5)| \quad &\text{when } b > 3a, \\ |l(a_5)| < |l(a_2) + l(a_{n-1})| < |l(a_6)| \quad &\text{when } 2a < b < 3a, \end{aligned}$$

so $l(a_2) + l(a_{n-1})$ is not a label by property (2).
(iv) $l(a_2) + l(a_n) = b - b = 0$ is not a label.

For a_3,
(i) By (2.1) and property(1) we have

$$l(a_3) + l(a_{n-3}) = a - b + a - l(a_{n-4}) = |l(a_{n-4})| - (b - 2a) < |l(a_{n-4})|.$$

On the other hand, by (2.1) and properties (1), (2) we have

$$\begin{aligned} l(a_3)+l(a_{n-3})-|l(a_{n-5})| &= -l(a_{n-4})-(b-2a)-l(a_{n-5}) = -l(a_{n-6})-(b-2a) \\ &= |l(a_{n-6})|-(b-2a) \geq |l(a_5)|-(b-2a) \\ &= 3b-2a-b+2a = 2b > 0. \end{aligned}$$

Therefore, $|l(a_{n-5})| < |l(a_3)+l(a_{n-3})| < |l(a_{n-4})|$, certainly $l(a_3)+l(a_{n-3})$ is not a label by property (2).
(ii) By property (1), $|l(a_3)+l(a_{n-2})| > |l(a_{n-2})|$, so $l(a_3)+l(a_{n-2})$ is not a label by property(2).
(iii) By (2.1), $l(a_3)+l(a_{n-1}) = a-b+a+b = 2a$, then we have

$$\begin{aligned} &|l(a_1)| < l(a_3)+l(a_{n-1}) < |l(a_3)| \quad \text{when } b > 3a. \\ &|l(a_3)| < l(a_3)+l(a_{n-1}) < |l(a_2)| \quad \text{when } 2a < b < 3a. \end{aligned}$$

So $l(a_3)+l(a_{n-1})$ is not a label by property (2).
(iv) By (2.1) $l(a_3)+l(a_n) = a-b-b = a-2b = -l(a_4)$, so $l(a_3)+l(a_{n-1})$ is not a label.

Thus the sum of labels of any two nonadjacent vertices between $\{a_{n-3}, a_{n-2}, a_{n-1}, a_n\}$ and $\{a_1, a_2, a_3\}$ is not a label.

Next we shall show that the sum of labels of any two nonadjacent vertices between $\{a_{n-3}, a_{n-2}, a_{n-1}, a_n\}$ and $\{a_i : 4 \leqslant i \leqslant n-4\}$ is not a label.

Case 1. When $l(a_i) > 0$, by property (1) i is even and $4 \leq i \leq n-5$.
(i) By property(1), $l(a_i)+l(a_{n-3}) > l(a_{n-3})$. As $l(a_{n-3})$ is the largest nonnegative integer in our labels, $l(a_i)+l(a_{n-3})$ is not a label.
(ii) For $l(a_i)+l(a_{n-2})$, by (2.1) and properties (1), (2) we have

$$\begin{aligned} |l(a_i)+l(a_{n-2})| &= -l(a_i)-l(a_{n-2}) = b+l(a_{n-3})-l(a_i) = b+a-l(a_{n-4})-l(a_i) \\ &\leq |l(a_{n-4})|+a+b-l(a_4) = |l(a_{n-4})|+a+b-2b+a \qquad (2.2) \\ &= |l(a_{n-4})|+2a-b < |l(a_{n-4})|. \end{aligned}$$

On the other hand, by (2.1) and properties (1), (2) we have

$$\begin{aligned} |l(a_i)+l(a_{n-2})|-|l(a_{n-5})| &= b+a-l(a_{n-4})-l(a_i)-l(a_{n-5}) \\ &= -l(a_{n-6})+a+b-l(a_i) \qquad (2.3) \\ &= |l(a_{n-6})|+a+b-l(a_i). \end{aligned}$$

When $4 \leq i < n-6$, by (2.3) and property (2) $|l(a_i)+l(a_{n-2})| > |l(a_{n-5})|$. Together with (2.2) we obtain $|l(a_{n-5})| < |l(a_i)+l(a_{n-2})| < |l(a_{n-4})|$. So $l(a_i)+l(a_{n-2})$ is not a label by property (2).

When $n-6 < i \leq n-5$, namely $i = n-5$, by (2.1),(2.3) and properties (1), (2) we have

$$\begin{aligned}|l(a_i)+l(a_{n-2})|-|l(a_{n-5})| &= |l(a_{n-6})|+a+b-l(a_i) = -l(a_{n-6})+a+b-l(a_{n-5})\\ &= -l(a_{n-7})+a+b \le -l(a_4)+a+b = a-2b+a+b\\ &= 2a-b<0.\end{aligned}$$

So we obtain $|l(a_i)+l(a_{n-2})| < |l(a_{n-5})|$. On the other hand, by (2.2),(2.1) and property (1) we have

$$\begin{aligned}|l(a_i)+l(a_{n-2})|-|l(a_{n-6})| &= b+a-l(a_{n-4})-l(a_i)-|l(a_{n-6})|\\ &= b+a-l(a_{n-4})-l(a_{n-5})+l(a_{n-6})\\ &= b+a-l(a_{n-6})+l(a_{n-6}) = a+b>0.\end{aligned}$$

Thus $|l(a_{n-6})| < |l(a_i)+l(a_{n-2})| < |l(a_{n-5})|$, certainly by property(2), $l(a_i)+l(a_{n-2})$ is not a label.

(iii) For $l(a_i)+l(a_{n-1})$.

When $i=4$, by (2.1) we have $l(a_i)+l(a_{n-1}) = 2b-a+a+b = 3b$, then we obtain $|l(a_5)| < |l(a_i)+l(a_{n-1})| < |l(a_6)|$, by property(2) $l(a_i)+l(a_{n-1})$ is not a label.

When $6 \le i \le n-5$, by property (1) we have $l(a_i)+l(a_{n-1}) > |l(a_i)|$. In addition, by (2.1) and properties (1), (2) we have

$$\begin{aligned}l(a_i)+l(a_{n-1})-|l(a_{i+1})| &= l(a_i)+a+b+l(a_{i+1}) = l(a_{i-1})+a+b\\ &\le l(a_5)+a+b = 2a-3b+a+b\\ &= 3a-2b<0.\end{aligned}$$

Therefore, $|l(a_i)| < l(a_i)+l(a_{n-1}) < |l(a_{i+1})|$, by property(2) $l(a_i)+l(a_{n-1})$ is not a label.

(iv) For $l(a_i)+l(a_n)$.

When $i=4$, by (2.1) we have $l(a_4)+l(a_n) = 2b-a-b = b-a = -l(a_3)$, which is not a label of C_n.

When $6 \le i \le n-5$, by (2.1) and property (1) we have $l(a_i)+l(a_n) = l(a_i)-b < |l(a_i)|$. In addition, by (2.1) and properties (1), (2) we have

$$\begin{aligned}l(a_i)+l(a_n)-|l(a_{i-1})| &= l(a_i)-b+l(a_{i-1}) = l(a_{i-2})-b \ge l(a_4)-b\\ &= 2b-a-b = b-a>0.\end{aligned}$$

Hence $|l(a_{i-1})| < |l(a_i)+l(a_n)| < |l(a_i)|$, by property(2) $l(a_i)+l(a_n)$ is not a label.

Case 2. When $l(a_i)<0$, by property (1) i is odd and $5 \le i \le n-4$.

(i) $l(a_i)+l(a_{n-3})$, here we only consider $5 \le i < n-4$, namely $5 \le i \le n-6$. By (2.1) and properties (1), (2) we have

$$l(a_i)+l(a_{n-3}) = l(a_i)+a-l(a_{n-4}) = |l(a_{n-4})|+l(a_i)+a < |l(a_{n-4})|.$$

On the other hand we have

$$\begin{aligned} l(a_i)+l(a_{n-3})-|l(a_{n-5})| &= l(a_i)+a-l(a_{n-4})-l(a_{n-5}) = l(a_i)+a-l(a_{n-6}) \\ &\geq l(a_{n-6})+a-l(a_{n-6}) = a > 0, \end{aligned}$$

then $|l(a_{n-5})| < l(a_i)+l(a_{n-3}) < |l(a_{n-4})|$, by property(2) $l(a_i)+l(a_{n-3})$ is not a label.

(ii) For $l(a_i)+l(a_{n-2})$, by property(1) we have $|l(a_i)+l(a_{n-2})| > |l(a_{n-2})|$, so $l(a_i)+l(a_{n-2})$ is not a label by property (2).

(iii) For $l(a_i)+l(a_{n-1})$.

When $i=5$, by (2.1) we obtain $|l(a_5)+l(a_{n-1})| = |2a-3b+a+b| = 2b-3a$, then we have

$$\begin{aligned} |l(a_{n-1})| < |l(a_5)+l(a_{n-1})| < |l(a_4)| &\quad \text{when } b > 4a. \\ |l(a_2)| < |l(a_5)+l(a_{n-1})| < |l(a_{n-1})| &\quad \text{when } 3a < b < 4a. \\ |l(a_3)| < |l(a_5)+l(a_{n-1})| < |l(a_2)| &\quad \text{when } 2a < b < 3a. \end{aligned}$$

For $b=4a$, by (2.1) we know $l(a_5)+l(a_{n-1}) = 2a-3b+a+b = 3a-2b = -5a = -l(a_{n-1})$, which is not a label, so $l(a_5)+l(a_{n-1})$ is not a label by property(2).

When $7 \leq i \leq n-4$, by properties (1), (2) we have $|l(a_i)+l(a_{n-1})| < |l(a_i)|$. In addition, by (2.1) and properties (1), (2) we have

$$\begin{aligned} |l(a_i)+l(a_{n-1})|-|l(a_{i-1})| &= -l(a_i)-(a+b)-l(a_{i-1}) = -l(a_{i-2})-(a+b) \\ &\geq -l(a_5)-(a+b) = 3b-2a-a-b = 2b-3a > 0. \end{aligned}$$

Then $|l(a_{i-1})| < |l(a_i)+l(a_{n-1})| < |l(a_i)|$, so $l(a_i)+l(a_{n-1})$ is not a label by property (2).

(iv) For $l(a_i)+l(a_n)$, by (2.1) we have $|l(a_i)+l(a_n)| = |l(a_i)-b| > |l(a_i)|$. In addition, by (2.1) and properties (1), (2) we have

$$\begin{aligned} |l(a_i)+l(a_n)|-|l(a_{i+1})| &= -l(a_i)+b-l(a_{i+1}) = -l(a_{i-1})+b \\ &\leq -l(a_4)+b = a-2b+b = a-b < 0. \end{aligned}$$

Then $|l(a_i)| < |l(a_i)+l(a_n)| < |l(a_{i+1})|$, so $l(a_i)+l(a_n)$ is not a label by property (2).

Therefore, Lemma 2.4 holds. ■

Up to now we obtain that the odd cycle C_n is an $\int\sum$-graph for $n \geq 11$, therefore Theorem 2.1 holds. ■

By Theorem 2.1, we obtain the following result.

Corollary 2.5. *If C_n is an odd cycle then $\xi(C_n) = 0$.*

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