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Global Existence of Solution for Semilinear Dissipative Wave Equation

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Abstract. In this paper, we consider an initial–boundary value problem for the semilinear dissipative wave equation in one space dimension of the type :

$$u_{tt} - u_{xx} + |u|^{m-1}u_t = V(t)|u|^{m-1}u + f(t,x) \quad \text{in} \quad (0,\infty)\times(a,b)$$

where initial data $u(0,x) = u_0(x) \in H_0^1(a,b)$, $u_t(0,x) = u_1(x) \in L^2(a,b)$ and boundary condition $u(t,a) = u(t,b) = 0$ for $t > 0$ with $m > 1$, on a bounded interval (a,b). The potential function $V(t)$ is smooth, positive and the source $f(t,x)$ is bounded. We investigate the global existence of solution as $t \to \infty$ under certain assumptions on the functions $V(t)$ and $f(t,x)$.

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1. Introduction and Results

In this paper, we consider an initial–boundary value problem for the semilinear dissipative wave equation in one space dimension

$$\begin{cases} u_{tt} - \Delta u + Q(u,u_t) = F(u) & \text{in } (0,\infty)\times(a,b), \\ u(0,x) = u_0(x), u_t(0,x) = u_1(x) & \text{for } x \in (a,b), \\ u(t,a) = u(t,b) = 0 & \text{for any } t > 0, \end{cases} \tag{1.1}$$

where the function $Q(u,u_t)=|u|^{m-1}u_t$ represents nonlinear damping and the function $F(u)=V(t)|u|^{m-1}u+f(t,x)$ represents source term with $m>1$, on a bounded interval (a,b). The potential function $V(t)$ is smooth, positive and $f(t,\cdot)$ is a source function, which is uniformly bounded as $t\to\infty$.

Georgiev–Todorova [3] treated the case when $Q(u,u_t)=|u_t|^{m-1}u_t$ and $F(u)=|u|^{p-1}u$, where $m>1$ and $p>1$. They proved that if $1<p\leqslant m$, a weak solution exists globally in time. On the other hand, they also proved that if $1<m<p$, the weak solution blows up in finite time for sufficiently negative initial energy

$$\mathcal{E}_1(0)=\|u_1\|^2_{L^2(\Omega)}+\|\nabla u_0\|^2_{L^2(\Omega)}-\frac{2}{p+1}\|u_0\|^{p+1}_{L^{p+1}(\Omega)}.$$

An extension of Georgiev–Todorova's blow up result was studied in Levine–Serrin [8], where, among other things, it was shown that if initial energy is negative, the solution is not global (blow up). Recently, the blow-up result of Georgiev–Todorova [3] has been improved also by Sheikh [11]. Ikehata [4] and Ikehata–Suzuki [5] considered the case when $Q(u,u_t)=u_t$ and $F(u)=|u|^{m-1}u$. They proved that the solution is global and local solution blows up in finite time by the concepts of stable and unstable sets due to Payne–Sattinger [10].

Lions–Strauss [9] considered the case when $Q(u,u_t)=k|u|^{m-1}u_t$ and $F(u)=f(t,x)$, where $m>1$ and k is a positive constant and proved that a solution exists globally in time. On the other hand, Katayama–Sheikh–Tarama [6] treated the Cauchy and mixed problems in one space dimensional case when $Q(u,u_t)=k|u|^{m-1}u_t$ and $F(u)=k_1|u|^{p-1}u$, where $m>1$, $p>1$ and k,k_1 are positive constants. They proved that if $1<p\leqslant m$, a weak solution exists globally in time for any initial data. They also proved that if $1<m<p$, the weak solution blows up in finite time for initial data with bounded support and negative initial energy

$$\mathcal{E}_2(0)=\|u_1\|^2_{L^2(a,b)}+\|u_{0,x}\|^2_{L^2(a,b)}-\frac{2k_1}{p+1}\|u_0\|^{p+1}_{L^{p+1}(a,b)}.$$

Here we remark that Levine–Serrin [8] considered some evolution equations with $Q(u,u_t)=|u|^{\kappa}|u_t|^m u_t$ and $F(u)=|u|^{p-1}u$ as an example. They proved that if $p>\kappa+m+1$, the solution is not global for negative initial energy (see also Levine–Pucci–Serrin [7]). Recently, Georgiev–Milani [2] treated the case when $Q(u,u_t)=|u_t|^{m-1}u_t$ and $F(u)=V(t)|u|^{m-1}u+f(t,x)$. They investigated the asymptotic behavior of solutions as time tends to infinity under suitable assumptions on the functions $V(t)$ and $f(t,x)$. With the exception of Katayama–Sheikh–Tarama [6], all of the above references were considered the problem on bounded domain $\Omega\in\mathbb{R}^n$ (i.e., Ω is a bounded domain $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$).

The main focus of our interest in this paper is to investigate the global existence of solution with different kind of nonlinear damping and nonlinear source terms $|u|^{m-1}u_t$ and $V(t)|u|^{m-1}u+f(t,x)$, respectively. However, until now there are very few results on this kind of nonlinear damping and nonlinear source terms.

Throughout this paper, the function spaces are the usual Lebesgue and Sobolev spaces. For convenience we use $\|\cdot\|_p$ instead of $\|\cdot\|_{L^p(a,b)}(1\leqslant p\leqslant\infty)$.

Also C will stand for various positive constants which may change line by line, even within the same inequality.

Now let us make the basic assumptions on the functions $V(t)$ and $f(t,x)$:

$$\sup_{t>0}|V(t)|<+\infty \quad \text{and} \quad \sup_{t>0}|V'(t)|<+\infty, \tag{1.2}$$

$$f(t,x)\in C^1\big([0,\infty);L^2(a,b)\big). \tag{1.3}$$

First of all, we have the following local existence of solution.

Theorem 1.1. *Let $m>1$ and for any initial data, $u_0(x)\in H_0^1(a,b)$ and $u_1(x)\in L^2(a,b)$. Suppose that the assumption* (1.2) *and* (1.3) *are satisfied. Then there exists some positive T such that the problem* (1.1) *admits a unique solution in the class*

$$u\in C([0,T);H_0^1(a,b))\cap C^1([0,T);L^2(a,b)).$$

Secondly, we state our global existence result of this paper.

Theorem 1.2. *Let $m>1$. Suppose that the assumption* (1.2) *and* (1.3) *are satisfied. Then there exists a unique global solution to the problem* (1.1) *in the class*

$$u\in C([0,\infty);H_0^1(a,b))\cap C^1([0,\infty);L^2(a,b)).$$

Using the idea of Katayama–Sheikh–Tarama [6], we can prove Theorems 1.1 and 1.2. The proofs of Theorem 1.1 and Theorem 1.2 will be given in Sec. 2 and Sec. 3, respectively.

Remark 1.3. The (local) existence of a solution relies heavily on the Sobolev embedding theorem $H_0^1(a,b)\hookrightarrow L^\infty(a,b)$. For this reason, we restrict our consideration to the problem in one dimensional space case.

2. Local Existence of a Solution

In this section, we shall prove the local existence Theorem 1.1. We define

$$\begin{aligned} X_T =& C\big([0,T];H_0^1(a,b)\big)\cap C^1\big([0,T];L^2(a,b)\big),\\ Y_T =& L^\infty\big(0,T;H_0^1(a,b)\big)\cap W^{1,\infty}\big(0,T;L^2(a,b)\big),\\ Y_{T,M} =& \Big\{u\in Y_T, f:\ \text{satisfy (1.3)};\\ & \sup_{0\leqslant t\leqslant T}\Big(\|u(t,\cdot)\|_{H_0^1}+\|u_t(t,\cdot)\|_{L^2}+\|f(t,\cdot)\|_{L^2}\Big)\leqslant M\Big\}, \end{aligned}$$

and $\quad X_{T,M}=Y_{T,M}\cap X_T$. $\quad$ Of course $X_T\subset Y_T$ and $X_{T,M}\subset Y_{T,M}$.

We set $G(u, u_t, f) = -|u|^{m-1}u_t + V(t)|u|^{m-1}u + f(t, \cdot)$. For any $v \in Y_T$, we define $\Phi[v] = u$, where $u \in X_T$ is a solution of

$$\begin{cases} u_{tt} - u_{xx} = G(v, v_t, f) & \text{in } (0,T) \times (a,b), \\ u(0,x) = u_0(x), u_t(0,x) = u_1(x) & \text{for } x \in (a,b), \\ u(t,a) = u(t,b) = 0 & \text{for any } t > 0. \end{cases} \tag{2.1}$$

Since we have $G(v, v_t, f) \in L^\infty(0,T; L^2(a,b))$ for any $v \in Y_T$ by the Sobolev embedding theorem, the existence and uniqueness of such solution $u \in X_T$ is guaranteed by the theory of mixed problem for linear wave equations.

Let $M = 4\big(\|u_0\|_{H_0^1} + \|u_1\|_{L^2}\big)$. We first claim that $v \in Y_{T,M}$ implies $\Phi[v] \in X_{T,M}$ for sufficiently small positive T.

Now multiplying the equation of (2.1) by $2u_t$ and integrating over (a, b), we have

$$\frac{d}{dt}\Big(\|u_t(t,\cdot)\|_2^2 + \|u_x(t,\cdot)\|_2^2\Big) \leqslant 2\|G(v, v_t, f)(t,\cdot)\|_2 \|u_t(t,\cdot)\|_2. \tag{2.2}$$

We define the energy identities for the equation of (2.1)

$$\begin{aligned} E(t) &= \|u_t(t,\cdot)\|_2^2 + \|u_x(t,\cdot)\|_2^2 \quad \text{and} \\ E_\varepsilon(t) &= \|u_t(t,\cdot)\|_2^2 + \|u_x(t,\cdot)\|_2^2 + \varepsilon \quad \text{for any } \varepsilon > 0. \end{aligned}$$

Then we have

$$\begin{aligned} \frac{d}{dt}E_\varepsilon^{\frac{1}{2}}(t) &= \frac{1}{2}E_\varepsilon^{-\frac{1}{2}}(t)E_\varepsilon^{'}(t) = \frac{1}{2}E_\varepsilon^{-\frac{1}{2}}(t)E^{'}(t) \\ &\leqslant \frac{1}{2}E_\varepsilon^{-\frac{1}{2}}(t) \cdot 2\|G(v, v_t, f)(t,\cdot)\|_2 \|u_t(t,\cdot)\|_2 \\ &\leqslant \|G(v, v_t, f)(t,\cdot)\|_2, \end{aligned} \tag{2.3}$$

here we have used the fact $E_\varepsilon(t) > \|u_t(t,\cdot)\|_2^2$ and inequality (2.2). Integrating (2.3) over $(0, T)$ and taking limit as $\varepsilon \downarrow 0$, we have

$$\begin{aligned} \sup_{0\leqslant t\leqslant T} \sqrt{\|u_t(t,\cdot)\|_2^2 + \|u_x(t,\cdot)\|_2^2} \leqslant &\sqrt{\|u_1\|_2^2 + \|u_{0,x}\|_2^2} \\ &+ \int_0^T \|G(v, v_t, f)(\tau,\cdot)\|_2 d\tau. \end{aligned} \tag{2.4}$$

From (2.4), we have

$$\begin{aligned} \sup_{0\leqslant t\leqslant T}\Big(\|u_t(t,\cdot)\|_2 + \|u_x(t,\cdot)\|_2\Big) \leqslant &\sqrt{2}\Big(\|u_1\|_2 + \|u_{0,x}\|_2\Big) \\ &+ \sqrt{2}\int_0^T \|G(v, v_t, f)(\tau,\cdot)\|_2 d\tau, \end{aligned} \tag{2.5}$$

here we have used the fact $\sqrt{|\zeta|^2 + |\eta|^2} \leqslant |\zeta| + |\eta| \leqslant \sqrt{2}\sqrt{|\zeta|^2 + |\eta|^2}$ for any $\zeta, \eta \in \mathbb{R}$.

By the Sobolev embedding inequality ($\|u\|_\infty \leqslant C\|u\|_{H_0^1}$), we have

$$
\begin{aligned}
\|G(v,v_t,f)(\tau,\cdot)\|_2 &= \|(-|v|^{m-1}v_t + V|v|^{m-1}v + f)(\tau,\cdot)\|_2 \\
&\leqslant C\Big(\||v(\tau,\cdot)|^{m-1}v_t(\tau,\cdot)\|_2 \\
&\quad + \sup_{0\leqslant\tau\leqslant T} V(\tau)\||v(\tau,\cdot)|^{m-1}v(\tau,\cdot)\|_2 + \|f(\tau,\cdot)\|_2\Big) \\
&\leqslant C\Big(\|v(\tau,\cdot)\|_\infty^{m-1}\|v_t(\tau,\cdot)\|_2 \\
&\quad + \|v(\tau,\cdot)\|_\infty^{m-1}\|v(\tau,\cdot)\|_2 + \|f(\tau,\cdot)\|_2\Big) \\
&\leqslant C\Big(\|v(\tau,\cdot)\|_{H_0^1}^{m-1}\|v_t(\tau,\cdot)\|_2 \\
&\quad + \|v(\tau,\cdot)\|_{H_0^1}^{m-1}\|v(\tau,\cdot)\|_2 + \|f(\tau,\cdot)\|_2\Big) \\
&\leqslant C\big(2M^m + M\big) \quad \text{for } 0\leqslant\tau\leqslant T.
\end{aligned} \tag{2.6}
$$

From (2.5) and (2.6), we have

$$
\begin{aligned}
\sup_{0\leqslant t\leqslant T}\|u_t(t,\cdot)\|_2 &\leqslant \sup_{0\leqslant t\leqslant T}\Big(\|u_t(t,\cdot)\|_2 + \|u_x(t,\cdot)\|_2\Big) \\
&\leqslant \sqrt{2}\Big(\frac{1}{4} + CT\big(2M^{m-1}+1\big)\Big)M.
\end{aligned} \tag{2.7}
$$

By the Schwarz inequality, we observe that

$$
\begin{aligned}
\frac{d}{dt}\|u(t,\cdot)\|_2 &= \frac{d}{dt}\Big(\int_a^b |u(t,\cdot)|^2 dx\Big)^{\frac{1}{2}} \\
&= \frac{1}{2}\Big(\int_a^b |u(t,\cdot)|^2 dx\Big)^{-\frac{1}{2}} \cdot 2\int_a^b u u_t dx \\
&\leqslant \frac{\|u(t,\cdot)\|_2\|u_t(t,\cdot)\|_2}{\|u(t,\cdot)\|_2} \leqslant \|u_t(t,\cdot)\|_2.
\end{aligned} \tag{2.8}
$$

From (2.8), we have

$$
\begin{aligned}
\|u(t,\cdot)\|_2 &\leqslant \|u(0,\cdot)\|_2 + \int_0^t \|u_t(\tau,\cdot)\|_2 d\tau \\
&\leqslant \frac{M}{4} + \sup_{0\leqslant\tau\leqslant T}\|u_t(\tau,\cdot)\|_2 \int_0^T d\tau \\
&\leqslant \frac{M}{4} + T\sup_{0\leqslant\tau\leqslant T}\|u_t(\tau,\cdot)\|_2.
\end{aligned} \tag{2.9}
$$

Therefore, the inequalities (2.7) and (2.9) imply

$$
\sup_{0\leqslant t\leqslant T}\|u(t,\cdot)\|_2 \leqslant \frac{M}{4} + T\sqrt{2}\Big(\frac{1}{4} + CT\big(2M^{m-1}+1\big)\Big)M. \tag{2.10}
$$

Finally from the inequalities (2.5) and (2.10), we arrive at

$$
\begin{aligned}
\sup_{0\leqslant t\leqslant T} &\Big(\|u(t,\cdot)\|_{H_0^1}+\|u_t(t,\cdot)\|_2\Big)\\
&\leqslant \sup_{0\leqslant t\leqslant T}\Big(\|u(t,\cdot)\|_2+\|u_t(t,\cdot)\|_2+\|u_x(t,\cdot)\|_2\Big)\\
&\leqslant \frac{M}{4}+T\sqrt{2}\Big(\frac{1}{4}+CT\big(2M^{m-1}+1\big)\Big)M\\
&\quad+\sqrt{2}\Big(\frac{1}{4}+CT\big(2M^{m-1}+1\big)\Big)M\\
&\leqslant C_{T,M}M,
\end{aligned}
\tag{2.11}
$$

where

$$
C_{T,M}=\frac{1}{4}+\sqrt{2}(T+1)\Big(\frac{1}{4}+CT\big(2M^{m-1}+1\big)\Big).
$$

Thus we can find $T_1(M)>0$ such that $C_{T,M}\leqslant 1$ for any $T\in(0,T_1]$. This implies that $u=\Phi[v]\in X_{T,M}$ and we complete the proof of the first claim. In the following, we always assume $T\in(0,T_1)$.

Next we claim that Φ is a contraction mapping in $X_{T,M}$ for small T by using the energy inequality and the mean value theorem. In the case $2\leqslant m<\infty$, we can apply the mean value theorem directly to the nonlinear damping term $|u|^{m-1}u_t$. But the function $|u|^{m-1}u_t$ is not Lipschitz continuous with respect to $(u,u_t)\in\mathbb{R}\times\mathbb{R}$ for $1<m<2$. For this reason, we modify the arguments by using the fact that $m|u|^{m-1}u_t=\frac{\partial}{\partial t}(|u|^{m-1}u)$, where $|u|^{m-1}u$ is Lipschitz continuous for $1<m<\infty$.

Suppose that $v_1,v_2\in Y_{T,M}$, then we have $\Phi[v_1],\Phi[v_2]\in X_{T,M}$. Let w_i and $\widetilde{w_i}$ $(i=1,2)$ be solutions to the following problems

$$
\begin{cases}
(w_i)_{tt}-(w_i)_{xx}=F(v_i,f) & \text{in } (0,T)\times(a,b),\\
w_i(0,x)=u_0(x),(w_i)_t(0,x)=u_1(x)+\frac{1}{m}|u_0|^{m-1}u_0 & \text{for } x\in(a,b),\\
w_i(t,a)=w_i(t,b)=0 & \text{for any } t>0,
\end{cases}
\tag{2.12}
$$

where $F(v_i,f)=V(t)|v_i|^{m-1}v_i+f(t,x)$ and

$$
\begin{cases}
(\widetilde{w}_i)_{tt}-(\widetilde{w}_i)_{xx}=-\frac{1}{m}|v_i|^{m-1}v_i & \text{in } (0,T)\times(a,b),\\
\widetilde{w}_i(0,x)=(\widetilde{w}_i)_t(0,x)=0 & \text{for } x\in(a,b),\\
\widetilde{w}_i(t,a)=(\widetilde{w}_i)(t,b)=0 & \text{for any } t>0,
\end{cases}
\tag{2.13}
$$

for $i=1,2$, respectively. Since $v_i\in Y_{T,M}$ implies that $F(v_i,f)$, $|v_i|^{m-1}v_i$ and $\frac{\partial}{\partial t}(|v_i|^{m-1}v_i)=m|v_i|^{m-1}(v_i)_t\in L^\infty\big(0,T;L^2(a,b)\big)$ by the Sobolev embedding theorem, we have $w_i\in X_T$ and $\widetilde{w}_i\in C\big([0,T];H^2\big)\cap C^1\big([0,T];H_0^1\big)\cap C^2\big([0,T];L^2\big)$ $(i=1,2)$. From the uniqueness of solution to the linear wave equations, we have

$$
\Phi[v_i]=w_i+(\widetilde{w}_i)_t\quad(i=1,2).
\tag{2.14}
$$

Since $|v|^{m-1}v$ with $m>1$ is a C^1 function, the mean value theorem implies

$$\big||v_1|^{m-1}v_1 - |v_2|^{m-1}v_2\big| \leqslant C\big(|v_1|^{m-1} + |v_2|^{m-1}\big)|v_1 - v_2|. \tag{2.15}$$

By (2.12), (2.15), the energy inequality and the Sobolev embedding inequality imply

$$\begin{aligned}
\|(w_1 - w_2)_t(t,\cdot)\|_2 &+ \|(w_1 - w_2)_x(t,\cdot)\|_2 \\
&\leqslant \int_0^t \|V(\tau)\big(|v_1|^{m-1}v_1 - |v_2|^{m-1}v_2\big)(\tau,\cdot)\|_2 d\tau \\
&\leqslant \sup_{0\leqslant\tau\leqslant T} V(\tau) \int_0^t \|\big(|v_1|^{m-1} + |v_2|^{m-1}\big)(v_1 - v_2)(\tau,\cdot)\|_2 d\tau \\
&\leqslant C \int_0^t \|\big(|v_1|^{m-1} + |v_2|^{m-1}\big)(\tau,\cdot)\|_\infty \|(v_1 - v_2)(\tau,\cdot)\|_2 d\tau \\
&\leqslant C \int_0^t \|\big(|v_1|^{m-1} + |v_2|^{m-1}\big)(\tau,\cdot)\|_{H_0^1} \|(v_1 - v_2)(\tau,\cdot)\|_2 d\tau \\
&\leqslant C \int_0^t \Big(\|v_1\|_{H_0^1}^{m-1} + \|v_2\|_{H_0^1}^{m-1}\Big) \|(v_1 - v_2)(\tau,\cdot)\|_2 d\tau \\
&\leqslant CTM^{m-1} \sup_{0\leqslant\tau\leqslant T} \|(v_1 - v_2)(\tau,\cdot)\|_2 \quad \text{for } 0 \leqslant t \leqslant T,
\end{aligned} \tag{2.16}$$

and in a similar manner we get from (2.13), (2.15), the energy inequality and the Sobolev embedding inequality

$$\begin{aligned}
\|(\widetilde{w}_1 - \widetilde{w}_2)_t(t,\cdot)\|_2 &+ \|(\widetilde{w}_1 - \widetilde{w}_2)_x(t,\cdot)\|_2 \\
&\leqslant CTM^{m-1} \sup_{0\leqslant\tau\leqslant T} \|(v_1 - v_2)(\tau,\cdot)\|_2 \quad \text{for } 0 \leqslant t \leqslant T.
\end{aligned} \tag{2.17}$$

We have also

$$\|(w_1 - w_2)(t,\cdot)\|_2 \leqslant T \sup_{0\leqslant\tau\leqslant T} \|(w_1 - w_2)_t(\tau,\cdot)\|_2 \quad \text{for } 0 \leqslant t \leqslant T. \tag{2.18}$$

Therefore, the inequalities (2.16), (2.17) and (2.18) lead to

$$
\begin{aligned}
\sup_{0\leqslant t\leqslant T} &\|\big(\Phi[v_1]- \Phi[v_2]\big)(t,\cdot)\|_2 \\
&= \sup_{0\leqslant t\leqslant T} \big\|\big((w_1+(\widetilde{w}_1)_t)(t,\cdot)-\big(w_2+(\widetilde{w}_2)_t\big)(t,\cdot)\big\|_2 \\
&\leqslant \sup_{0\leqslant t\leqslant T} \|(w_1-w_2)(t,\cdot)\|_2 + \sup_{0\leqslant t\leqslant T} \|(\widetilde{w}_1-\widetilde{w}_2)_t(t,\cdot)\|_2 \\
&\leqslant T \sup_{0\leqslant t\leqslant T} \|(w_1-w_2)_t(t,\cdot)\|_2 + \sup_{0\leqslant t\leqslant T} \|(\widetilde{w}_1-\widetilde{w}_2)_t(t,\cdot)\|_2 \\
&\leqslant CT^2M^{m-1} \sup_{0\leqslant t\leqslant T} \|(v_1-v_2)(t,\cdot)\|_2 \\
&\qquad + CTM^{m-1} \sup_{0\leqslant t\leqslant T} \|(v_1-v_2)(t,\cdot)\|_2 \\
&\leqslant CTM^{m-1}\big(T+1\big) \sup_{0\leqslant t\leqslant T} \|(v_1-v_2)(t,\cdot)\|_2
\end{aligned} \tag{2.19}
$$

In the following, we fix $T\in(0,T_1]$ which is small enough to satisfy $CTM^{m-1}(T+1)<1/2$. Then we have

$$
\sup_{0\leqslant t\leqslant T} \|\big(\Phi[v_1]-\Phi[v_2]\big)(t,\cdot)\|_2 \leqslant \frac{1}{2} \sup_{0\leqslant t\leqslant T} \|(v_1-v_2)(t,\cdot)\|_2 \tag{2.20}
$$

for such T.

Finally, we define

$$
\begin{cases} u^{(0)}(t,x)=u_0(x), \\ u^{(n)}(t,x)=\Phi[u^{(n-1)}] \qquad (n=1,2,3,\cdots). \end{cases}
$$

By the inequality (2.20), there exists some $u\in C([0,T];L^2)$ such that $u^{(n)}\to u$ in $C([0,T];L^2)$ as $n\to\infty$.

Now, we will show that this solution u belongs to X_T and this u is a solution to (1.1). Since $u^{(n)}\in X_{T,M}$, $\{u^{(n)}\}$ (resp. $\{u_t^{(n)}\}$) has a weak-$*$ convergent subsequence in $L^\infty(0,T;H_0^1)$ (resp. in $L^\infty(0,T;L^2)$) and $u^{(n)}\to u$ in $C([0,T],;L^2)$, the above subsequence of $\{u^{(n)}\}$ (resp. of $\{u_t^{(n)}\}$) converges weakly-$*$ to u (resp. to u_t) in $L^\infty(0,T;H_0^1)$ (resp. in $L^\infty(0,T;L^2)$), and consequently we see that $u\in L^\infty(0,T;H_0^1)$ and $u_t\in L^\infty(0,T;L^2)$. Therefore we can see that $u\in Y_{T,M}$ and then we have $\Phi[u]\in X_{T,M}$. Hence we can apply (2.20) to have

$$
\sup_{0\leqslant t\leqslant T} \big\|\big(\Phi[u]-\Phi[u^{(n)}]\big)(t,\cdot)\|_2 \leqslant \frac{1}{2} \sup_{0\leqslant t\leqslant T} \|\big(u-u^{(n)}\big)(t,\cdot)\big\|_2. \tag{2.21}
$$

Since the right-hand side of (2.21) tends to 0 as $n\to\infty$, we get $\Phi[u^{(n)}]\to\Phi[u]$ in $C([0,T];L^2)$. Since we have proved $u^{(n)}\to u$ in $C([0,T];L^2)$, passing to the limit in $u^{(n+1)}=\Phi[u^{(n)}]$, we obtain $u=\Phi[u]\in X_{T,M}$. This u is apparently the desired solution. What is left to prove is the uniqueness of solutions in $X_{T,M}$, it follows that from (2.20)

$$
\|u-v\|_{X_{T,M}} \leqslant \frac{1}{2}\|u-v\|_{X_{T,M}} \tag{2.22}
$$

with $u_1 = u$ and $u_2 = v$. Then we have $\|u - v\|_{X_{T,M}} \leqslant 0$. Hence we see that $u = v \in X_{T,M}$. This completes the proof of Theorem 1.1. ■

3. Existence of a Global Solution

In this section, we will prove the global existence Theorem 1.2. Before proving the global existence result, first we introduce well-known Gronwall lemma due to Alain Haraux [1].

Lemma 3.1. *Let T be positive, $\alpha(t) \in L^1(0,T)$ with $\alpha(t) > 0$ and $f(t) \in L^1(0,T)$ with f being a nonnegative function almost everywhere on $(0,T)$. Assume that $w(t) \in W^{1,1}(0,T)$ satisfies $w(t) \geq 0$ on $[0,T]$ and*

$$\frac{d}{dt}w(t) \leqslant \alpha(t)w(t) + f(t) \qquad a.e. \text{ on } (0,T). \tag{3.1}$$

Then we have

$$w(t) \leqslant \exp\Big(\int_0^t \alpha(s)ds\Big)w(0) + \int_0^t \exp\Big(\int_s^t \alpha(\tau)d\tau\Big)f(s)ds \tag{3.2}$$

for any $t \in [0,T]$.

Now we are in a position to prove the global existence Theorem 1.2. Let $u(t,x)$ be a solution to the problem (1.1) in the class

$$C\big([0,T]; H_0^1(a,b)\big) \cap C^1\big([0,T]; L^2(a,b)\big).$$

We define the energy identity for the equation to the problem (1.1)

$$\mathcal{E}(t) = M + \frac{1}{2}\Big(\|u_t\|_2^2 + \|u_x\|_2^2\Big) + \frac{1}{m+1}\int_a^b V(t)|u|^{m+1}dx. \tag{3.3}$$

Multiplying the equation to the problem (1.1) by u_t and integrating over (a,b), we have

$$\begin{aligned}
&\frac{d}{dt}\Big\{\frac{1}{2}\big(\|u_t\|_2^2 + \|u_x\|_2^2\big) + \frac{1}{m+1}\int_a^b V(t)|u|^{m+1}dx\Big\} \\
&\qquad = -\int_a^b |u|^{m-1}|u_t|^2dx + 2\int_a^b V(t)|u|^{m-1}uu_tdx \qquad (3.4)\\
&\qquad\quad + \frac{1}{m+1}V'(t)\int_a^b |u|^{m+1}dx + \int_a^b fu_tdx,
\end{aligned}$$

or

$$
\begin{aligned}
\mathcal{E}'(t) \leqslant & -\int_a^b |u|^{m-1}|u_t|^2 dx + 2\sup_{t>0}|V(t)| \int_a^b |u|^{m-1} u u_t dx \\
& + \frac{1}{m+1}\sup_{t>0}|V'(t)| \|u\|_{m+1}^{m+1} + \int_a^b f u_t dx \\
\leqslant & -\int_a^b |u|^{m-1}|u_t|^2 dx + C\int_a^b |u|^{m-1} u u_t dx \\
& + C\|u\|_{m+1}^{m+1} + \frac{1}{2}\|f\|_2^2 + \frac{1}{2}\|u_t\|_2^2 \\
\leqslant & -\int_a^b |u|^{m-1}|u_t|^2 dx + C\int_a^b |u|^{m-1} u u_t dx \\
& + C\|u\|_{m+1}^{m+1} + \frac{1}{2}\Big(\sup_{t>0}\|f\|_2\Big)^2 + \frac{1}{2}\|u_t\|_2^2 \\
\leqslant & -\int_a^b |u|^{m-1}|u_t|^2 dx + C\int_a^b |u|^{m-1} u u_t dx \\
& + C\|u\|_{m+1}^{m+1} + CM + \frac{1}{2}\|u_t\|_2^2,
\end{aligned}
\tag{3.5}
$$

here we have used the Young inequality. Since $2|\zeta\eta| \leqslant \zeta^2 + \eta^2$ for $\zeta, \eta \in \mathbb{R}$, we have

$$
\big||u|^{m-1} u u_t\big| \leqslant \varepsilon |u|^{m-1}|u_t|^2 + \frac{1}{4\varepsilon}|u|^{m+1} \tag{3.6}
$$

for any positive ε. From (3.5)and (3.6), we have

$$
\begin{aligned}
\mathcal{E}'(t) \leqslant & -\int_a^b |u|^{m-1}|u_t|^2 dx + C\varepsilon \int_a^b |u|^{m-1}|u_t|^2 dx \\
& + \frac{C}{4\varepsilon}\int_a^b |u|^{m+1} dx + C\|u\|_{m+1}^{m+1} + CM + \frac{1}{2}\|u_t\|_2^2.
\end{aligned}
\tag{3.7}
$$

Therefore, if we choose sufficiently small ε, (3.7) leads to

$$
\mathcal{E}'(t) \leqslant C\mathcal{E}(t). \tag{3.8}
$$

Now we apply the Gronwall Lemma 3.1 with $C = \alpha$ and $f = 0$, we arrive at

$$
\mathcal{E}(t) \leqslant \exp(Ct)\mathcal{E}(0) \quad \text{for any } t > 0. \tag{3.9}
$$

The local existence Theorem 1.1 and usual continuation arguments will give the global existence theorem. This completes the proof of Theorem 1.2. ■

References

1. Alain Haraux, *Semi-linear Hyperbolic Problem in Bounded Domains*, Mathematical Reports, Vol. 3 , Part. 1, Harwood academic publishers, 1987.
2. V. Georgiev and A. Milani, On the asymptotic behavior of semilinear wave equations with degenerate dissipation and source terms, *Nonlinear Diff. Eqs. and Appl.* **5** (1998) 53–68.
3. V. Georgiev and G. Todorova, Existence of a solution of the wave equation with nonlinear damping and source terms, *J. Diff. Eqns.* **109** (1994) 295–308.
4. R. Ikehata, Some remarks on the wave equation with nonlinear damping and source terms, *Nonlinear Anal., Meth. and Appl.* **27** (1996) 1165–1175.
5. R. Ikehata and T. Suzuki, Stable and unstable sets for evalution equations of parabolic and hyperbolic type, *Hiroshima Math. J.* **26** (1996) 475–491.
6. S. Katayama, M. A. N. Sheikh, and S. Tarama, The Cauchy and mixed problems for semilinear wave equations with nonlinear damping terms, *Math. Japonica* **51** (2000) 456-466.
7. H. A. Levine, P. Pucci, and J. Serrin, Some remarks on global nonexistence for nonautonomous abstract evolution equations, *Contem. Math.* **208** (1997) 253–263.
8. H. A. Levine and J. Serrin, Global nonexistence theorem for quasilinear evalution equations with dissipation, *Arch. Rational Mech. Anal.* **137** (1997) 341–361.
9. J. L. Lions and W. A. Strauss, Some nonlinear evolution equations, *Bull. Soc. Math. France* **93** (1965) 43–96.
10. L. E. Payne and D. H. Sattinger, Saddle points and instability of nonlinear hyperbolic equations, *Israel J. Math.* **49** (1975) 43–303.
11. M. A. N. Sheikh, Remarks on the semilinear wave equations, *Vietnam J. Math.* **28** (2000) 17–24.