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Polar Coordinates on H-type Groups and Applications*

Junqiang Han and Pengcheng Niu

Department of Applied Math., Northwestern Polytechnical University
Xi'an, Shaanxi, 710072, China

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Abstract. In this paper we construct polar coordinates on H-type groups. As applications, we explicitly compute the volume of the ball in the sense of the distance and the constant in the fundamental solution of p-sub-Laplacian on the H-type group. Also, we prove some nonexistence results of weak solutions for a degenerate elliptic inequality on the H-type group.

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1. Introduction

The polar coordinates for the Heisenberg group $\mathbf{H}^1$ and $\mathbf{H}^n$ were defined by Greiner [8] and D'Ambrosio [3], respectively. Using their introduction as in [3] we can explicitly compute the volume of the Heisenberg ball (see [6]) and the constant in the fundamental solution of $\triangle_{\mathbf{H}^n}$ (see [4, 5]). In this paper we will construct polar coordinates on H-type groups. In [1], the polar coordinates were given in Carnot groups and groups of H-type, but the expression here is slightly different. As an application, we will explicitly calculate the volume of the ball in the sense of the distance and the constant in the fundamental solution of

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p-sub-Laplacian on the H-type groups.

Nonexistence results of weak solutions for some degenerate and singular elliptic, parabolic and hyperbolic inequalities on the Euclidean space $\mathbb{R}^n$ have been largely considered, see [13, 14] and their references. The singular sub-Laplace inequality and related evolution inequalities on the Heisenberg group $\mathbf{H}^n$ were studied in [3, 6]. In this paper we will discuss the nonexistence of weak solutions for some degenerate elliptic inequality on the H-type groups.

We recall some known facts about the H-type group.

H-type groups form an interesting class of Carnot groups of step two in connection with hypoellipticity questions. Such groups, which were introduced by Kaplan [9] in 1980, constitute a direct generalization of Heisenberg groups and are more complicated. There has been subsequently a considerable amount of work in the study of such groups.

Let $\mathbf{G}$ be a Carnot group of step two whose Lie algebra $\mathbf{g} = V_1 \oplus V_2$. Suppose that a scalar product $< \cdot, \cdot >$ is given on $\mathbf{g}$ for which V_1, V_2 are orthogonal. With $m = dimV_1, k = dimV_2$, let $X = \{X_1, \dots, X_m\}$ and $Y = \{Y_1, \dots, Y_k\}$ be a basis of V_1 and V_2, respectively. Assume that ξ_1 and ξ_2 are the projections of $\xi \in \mathbf{g}$ in V_1 and V_2, respectively. The coordinate of ξ_1 in the basis $\{X_1, \dots, X_m\}$ is denoted by $x = (x_1, \dots, x_m) \in \mathbb{R}^m$; the coordinate of ξ_2 in the basis $\{Y_1, \dots, Y_k\}$ is denoted by $y = (y_1, \dots, y_k) \in \mathbb{R}^k$.

Define a linear map $J : V_2 \to End(V_1)$:

$$< J(\xi_2)\xi_1', \xi_1'' >=< \xi_2, [\xi_1', \xi_1''] >, \quad \xi_1', \xi_1'' \in V_1, \xi_2 \in V_2.$$

A Carnot group of step two, $\mathbf{G}$, is said of H-type if for every $\xi_2 \in V_2$, with $|\xi_2| = 1$, the map $J(\xi_2) : V_1 \to V_1$ is orthogonal (see [9]).

As stated in [7], it has

$$X_j = \frac{\partial}{\partial x_j} + \frac{1}{2} \sum_{i=1}^{k} \langle [\xi, X_j], Y_i \rangle \frac{\partial}{\partial y_i}, \quad j = 1, \dots, m. \tag{1}$$

For a function u on $\mathbf{G}$, we denote the horizontal gradient by $Xu = (X_1 u, \dots, X_m u)$ and let $|Xu| = \left(\sum_{j=1}^m |X_j u|^2 \right)^{\frac{1}{2}}$. The sub-Laplacian on the group of H-type $\mathbf{G}$ is given by

$$\triangle_{\mathbf{G}} = - \sum_{j=1}^{m} X_j^2. \tag{2}$$

and the p-sub-Laplacian on $\mathbf{G}$ is

$$\Delta_{\mathbf{G},p} u = - \sum_{j=1}^{m} X_j \left(|Xu|^{p-2} X_j u \right) \tag{3}$$

for a function u on $\mathbf{G}$.

A family of non-isotropic dilations on $\mathbf{G}$ is

$$\delta_\lambda(x, y) = (\lambda x, \lambda^2 y), \qquad \lambda > 0, (x, y) \in \mathbf{G}. \tag{4}$$

The homogeneous dimension of $\mathbf{G}$ is $Q = m + 2k$.

Let

$$d(x,y) = (|x|^4 + 16|y|^2)^{\frac{1}{4}}. \tag{5}$$

Then d is a homogeneous norm on $\mathbf{G}$. The open ball of radius R and centered at $(0,0) \in \mathbf{G}$ is denoted by

$$B_R = \{(x,y) \in \mathbf{G} | d(x,y) < R\}.$$

Let $\psi = \frac{|x|^2}{d^2}$, a direct computation shows

$$|Xd|^2 = \psi. \tag{6}$$

As in [3], we need the following concepts. A function $u : \Omega \subset \mathbf{G} \to \mathbb{R}$ is said to be cylindrical, if $u(x,y) = u(|x|,|y|)$, and in particular, u is said to be radial, if $u(x,y) = u(d(x,y))$, that is u depends only on d.

Let $u \in C^2(\Omega)$. If u is radial, then it is easy to check that

$$|Xu|^2 = \psi|u'|^2 \tag{7}$$

and

$$\triangle_{\mathbf{G}} u = \psi \left(u'' + \frac{Q-1}{d} u' \right). \tag{8}$$

The following definitions are extensions of those introduced in [6].

Definition 1.1. *For $R > 0$ and $1 < p < \infty$ we define the volume of the ball B_R as*

$$|B_R|_p = \int_{B_R} |Xd|^p, \tag{9}$$

and the area of spherical surface ∂B_R as

$$|\partial B_R|_p = \frac{d}{dR} |B_R|_p. \tag{10}$$

We refer the following proposition to [2].

Proposition 1.1. *Let $1 < p < Q$ and*

$$C_{p,Q}^{-1} = \left(\frac{Q-p}{p-1} \right)^{p-1} (Q + 3p - 4) \int_{\mathbf{G}} \frac{|x|^p d^{2(p-2)}}{(1+d^4)^{\frac{3p+Q}{4}}}. \tag{11}$$

The function

$$\Gamma_p = C_{p,Q} d^{\frac{p-Q}{p-1}} \tag{12}$$

is a fundamental solution of (3) with singularity of the identity element $(0,0) \in \mathbf{G}$.

Here the integral in (11) is convergent, but it is not computed explicitly.

We will give a description of polar coordinates on the H-type group $\mathbf{G}$, and then compute explicitly $|B_R|_p, |\partial B_R|_p$ and $C_{p,Q}$ in Sec. 2. In Sec. 3, we study some degenerate elliptic inequality on the H-type group. The main technique will be the so called test functions method introduced in [10, 11] and developed in [12]. Roughly speaking, this approach is based on the derivation of suitable a priori bounds of the weak solutions by carefully choosing special test functions and scaling argument.

In the sequel we shall use a function $\varphi_0 \in C_0^2(\mathbb{R})$ meeting the property

$$0 \le \varphi_0 \le 1 \quad \text{and} \quad \varphi_0(\eta) = \begin{cases} 1, & \text{if } |\eta| \le 1, \\ 0, & \text{if } |\eta| \ge 2. \end{cases} \tag{13}$$

The quantities

$$\int_{\mathbb{R}} \frac{|\varphi_0''(\eta)|^q}{\varphi_0(\eta)^{q-1}} d\eta \quad \text{or} \quad \int_{\mathbb{R}} \frac{|\varphi_0'(\eta)|^q}{\varphi_0(\eta)^{q-1}} d\eta$$

where $q > 1$, are said to be finite, if there exists a suitable φ_0 with the property (13) such that the integrals are finite. Such a function φ_0 satisfying above hypotheses is called an admissible function.

For $q > 1$, $q' = \frac{q}{q-1}$ is the Hölder exponent relative to q.

2. Polar Coordinates and Applications

Assume $\Omega = B_{R_2} \backslash \overline{B}_{R_1}$, with $0 \le R_1 < R_2 \le +\infty$, $u \in L^1(\Omega)$ is a cylindrical function. To compute $\int_\Omega u$, we consider the change of the variables $(x_1, \ldots, x_m, y_1, \ldots, y_k)$
$\to (\rho, \theta, \theta_1, \ldots, \theta_{m-1}, \gamma_1, \ldots, \gamma_{k-1})$ defined by

$$\begin{cases} x_1 = \rho(\sin\theta)^{\frac{1}{2}} \cos\theta_1; \quad x_2 = \rho(\sin\theta)^{\frac{1}{2}} \sin\theta_1 \cos\theta_2; \\ x_3 = \rho(\sin\theta)^{\frac{1}{2}} \sin\theta_1 \sin\theta_2 \cos\theta_3; \\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots \\ x_{m-1} = \rho(\sin\theta)^{\frac{1}{2}} \sin\theta_1 \sin\theta_2 \ldots \sin\theta_{m-2} \cos\theta_{m-1}; \\ x_m = \rho(\sin\theta)^{\frac{1}{2}} \sin\theta_1 \sin\theta_2 \ldots \sin\theta_{m-2} \sin\theta_{m-1}; \\ y_1 = \frac{1}{4}\rho^2 \cos\theta \cos\gamma_1; \quad y_2 = \frac{1}{4}\rho^2 \cos\theta \sin\gamma_1 \cos\gamma_2; \\ y_3 = \frac{1}{4}\rho^2 \cos\theta \sin\gamma_1 \sin\gamma_2 \cos\gamma_3; \\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots \\ y_{k-1} = \frac{1}{4}\rho^2 \cos\theta \sin\gamma_1 \sin\gamma_2 \ldots \sin\gamma_{k-2} \cos\gamma_{k-1}; \\ y_k = \frac{1}{4}\rho^2 \cos\theta \sin\gamma_1 \sin\gamma_2 \ldots \sin\gamma_{k-2} \sin\gamma_{k-1} \end{cases} \tag{14}$$

where $R_1 < \rho < R_2, \theta \in (0, \pi), \theta_1, \ldots, \theta_{m-2}, \gamma_1, \ldots, \gamma_{k-2} \in (0, \pi)$ and $\theta_{m-1}, \gamma_{k-1} \in (0, 2\pi)$. One easily sees that

$$r = |x| = \rho(\sin\theta)^{\frac{1}{2}}, \qquad s = |y| = \frac{1}{4}\rho^2 |\cos\theta|. \tag{15}$$

Using the ordinary spherical coordinates in $\mathbb{R}^m$ and $\mathbb{R}^k$ leads to

$$dx = r^{m-1} dr d\omega_m, \qquad dy = s^{k-1} ds d\omega_k, \tag{16}$$

where $d\omega_m$ and $d\omega_k$ denote the Lebesgue measures on S^{m-1} in $\mathbb{R}^m$ and S^{k-1} in $\mathbb{R}^k$, respectively. From (14) and (15), we have

$$dr\,ds = \frac{1}{4}\rho^2(\sin\theta)^{-\frac{1}{2}}d\rho\,d\theta, \tag{17}$$

and then

$$dx\,dy = \frac{1}{4^k}\rho^{Q-1}(\sin\theta)^{\frac{m-2}{2}}|\cos\theta|^{k-1}d\rho\,d\theta\,d\omega_m\,d\omega_k.$$

Therefore the following formula holds

$$\int_\Omega u(r,s) = \omega_m\omega_k\int_0^\pi d\theta\int_{R_1}^{R_2}\frac{1}{4^k}\rho^{Q-1}(\sin\theta)^{\frac{m-2}{2}}|\cos\theta|^{k-1}u\Big(\rho(\sin\theta)^{\frac{1}{2}},\\ \frac{1}{4}\rho^2\cos\theta\Big)d\rho,$$

where

$$\omega_m = \int_0^\pi d\theta_1\int_0^\pi d\theta_2\ldots\int_0^\pi d\theta_{m-2}\int_0^{2\pi} d\theta_{m-1}\\ \sin^{m-2}\theta_1\ldots\sin^2\theta_{m-3}\sin\theta_{m-2},$$

$$\omega_k = \int_0^\pi d\gamma_1\int_0^\pi d\gamma_2\ldots\int_0^\pi d\gamma_{k-2}\int_0^{2\pi} d\gamma_{k-1}\\ \sin^{k-2}\gamma_1\ldots\sin^2\gamma_{k-3}\sin\gamma_{k-2}$$

are the Lebesgue measures of the unitary Euclidean spheres in $\mathbb{R}^m$ and $\mathbb{R}^k$, respectively.

Furthermore, if u is of the form $u(x,y) = \psi v(d)$, then

$$\begin{aligned}\int_\Omega \psi v(d) &= \omega_m\omega_k\int_0^\pi d\theta\int_{R_1}^{R_2}\frac{1}{4^k}\rho^{Q-1}(\sin\theta)^{\frac{m-2}{2}}|\cos\theta|^{k-1}\frac{\rho^2\sin\theta}{\rho^2}v(\rho)d\rho\\ &= s_{m,k}\int_{R_1}^{R_2}\rho^{Q-1}v(\rho)d\rho,\end{aligned} \tag{19}$$

where $s_{m,k} = \frac{1}{4^k}\omega_m\omega_k\int_0^\pi(\sin\theta)^{\frac{m}{2}}|\cos\theta|^{k-1}d\theta$.

Theorem 2.1. *We have the following formulae:*

$$(1)\quad |B_R|_p = \frac{R^Q}{4^{k-1}Q}\frac{\pi^{\frac{m+k}{2}}}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{k}{2}\right)}\,\mathrm{B}\Big(\frac{k}{2},\frac{p+m}{4}\Big); \tag{20}$$

$$(2)\quad |\partial B_R|_p = \frac{R^{Q-1}}{4^{k-1}}\frac{\pi^{\frac{m+k}{2}}}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{k}{2}\right)}\,\mathrm{B}\Big(\frac{k}{2},\frac{p+m}{4}\Big). \tag{21}$$

Proof. (1) By (9) and (14),

$$|B_R|_p = \int_{B_R}|Xd|^p = \int_{B_R}\frac{|x|^p}{d^p}$$

$$= \int\limits_{B_R} \frac{[\rho(\sin\theta)^{\frac{1}{2}}]^p}{\rho^p} \cdot \frac{1}{4^k}\rho^{Q-1}(\sin\theta)^{\frac{m-2}{2}}|\cos\theta|^{k-1} d\rho d\theta d\omega_m d\omega_k$$

$$= \omega_m\omega_k \cdot \frac{1}{4^k}\frac{1}{Q}R^Q \int\limits_0^{\pi} (\sin\theta)^{\frac{p+m-2}{2}}|\cos\theta|^{k-1}d\theta$$

$$= \frac{2\pi^{\frac{m}{2}}}{\Gamma\left(\frac{m}{2}\right)} \cdot \frac{2\pi^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right)} \cdot \frac{R^Q}{4^k Q} \cdot$$

$$\Big[\int_0^{\frac{\pi}{2}} (\sin\theta)^{\frac{p+m-2}{2}}(\cos\theta)^{k-1}d\theta + \int_0^{\frac{\pi}{2}} (\cos\theta)^{\frac{p+m-2}{2}}(\sin\theta)^{k-1}d\theta\Big]$$

$$= \frac{R^Q}{4^{k-1}Q}\frac{\pi^{\frac{m+k}{2}}}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{k}{2}\right)}\Big[\frac{\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{p+m}{4}\right)}{2\Gamma\left(\frac{k}{2}+\frac{p+m}{4}\right)} + \frac{\Gamma\left(\frac{p+m}{4}\right)\Gamma\left(\frac{k}{2}\right)}{2\Gamma\left(\frac{k}{2}+\frac{p+m}{4}\right)}\Big]$$

$$= \frac{R^Q}{4^{k-1}Q}\frac{\pi^{\frac{m+k}{2}}}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{k}{2}\right)}\frac{\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{p+m}{4}\right)}{\Gamma\left(\frac{k}{2}+\frac{p+m}{4}\right)} = \frac{R^Q}{4^{k-1}Q}\frac{\pi^{\frac{m+k}{2}}}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{k}{2}\right)}\mathrm{B}\Big(\frac{k}{2}, \frac{p+m}{4}\Big).$$

(2) From (1.10), the conclusion is obvious. ∎

Remark 1. On Heisenberg groups we can analogously obtain $|B_R|_p = \frac{2\pi^{n+\frac{1}{2}}R^Q\Gamma\left(\frac{n}{2}+\frac{p}{4}\right)}{Q\Gamma(n)\Gamma\left(\frac{1}{2}+\frac{n}{2}+\frac{p}{4}\right)}$ by using the polar coordinates introduced in [3].

Next we compute explicitly $C_{p,Q}^{-1}$ in Prpposition 1.1.

Theorem 2.2. *We have*

$$C_{p,Q}^{-1} = \left(\frac{Q-p}{p-1}\right)^{p-1}\frac{\pi^{\frac{m+k}{2}}}{4^{k-1}}\frac{B\left(\frac{k}{2},\frac{m+p}{4}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{k}{2}\right)}. \tag{22}$$

Proof. By (14), it follows that

$$\int_{\mathbf{G}} \frac{|x|^p d^{2(p-2)}}{(1+d^4)^{\frac{3p+Q}{4}}}$$

$$= \omega_m\omega_k \int_0^{\pi} d\theta \int_0^{+\infty} \frac{1}{4^k}\rho^{Q-1}(\sin\theta)^{\frac{m-2}{2}}\,|\cos\theta|^{k-1}\,\frac{\rho^p(\sin\theta)^{\frac{p}{2}}\cdot\rho^{2(p-2)}}{(1+\rho^4)^{\frac{3p+Q}{4}}}d\rho$$

$$= \omega_m\omega_k\frac{1}{4^k}\int_0^{\pi}(\sin\theta)^{\frac{m+p-2}{2}}\,|\cos\theta|^{k-1}\,d\theta\int_0^{+\infty}\frac{\rho^{Q+3p-5}}{(1+\rho^4)^{\frac{3p+Q}{4}}}d\rho$$

$$= \frac{2\pi^{\frac{m}{2}}}{\Gamma\left(\frac{m}{2}\right)} \cdot \frac{2\pi^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right)} \cdot \frac{1}{4^k}\frac{\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{m+p}{4}\right)}{\Gamma\left(\frac{2k+m+p}{4}\right)} \cdot \frac{1}{-4+3p+Q}$$

$$= \frac{1}{Q+3p-4} \frac{\pi^{\frac{m+k}{2}}}{4^{k-1}} \frac{\mathrm{B}\left(\frac{k}{2}, \frac{m+p}{4}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{k}{2}\right)},$$

and so

$$\begin{aligned} C_{p,Q}^{-1} &= \left(\frac{Q-p}{p-1}\right)^{p-1} (Q+3p-4) \int_{\mathbb{R}^{m+k}} \frac{|x|^p d^{2(p-2)}}{(1+d^4)^{\frac{3p+Q}{4}}} \\ &= \left(\frac{Q-p}{p-1}\right)^{p-1} \frac{\pi^{\frac{m+k}{2}}}{4^{k-1}} \frac{\mathrm{B}\left(\frac{k}{2}, \frac{m+p}{4}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{k}{2}\right)}. \end{aligned}$$

■

Remark 2. In [15] the fundamental solution of p-sub-Laplacian on the Heisenberg group is $C_{p,Q} d^{\frac{p-Q}{p-1}}$, where $C_{p,Q}^{-1} = \left(\frac{Q-p}{p-1}\right)^{p-1} (Q+3p-4) \cdot \int_{\mathbf{H}^n} \frac{|z|^p d^{2(p-2)}}{(1+d^4)^{\frac{3p+Q}{4}}} dz dt$. One deduces easily by using the polar coordinates in [3] $C_{p,Q}^{-1} = \left(\frac{Q-p}{p-1}\right)^{p-1} \cdot \frac{2\pi^{n+\frac{1}{2}}\Gamma\left(\frac{2n+p}{4}\right)}{\Gamma(n)\Gamma\left(\frac{2+2n+p}{4}\right)}$. Especially when $p=2$, the constant appears in the fundamental solution of the sub-Laplacian in [4].

3. A Degenerate Elliptic Inequality

The target of this section is to deal with the inequality

$$-\frac{d^2}{\psi} \Delta_{\mathbf{G}}(au) \geq |u|^q \quad \text{on } \mathbf{G} \backslash \{(0,0)\}, \tag{23}$$

where $a \in L^\infty(\mathbf{G})$.

Definition 3.1. *Let $q \geq 1$. A function u is called a weak solution of* (23), *if $u \in L^q_{loc}(\mathbf{G} \backslash \{(0,0)\})$ and*

$$\int_{\mathbf{G}} \frac{|u|^q}{d^Q} \psi \varphi \, dx dy \leq -\int_{\mathbf{G}} a u \Delta_{\mathbf{G}}(d^{2-Q}\varphi) \, dx dy \tag{24}$$

for any nonnegative $\varphi \in C_0^2(\mathbf{G} \backslash \{(0,0)\})$.

Theorem 3.1. *For any $q > 1$,* (23) *has no nontrivial weak solutions.*

Proof. Let u be a nontrivial weak solution of (23) and $\varphi \in C_0^2(\mathbf{G} \backslash \{(0,0)\})$, $\varphi \geq 0$. We set

$$F = 2(2-Q) d < Xd, X\varphi > + d^2 \Delta_{\mathbf{G}} \varphi.$$

Using (6) and (8), we have

$$\Delta_{\mathbf{G}}(d^{2-Q}\varphi) = \frac{1}{d^Q}[2(2-Q)d < Xd, X\varphi > + d^2 \Delta_{\mathbf{G}} \varphi] = \frac{F}{d^Q}. \tag{25}$$

By (24), (25) and Hölder's inequality, we get

$$\begin{aligned}\int\limits_{\mathbf{G}} \frac{|u|^q}{d^Q}\psi\varphi\,dxdy &\le -\int_{\mathbf{G}} au\Delta_{\mathbf{G}}(d^{2-Q}\varphi)\,dxdy\\ = -\int_{\mathbf{G}} \frac{auF}{d^Q}dxdy &\le \|a\|_\infty \int_{\mathbf{G}} \frac{|u|\,|F|}{d^Q}\,dxdy\\ \le \|a\|_\infty\Big(\int\limits_{\mathbf{G}} \frac{|u|^q}{d^Q}\psi\varphi\,dxdy\Big)^{\frac{1}{q}}&\Big(\int\limits_{\mathbf{G}} \frac{|F|^{q'}}{d^Q\psi^{q'-1}\varphi^{q'-1}}\,dxdy\Big)^{\frac{1}{q'}},\end{aligned}$$

and therefore

$$\int\limits_{\mathbf{G}} \frac{|u|^q}{d^Q}\psi\varphi\,dxdy \le \|a\|_\infty^{q'}\int_{\mathbf{G}} \frac{|F|^{q'}}{d^Q\psi^{q'-1}\varphi^{q'-1}}\,dxdy = \|a\|_\infty^{q'} I_1, \tag{26}$$

where $I_1 = \int_{\mathbf{G}} \frac{|F|^{q'}}{d^Q\psi^{q'-1}\varphi^{q'-1}}\,dxdy$.

We select the function φ by letting $\varphi = \varphi(d)$. Clearly, F becomes

$$\begin{aligned}F &= 2(2-Q)d\varphi'(d)\psi + d^2\psi\left[\varphi''(d) + \frac{Q-1}{d}\varphi'(d)\right]\\ &= \psi[d^2\varphi''(d) + (3-Q)d\varphi'(d)].\end{aligned}$$

Hence, we have from (19)

$$\begin{aligned}I_1 &= \int\limits_{\mathbf{G}} \psi\frac{|d^2\varphi''(d) + (3-Q)d\varphi'(d)|^{q'}}{d^Q\varphi^{q'-1}}\,dxdy\\ &= s_{m,k}\int_0^{+\infty} \frac{|\rho^2\varphi''(\rho) + (3-Q)\rho\varphi'(\rho)|^{q'}}{\rho\varphi^{q'-1}}\,d\rho.\end{aligned}$$

Letting $s = \ln\rho$ and $\widetilde{\varphi}(s) = \varphi(\rho)$, leads to

$$I_1 = s_{m,k}\int_{-\infty}^{+\infty} \frac{|\widetilde{\varphi}''(s) + (2-Q)\widetilde{\varphi}'(s)|^{q'}}{\widetilde{\varphi}(s)^{q'-1}}\,ds.$$

We perform our choice of φ by taking $\widetilde{\varphi}(s) = \varphi_0(\frac{s}{R})$ with φ_0 as in (13) and obtain

$$\begin{aligned}I_1 &= s_{m,k}\int_{R\le|s|\le 2R} \frac{|\frac{1}{R^2}\varphi_0''(\frac{s}{R}) + (2-Q)\frac{1}{R}\varphi_0'(\frac{s}{R})|^{q'}}{\varphi_0(\frac{s}{R})^{q'-1}}\,ds\\ &= s_{m,k}\int_{1\le|\tau|\le 2} \frac{|\frac{\varphi_0''(\tau)}{R} + (2-Q)\varphi_0'(\tau)|^{q'}}{\varphi_0(\tau)^{q'-1}}R^{1-q'}\,d\tau\\ &= s_{m,k}R^{1-q'}I_2,\end{aligned} \tag{27}$$

where

$$I_2 = \int_{1\le|\tau|\le 2} \frac{|\frac{\varphi_0''(\tau)}{R} + (2-Q)\varphi_0'(\tau)|^{q'}}{\varphi_0(\tau)^{q'-1}}\,d\tau.$$

Let φ_0 be an admissible function. For $R > 1$, it follows that

$$
\begin{aligned}
I_2 &\le \int\limits_{1\le|\tau|\le 2} \frac{\left(\frac{|\varphi_0''(\tau)|}{R} + (Q-2)|\varphi_0'(\tau)|\right)^{q'}}{\varphi_0(\tau)^{q'-1}}\, d\tau \\
&\le \int\limits_{1\le|\tau|\le 2} \frac{2^{q'-1}\left[\left(\frac{|\varphi_0''(\tau)|}{R}\right)^{q'} + ((Q-2)|\varphi_0'(\tau)|)^{q'}\right]}{\varphi_0(\tau)^{q'-1}}\, d\tau \\
&= \frac{2^{q'-1}}{R^{q'}} \int_{1\le|\tau|\le 2} \frac{|\varphi_0''(\tau)|^{q'}}{\varphi_0(\tau)^{q'-1}} d\tau + 2^{q'-1}(Q-2)^{q'} \int_{1\le|\tau|\le 2} \frac{|\varphi_0'(\tau)|^{q'}}{\varphi_0(\tau)^{q'-1}} d\tau \\
&\le M < +\infty,
\end{aligned}
$$

with M independent of R. Merging (26) into (27) and considering $\varphi(x,y) = \widetilde{\varphi}(\ln d) = \varphi_0(\frac{\ln d}{R})$, we have

$$
\int_{e^{-R}\le d\le e^R} \frac{|u|^q}{d^Q}\psi\, dxdy \le M \parallel a \parallel_\infty^{q'} s_{m,k} R^{1-q'} = CR^{1-q'}.
$$

Letting $R \to +\infty$, it induces $u = 0$. This contradiction completes the proof. ∎

Remark 3. Arguing as in [3], we can treat the evolution inequalities

$$
\begin{cases}
u_t - \frac{d^2}{\psi}\Delta_{\mathbf{G}}(au) \ge |u|^q & \text{on } \mathbf{G}\backslash\{(0,0)\} \times (0,+\infty), \\
u(x,y,0) = u_0(x,y) & \text{on } \mathbf{G}\backslash\{(0,0)\},
\end{cases}
$$

where $a \in \mathbb{R}$, and

$$
\begin{cases}
u_{tt} - \frac{d^2}{\psi}\Delta_{\mathbf{G}}(au) \ge |u|^q & \text{on } \mathbf{G}\backslash\{(0,0)\} \times (0,+\infty), \\
u(x,y,0) = u_0(x,y) & \text{on } \mathbf{G}\backslash\{(0,0)\}, \\
u_t(x,y,0) = u_1(x,y) & \text{on } \mathbf{G}\backslash\{(0,0)\},
\end{cases}
$$

where $a \in L^\infty(\mathbf{G} \times [0,+\infty))$, in the setting of the H-type group.

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