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On the Functional Equation $P(f)=Q(g)$ in Complex Numbers Field*

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Abstract. In this paper, we study the existence of non-constant meromorphic solutions f and g of the functional equation $P(f) = Q(g)$, where $P(z)$ and $Q(z)$ are given nonlinear polynomials with coefficients in the complex field $\mathbb{C}$.

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1. Introduction

Let $\mathbb{C}$ be the complex number field. In [3], Li and Yang introduced the following definition.

Definition. *A non-constant polynomial $P(z)$ defined over $\mathbb{C}$ is called a uniqueness polynomial for entire (or meromorphic) functions if the condition $P(f) = P(g)$, for entire (or meromorphic) functions f and g, implies that $f \equiv g$. $P(z)$ is called a strong uniqueness polynomial if the condition $P(f) = CP(g)$, for entire (or meromorphic) functions f and g, and some non-zero constant C, implies that $C = 1$ and $f \equiv g$.*

Recently, there has been considerable progress in the study of uniqueness polynomials, Boutabaa, Escassut and Hadadd [10] showed that a complex poly-

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nomial P is a strong uniqueness polynomial for the family of complex polynomials if and only if no non-trivial affine transformation preserves its set of zeros. As for the case of complex meromorphic functions, some sufficient conditions were found by Fujimoto in [8]. When P is injective on the roots of its derivative P', necessary and sufficient conditions were given in [5]. Recently, Khoai and Yang generalized the above studies by considering a pair of two nonlinear polynomials $P(z)$ and $Q(z)$ such that the only meromorphic solutions f, g satisfying $P(f) = Q(g)$ are constants. By using the singularity theory and the calculation of the genus of algebraic curves based on Newton polygons as the main tools, they gave some sufficient conditions on the degrees of P and Q for the problem (see [1]). After that, by using value distribution theory, in [2], Yang-Li gave more sufficient conditions related to this problem in general, and also gave some more explicit conditions for the cases when the degrees of P and Q are $2, 3, 4$.

In this paper, we solve this functional equation by studying the hyperbolicity of the algebraic curve $\{P(x) - Q(y) = 0\}$. Using different from Khoai and Yang's method, we estimate the genus by giving sufficiently many linear independent regular 1-forms of Wronskian type on that curve. This method was first introduced in [4] by An-Wang-Wong.

2. Main Theorems

Definition. *Let $P(z)$ be a nonlinear polynomial of degree n whose derivative is given by*

$$P'(z) = c(z - \alpha_1)^{n_1} \dots (z - \alpha_k)^{n_k},$$

where $n_1 + \dots + n_k = n - 1$ and $\alpha_1, \dots, \alpha_k$ are distinct zeros of P'. The number k is called the derivative index of P.

The polynomial $P(z)$ is said to satisfy *the condition separating the roots of P' (separation condition)* if $P(\alpha_i) \neq P(\alpha_j)$ for all $i \neq j,\ i, j = 1, 2. \dots, k$.

Here we only consider two nonlinear polynomials of degrees n and m, respectively

$$P(x) = a_n x^n + \dots + a_1 x + a_0, \quad Q(y) = b_m y^m + \dots + b_1 y + b_0, \tag{1}$$

in $\mathbb{C}$ so that $P(x) - Q(y)$ has no linear factors of the form $ax + by + c$.

Assume that

$$P'(x) = na_n(x - \alpha_1)^{n_1} \dots (x - \alpha_k)^{n_k},$$
$$Q'(y) = mb_m(y - \beta_1)^{m_1} \dots (y - \beta_l)^{m_l},$$

where $n_1 + \dots + n_k = n - 1,\ m_1 + \dots + m_l = m - 1, \alpha_1, \dots, \alpha_k$ are distinct zeros of P' and $\beta_1, \dots, \beta_l$ are distinct zeros of Q'. Let

$$\Delta := \{\alpha_i | \text{ there exist } \beta_j \text{ such that } P(\alpha_i) = Q(\beta_j)\},$$

and

$$\Lambda := \{\beta_j | \text{ there exist } \alpha_i \text{ such that } P(\alpha_i) = Q(\beta_j)\}.$$

Put

$$I = \#\Delta, \ J = \#\Lambda,$$

then $k \geq I$ and $l \geq J$. We obtain the following results.

Theorem 2.1. *Let $P(x)$ and $Q(y)$ be nonlinear polynomials of degree n and m, respectively, $n \geq m$. Assume that $P(x) - Q(y)$ has no linear factor, and I, J, n_i, m_j be defined as above. Then there exist no non-constant meromorphic functions f and g such that $P(f) = Q(g)$ provided that P and Q satisfy one of the following conditions*

(i) $\sum_{i|\alpha_i \notin \Delta} n_i \geq n - m + 3$,

(ii) $\sum_{j|\beta_j \notin \Lambda} m_j \geq 3$.

Corollary 2.2. *Let $P(x)$ and $Q(y)$ be nonlinear polynomials of degree n and m, respectively, $n \geq m$. Assume that $P(x) - Q(y)$ has no linear factor. Let k, l be the derivative indices of P, Q, respectively and Δ, Λ, I, J be defined as above. Then there exist no non-constant meromorphic functions f and g such that $P(f) = Q(g)$ provided that P and Q satisfy one of the following conditions*

(i) $k - I \geq n - m + 3$,

(ii) $l - J \geq 3$,

(iii) $k - I = 2$ *and* $n_1 + n_2 \geq n - m + 3$, *where* $n_1, \ n_2$ *are multiplicities of distinct zeros* α_1, α_2 *of* P', *respectively, such that* $\alpha_1, \alpha_2 \notin \Delta$,

(iv) $l - J = 2$ *and* $m_1 + m_2 \geq 3$, *where* $m_1, \ m_2$ *are multiplicities of distinct zeros* β_1, β_2 *of* Q', *respectively, such that* $\beta_1, \beta_2 \notin \Lambda$,

(v) $k - I = 1$ *and* $n_1 \geq n - m + 3$, *where* n_1 *is the multiplicity of zero* α_1 *of* P' *such that* $\alpha_1 \notin \Delta$,

(vi) $l - J = 1$ *and* $m_1 \geq 3$, *where* m_1 *is the multiplicity of zero* β_1 *of* Q' *such that* $\beta_1 \notin \Lambda$.

Corollary 2.3. *Let $P(z)$ and $Q(z)$ be two nonlinear polynomials of degrees n and m, respectively, $n \geq m$. Suppose that $P(\alpha) \neq Q(\beta)$ for all zeros α of P' and β of Q'. If $m \geq 4$, then there exists no non-constant meromorphic functions f and g such that $P(f) = Q(g)$.*

Theorem 2.4. *Let $P(z), Q(z)$ be nonlinear polynomials of degree n and m, respectively, $n \geq m$, and $\Lambda, \ J, \ n_i, \ m_j$ are defined as above. Rearrange $\beta_j \in \Lambda$ so that $m_1 \geq m_2 \geq \ldots \geq m_J$.*

Assume that P satisfies the separation condition, $J \geq 2$ and $P(\alpha_t) = Q(\beta_t)$, with $t = 1, 2$. Then there exists no pair of non-constant meromorphic functions f, g such that $P(f) = Q(g)$ if one of the following conditions is satisfied

(i) $m_1 \geq m_2 \geq 3, m_1 \geq n_1, m_2 \geq n_2$, *or*

(ii) $m_1 \geq n_1, m_1 > 3, n_2 > m_2 \geq 3, \frac{m_2+1}{m_2} \geq \frac{n_2-m_2}{m_1-3}$, *or*

(iii) $n_1 > m_1 \geq m_2 > 3, m_2 \geq n_2, \frac{m_1+1}{m_1} \geq \frac{n_1-m_1}{m_2-3}$, *or*

(iv) $n_1 > m_1 \geq m_2 > 3, n_2 > m_2, \ \frac{m_1+1}{m_1} \geq \frac{n_1-m_1}{m_2-3}$ *and* $\frac{m_2+1}{m_2} \geq \frac{n_2-m_2}{m_1-3}$.

If $k = I = J = l = 1$, then there exist non-constant meromorphic functions f, g such that $P(f) = Q(g)$.

Corollary 2.5. *Under the hypotheses of Theorem* 2.4, *then there exists no pair of non-constant meromorphic functions* f *and* g *such that* $P(f) = Q(g)$ *if* $J \geq 2,\ m_1 + m_2 - 4 \geq \max\{n_1,\ n_2\}$ *and* $m_1, m_2 \geq 3$.

Remark. In the case $n = m = 2$, the equation $P(f) = Q(g)$ has some non-constant meromorphic function solutions. Indeed, in this case we can rewrite the equation $P(f) = Q(g)$ in the form:

$$(f-a)^2 = (bg-c)^2 + d,$$

where $a, b, c, d \in \mathbb{C}$ and $b \neq 0$. Assume that h is a non-constant meromorphic function. Let

$$f = \frac{1}{2}(h + \frac{d}{h}) + a, \quad g = \frac{1}{2b}(-h + \frac{d}{h}) + \frac{c}{b}.$$

Then f and g are non-constant meromorphic solutions of the equation $P(f) = Q(g)$. ∎

3. Proofs of the Main Theorems

Suppose that $H(X, Y, Z)$ is a homogeneous polynomial of degree n. Let

$$C := \{(X : Y : Z) \in \mathbb{P}^2(\mathbb{C}) | H(X, Y, Z) = 0\}.$$

Put

$$W(X,Y) := \begin{vmatrix} X & Y \\ dX & dY \end{vmatrix}, \ W(Y,Z) := \begin{vmatrix} Y & Z \\ dY & dZ \end{vmatrix}, \ W(X,Z) := \begin{vmatrix} X & Z \\ dX & dZ \end{vmatrix}.$$

Definition. *Let* C *be an algebraic curve in* $\mathbb{P}^2(\mathbb{C})$. *A 1-form* ω *on* C *is said to be regular if it is the pull-back of a rational* 1*-form on* $\mathbb{P}^2(\mathbb{C})$ *such that the set of poles of* ω *does not intersect* C. *A well-defined rational regular 1-form on* C *is said to be a* 1*-form of Wronskian type.*

Notice that to solve the functional equation $P(f) = Q(g)$, is similar to find meromorphic functions f, g on $\mathbb{C}$ such that $(f(z), g(z))$ lies in curve $\{P(x) - Q(y) = 0\}$. On the other hand, if C is hyperbolic on $\mathbb{C}$ and suppose that f, g are meromorphic functions such that $(f(z), g(z)) \in C$, where $z \in \mathbb{C}$, then f and g are constant. Therefore, to prove that a functional equation $P(f) = Q(g)$ has no non-constant meromorphic function solution, it suffices to show that any irreducible component of the curves $\{F(X, Y, Z) = 0\}$ has genus at least 2, where $F(X, Y, Z)$ is the homogenization of the polynomial $P(x) - Q(y)$ in $\mathbb{P}^2(\mathbb{C})$.

It is well-known that the genus g of an algebraic curve C is equal to the dimension of the space of regular 1-forms on C. Therefore, to compute the genus, we have to construct a basis of the space of regular 1-forms on C.

Now, let $P(x)$ and $Q(y)$ be two nonlinear polynomials of degrees n and m, respectively, in $\mathbb{C}$, defined by (1). Without loss of generality, we assume that $n \geq m$. Set

$$F_1(x,y) := P(x) - Q(y),$$
$$F(X,Y,Z) := Z^n\left\{P(\frac{X}{Z}) - Q(\frac{Y}{Z})\right\}, \tag{2}$$
$$C := \{(X:Y:Z) \in \mathbb{P}^2(\mathbb{C})\ |F(X,Y,Z) = 0\}. \tag{3}$$

We define

$$P'(X,Z) := Z^{n-1}P'(\frac{X}{Z}),$$
$$Q'(Y,Z) := Z^{m-1}Q'(\frac{Y}{Z}),$$

then

$$\frac{\partial F}{\partial X} = P'(X,Z),$$
$$\frac{\partial F}{\partial Y} = -Z^{n-m}Q'(Y,Z),$$
$$\frac{\partial F}{\partial Z} = \sum_{i=0}^{n-1}(n-i)a_iX^iZ^{n-1-i} - \sum_{j=0}^{m'}(n-j)b_jY^jZ^{n-1-j},$$

where

$$m' = \begin{cases} n-1 & \text{if} \quad n = m \\ m & \text{if} \quad n > m. \end{cases}$$

It is known that (see [4] for details)

$$\frac{W(Y,Z)}{\frac{\partial F}{\partial X}} = \frac{W(Z,X)}{\frac{\partial F}{\partial Y}} = \frac{W(X,Y)}{\frac{\partial F}{\partial Z}}. \tag{4}$$

Therefore,

$$\frac{W(Y,Z)}{P'(X,Z)} = \frac{W(X,Z)}{Z^{n-m}Q'(Y,Z)}$$
$$= \frac{W(X,Y)}{\sum_{i=0}^{n-1}(n-i)a_iX^iZ^{n-1-i} - \sum_{j=0}^{m'}(n-j)b_jY^jZ^{n-1-j}}. \tag{5}$$

We recall the following notation. Assume that $\varphi(x,y)$ is an analytic function in x, y and is singular at (a,b). The Puiseux expansion of $\varphi(x,y)$ at $\rho := (a,b)$ is given by

$$[x = a + a_\alpha t^\alpha + \text{higher terms}, \quad y = b + b_\beta t^\beta + \text{higher terms}],$$

where $\alpha, \beta \in \mathbb{N}^*$ and $a_\alpha, b_\beta \neq 0$. The α (respectively, β) is the order (also the multiplicity number) of x at ρ, (respectively, the order of y at ρ) for φ and is denoted by

$$\alpha := \mathrm{ord}_{\rho,\varphi}(x) \text{ (respectively, } \beta := \mathrm{ord}_{\rho,\varphi}(y)).$$

In order to prove the main results, we need the following lemmas.

Lemma 3.1. *Let P and Q be two nonlinear polynomials of degrees n and m, respectively, $n \geq m$, and C be a projective curve, defined by (3). If $P(\alpha_i) \neq Q(\beta_j)$ for all zeros α_i of P' and β_j of Q', then we have the following assertions*

(i) *If $n = m$ or $n = m+1$ then C is non-singular in $\mathbb{P}^2(\mathbb{C})$.*

(ii) *If $n - m \geq 2$ then the point $(0:1:0)$ is a unique singular point of C in $\mathbb{P}^2(\mathbb{C})$.*

Proof. By assumption, the curve C is non-singular in $\mathbb{P}^2(\mathbb{C}) \setminus [Z=0]$. Now we consider the singularity of C in $[Z=0]$. Assume that $(X:Y:0)$ is a singular point of C. We obtain

$$\frac{\partial F}{\partial X}(X,Y,0) = \frac{\partial F}{\partial Y}(X,Y,0) = \frac{\partial F}{\partial Z}(X,Y,0) = 0.$$

If $n = m$ or $n = m+1$, then the above system has no root in $\mathbb{P}^2(\mathbb{C})$.

If $n - m \geq 2$, then the system has a unique root $(0:1:0)$ in $\mathbb{P}^2(\mathbb{C})$. Thus, if $n = m$ or $n = m+1$ then C is a smooth curve. If $n - m \geq 2$ then C is singular with a unique singular point at $(0:1:0)$. ∎

Remark 3.2.

(i). We also require that the 1-form, defined by (5), is non trivial when it restricts to a component of $\mathbb{C}$. This is equivalent to the condition that the nominators are not identically zero when they restrict to a component of $\mathbb{C}$ i.e., the Wronskians $W(X,Y), W(X,Z), W(Y,Z)$ are not identically zero. It means that the homogeneous polynomial defining C has no linear factors of the forms $aX - bY,\ aY - bZ,$ or $aX - bZ$, with $a, b \in \mathbb{C}$ if $P \neq Q$. Indeed, suppose on the contrary that, $aX - bZ$ is a factor of the curve C defined by (3). Without loss of generality, we can take $a \neq 0$. Since $aX - bZ$ is a factor of $F(X,Y,Z)$, we have

$$0 = F(\frac{b}{a}Z, Y, Z) = Z^n\{P(\frac{\frac{b}{a}Z}{Z}) - Q(\frac{Y}{Z})\} = Z^n\{P(\frac{b}{a}) - Q(\frac{Y}{Z})\},$$

this gives $P(\frac{b}{a}) \equiv Q(\frac{Y}{Z})$ for all Y, Z, a contradiction.

(ii). Assume that $P(\alpha_i) \neq Q(\beta_j)$ for all zeros α_i of P' and β_j of Q' and $m > n$. If $m = n+1$ then C is non-singular in $\mathbb{P}^2(\mathbb{C})$. If $m - n \geq 2$ then the point $(1:0:0)$ is a unique singular point of C in $\mathbb{P}^2(\mathbb{C})$. ∎

From Lemma 3.1, the only possible singularities of the curve C in $\mathbb{P}^2(\mathbb{C}) \setminus [Z = 0]$ are at $(\alpha_i : \beta_j : 1)$, where $\alpha_1, \ldots, \alpha_k$ are distinct zeros of P' and $\beta_1, \ldots, \beta_l$ are distinct zeros of Q'. Assume that the distinct zeros $\alpha_1, \ldots, \alpha_k$ of P' with multiplicities $n_1, \ldots, n_k$, and the distinct zeros $\beta_1, \ldots, \beta_l$ of Q' with multiplicities $m_1, \ldots, m_l$, respectively. Let

$$\Gamma := \{(\alpha_i : \beta_j : 1) \mid (\alpha_i : \beta_j : 1) \text{ is a singularity of } C\}, \tag{6}$$
$$\Delta := \{\alpha_i \mid (\alpha_i : \beta_j : 1) \text{ is a singularity of } C\}, \tag{7}$$
$$\Lambda := \{\beta_j \mid (\alpha_i : \beta_j : 1) \text{ is a singularity of } C\}. \tag{8}$$

Setting $I = \#\Delta$, $J = \#\Lambda$, then we have $k \geq I$ and $l \geq J$. Without loss of generality, we can take

$$\Lambda = \{\beta_1, \dots, \beta_J\} \text{ and } m_1 \geq m_2 \geq \dots \geq m_J.$$

Lemma 3.3. *Suppose that Λ, β_t, m_t are defined as above. Then, the 1-form*

$$\theta := \frac{W(X, Z)}{\prod_{t|\beta_t \notin \Lambda} (Y - \beta_t Z)^{m_t}},$$

is regular on C.

Proof. By the hypotheses, θ is regular on C because no point of the set $\{(\alpha_i : \beta_t : 1) \mid \beta_t \notin \Lambda\}$ is in C. ∎

Lemma 3.4. *Assume that Δ, α_i, n_i are defined as above. Then, the 1-form*

$$\sigma := \frac{Z^{n-m}}{\prod_{i|\alpha_i \notin \Delta} (X - \alpha_i Z)^{n_i}} W(Y, Z),$$

is regular on C.

Proof. By (5) and the hypotheses of the Lemma, we have

$$\begin{aligned}
\sigma &= \frac{Z^{n-m}}{\prod_{i|\alpha_i \notin \Delta} (X - \alpha_i Z)^{n_i}} W(Y, Z) \\
&= \frac{p Z^{n-m} \prod_{i|\alpha_i \in \Delta} (X - \alpha_i Z)^{n_i}}{p \prod_{i=1}^{k} (X - \alpha_i Z)^{n_i}} W(Y, Z) \\
&= \frac{p \prod_{i=1}^{I} (X - \alpha_i Z)^{n_i}}{Q'(Y, Z)} W(X, Z),
\end{aligned}$$

where $p = n a_n \neq 0$. By the definition of the set Δ, we have σ is regular on C. ∎

Proposition 3.5. *Assume that $n \geq m$, $P(x) - Q(y)$ has no linear factor and $k, l, \Delta, J, n_i, m_j$ are defined as above. Then the curve C is Brody hyperbolic if one of following conditions is satisfied*

(i) $\sum_{i|\alpha_i \notin \Delta} n_i \geq n - m + 3$.

(ii) $\sum_{j|\beta_j \notin \Lambda} m_j \geq 3$.

Proof. By Lemma 3.3, set

$$\vartheta := Z^{\sum_{j|\beta_j \notin \Lambda} m_j - 2} \theta.$$

Then ϑ is a well-defined regular 1-form of Wronskian type on C if $\sum_{j|\beta_j\notin\Lambda} m_j \geq 2$. Let $p := \sum_{j|\beta_j\notin\Lambda} m_j - 2$. If $p \geq 1$, we take $\{R_1, R_2, \ldots, R_{\frac{(p+1)(p+2)}{2}}\}$ as a basis of monomials of degree p in $\{X, Y, Z\}$. Then

$$\{R_i\ \theta|\ i = 1, 2, \ldots, \frac{(p+1)(p+2)}{2}\}$$

are linearly independent and are global regular 1-forms of Wronskian type on the curve C. Thus, the genus g_C of C is

$$g_C \geq \frac{(p+1)(p+2)}{2}.$$

Therefore, C is Brody hyperbolic if $p \geq 1$, that means, $\sum_{j|\beta_j\notin\Lambda} m_j \geq 3$.

By Lemma 3.4, we set

$$\varsigma := Z^{\sum_{i|\alpha_i\notin\Delta} n_i - (n-m+2)}\sigma.$$

By a similar argument as above, the curve C is Brody hyperbolic if

$$q = \sum_{i|\alpha_i\notin\Delta} n_i - (n - m + 2) \geq 1,$$

that means,

$$\sum_{i|\alpha_i\notin\Delta} n_i \geq n - m + 3.$$

∎

Assume that $(\alpha_i : \beta_j : 1)$ is a singular point of C. Then, we obtain

$$P(x) - P(\alpha_i) = \sum_{t=n_i+1}^{n} (x - \alpha_i)^t,$$

$$Q(y) - Q(\beta_j) = \sum_{t=m_j+1}^{m} (y - \beta_j)^t,$$

with $P(\alpha_i) = Q(\beta_j)$, hence

$$\begin{aligned} F(X,Y,Z) &= Z^n\{P(\frac{X}{Z}) - Q(\frac{Y}{Z})\} = Z^n\{\{P(\frac{X}{Z}) - P(\alpha_i)\} - \{Q(\frac{Y}{Z}) - Q(\beta_j)\}\} \\ &= \sum_{t=n_i+1}^{n} (X - \alpha_i Z)^t - Z^{n-m} \sum_{t=m_j+1}^{m} (Y - \beta_j Z)^t. \end{aligned}$$

Using Puiseux expansion of $F(X, Y, Z)$ at $\rho_{ij} = (\alpha_i : \beta_j : 1)$, we have

$$(n_i + 1)\mathrm{ord}_{\rho_{ij},F}(X - \alpha_i Z) = (m_j + 1)\mathrm{ord}_{\rho_{ij},F}(Y - \beta_j Z). \tag{9}$$

Suppose that $\rho_1 = (\alpha_{i_1} : \beta_{j_1} : 1)$ and $\rho_2 = (\alpha_{i_2} : \beta_{j_2} : 1)$ are two distinct finite singular points of C. Setting

$$L_{12} := \begin{cases} (X - \alpha_{i_1}Z) - \frac{\alpha_{i_2}-\alpha_{i_1}}{\beta_{j_2}-\beta_{j_1}}(Y - \beta_{j_1}Z) & \text{if } \beta_{j_1} \neq \beta_{j_2} \\ (Y - \beta_{j_2}Z) - \frac{\beta_{j_2}-\beta_{j_1}}{\alpha_{i_2}-\alpha_{i_1}}(X - \alpha_{i_2}Z) & \text{if } \alpha_{i_1} \neq \alpha_{i_2}. \end{cases}$$

Then

$$L_{12}(\alpha_{i_1}, \beta_{j_1}, 1) = L_{12}(\alpha_{i_2}, \beta_{j_2}, 1) = 0,$$

and

$$\mathrm{ord}_{\rho_t,F} L_{12} \geq \min\{\mathrm{ord}_{\rho_t,F}(X - \alpha_{i_t} Z), \mathrm{ord}_{\rho_t,F}(Y - \beta_{j_t} Z)\}.$$

Hence,

$$\mathrm{ord}_{\rho_t,F} L_{12} \geq \begin{cases} \mathrm{ord}_{\rho_t,F}(X - \alpha_{i_t} Z) & \text{if } m_{j_t} < n_{i_t} \\ \mathrm{ord}_{\rho_t,F}(Y - \beta_{j_t} Z) & \text{if } m_{j_t} \geq n_{i_t}, \end{cases} \tag{10}$$

for $t = 1, 2$.

Now, assume that P satisfies the separation condition. Then $J \geq I$ and for every $\beta_j \in \Lambda$, there exists a unique value $\alpha_{i_j} \in \Delta$ such that $(\alpha_{i_j} : \beta_j : 1)$ is singular point of C (these α_{i_j} can be equal to each other). Therefore,

$$\Gamma = \{(\alpha_{i_j} : \beta_j : 1) | \beta_J \in \Lambda\} \tag{11}$$

is the set of singular points of C, with $l \geq J$. We have the following proposition.

Proposition 3.6. *Let P, Q be nonlinear polynomials and C is a projective curve defined by* (3)*. Assume that $\Gamma = \{(\alpha_i : \beta_j : 1)\}$ is the set of all finite singular points of C. Let $\Lambda = \{\beta_1, \ldots, \beta_J\}$, defined by (8), where $m_1 \geq m_2 \geq \ldots \geq m_J$ and $(\alpha_1 : \beta_1 : 1)$, $(\alpha_2 : \beta_2 : 1) \in \Gamma$. Furthermore, suppose that P satisfies the separation condition. Then, the curve C is Brody hyperbolic if $J \geq 2$ and one of the following conditions is satisfied*

(i) $m_1 \geq m_2 \geq 3, m_1 \geq n_1, m_2 \geq n_2$, *or*
(ii) $m_1 \geq n_1, m_1 > 3, n_2 > m_2 \geq 3, \frac{m_2+1}{m_2} \geq \frac{n_2-m_2}{m_1-3}$, *or*
(iii) $n_1 > m_1 \geq m_2 > 3, m_2 \geq n_2, \frac{m_1+1}{m_1} \geq \frac{n_1-m_1}{m_2-3}$, *or*
(iv) $n_1 > m_1 \geq m_2 > 3, n_2 > m_2, \frac{m_1+1}{m_1} \geq \frac{n_1-m_1}{m_2-3}$ *and* $\frac{m_2+1}{m_2} \geq \frac{n_2-m_2}{m_1-3}$.

Proof. By the hypotheses, if $\rho_1 = (\alpha_1 : \beta_1 : 1) \neq \rho_2 = (\alpha_2 : \beta_2 : 1)$, then $\beta_1 \neq \beta_2$. Indeed, assume on the contrary that $\beta_1 = \beta_2$. Since $\rho_1 \neq \rho_2$, we obtain $\alpha_1 \neq \alpha_2$. Hence $P(\alpha_1) = Q(\beta_1) = Q(\beta_2) = P(\alpha_2)$, which is a contradiction. Let

$$L := (X - \alpha_1 Z) - \frac{\alpha_2 - \alpha_1}{\beta_2 - \beta_1}(Y - \beta_1 Z).$$

By (9) and (10), we get

$$\mathrm{ord}_{\rho_t,F} L \geq \begin{cases} \mathrm{ord}_{\rho_t,F}(X - \alpha_t Z) & \text{if } m_t < n_t \\ \mathrm{ord}_{\rho_t,F}(Y - \beta_t Z) & \text{if } m_t \geq n_t, \end{cases} \tag{12}$$

for $t = 1, 2$. The rational 1-forms

$$\omega_1 := \frac{L^{m_1+m_2-3}}{(Y - \beta_1 Z)^{m_1-1}(Y - \beta_2 Z)^{m_2}} W(X, Z),$$
$$\omega_2 := \frac{L^{m_1+m_2-3}}{(Y - \beta_1 Z)^{m_1}(Y - \beta_2 Z)^{m_2-1}} W(X, Z),$$

are well-defined if $m_1 + m_2 \geq 3$. We claim that $\omega_1,\ \omega_2$ are regular. To prove this problem we only need to check the regularity at $\rho_t = (\alpha_t : \beta_t : 1)$ (for $t = 1, 2$), since P satisfies the separation condition, we have for every $u \neq t$ then

$(\alpha_u : \beta_t : 1) \notin C$, with $t = 1, 2$, respectively. $\omega_i,\ i = 1, 2$ are regular at ρ_t if the 1-forms

$$\chi_{11} := \frac{L^{m_1+m_2-3}}{(Y-\beta_1 Z)^{m_1-1}} W(X, Z),$$
$$\chi_{12} := \frac{L^{m_1+m_2-3}}{(Y-\beta_2 Z)^{m_2}} W(X, Z),$$
$$\chi_{21} := \frac{L^{m_1+m_2-3}}{(Y-\beta_1 Z)^{m_1}} W(X, Z),$$
$$\chi_{22} := \frac{L^{m_1+m_2-3}}{(Y-\beta_2 Z)^{m_2-1}} W(X, Z),$$

are regular at ρ_t with $t = 1, 2$. From (12), we have

$$\mathrm{ord}_{\rho_1,F} \frac{L^{m_1+m_2-3}}{(Y-\beta_1 Z)^{m_1-1}} \geq \begin{cases} (m_2-2)\ \mathrm{ord}_{\rho_1,F}(Y-\beta_1 Z) & \text{if } m_1 \geq n_1 \\ \frac{(m_1+1)(m_2-2)-(m_1-1)(n_1-m_1)}{m_1+1} \mathrm{ord}_{\rho_1,F}(X-\alpha_1 Z) & \text{if } m_1 < n_1, \end{cases}$$
$$\mathrm{ord}_{\rho_2,F} \frac{L^{m_1+m_2-3}}{(Y-\beta_2 Z)^{m_2}} \geq \begin{cases} (m_1-3)\mathrm{ord}_{\rho_2,F}(Y-\beta_2 Z) & \text{if } m_2 \geq n_2 \\ \frac{(m_2+1)(m_1-3)-m_2(n_2-m_2)}{m_2+1}\ \mathrm{ord}_{\rho_2,F}(X-\alpha_2 Z) & \text{if } m_2 < n_2, \end{cases}$$
$$\mathrm{ord}_{\rho_1,F} \frac{L^{m_1+m_2-3}}{(Y-\beta_1 Z)^{m_1}} \geq \begin{cases} (m_2-3)\mathrm{ord}_{\rho_1,F}(Y-\beta_1 Z) & \text{if } m_1 \geq n_1 \\ \frac{(m_1+1)(m_2-3)-m_1(n_1-m_1)}{m_1+1}\ \mathrm{ord}_{\rho_1,F}(X-\alpha_1 Z) & \text{if } m_1 < n_1, \end{cases}$$
$$\mathrm{ord}_{\rho_2,F} \frac{L^{m_1+m_2-3}}{(Y-\beta_2 Z)^{m_2-1}} \geq \begin{cases} (m_1-2)\mathrm{ord}_{\rho_2,F}(Y-\beta_1 Z) & \text{if } m_2 \geq n_2 \\ \frac{(m_2+1)(m_1-2)-(m_2-1)(n_2-m_2)}{m_2+1} \mathrm{ord}_{\rho_2,F}(X-\alpha_2 Z) & \text{if } m_2 < n_2. \end{cases}$$

Thus, the 1- form χ_{11} is regular at ρ_1 if one of the following conditions is satisfied

(r_1) $m_1 \geq n_1$ and $m_2 \geq 2$, or
(r_2) $m_1 < n_1$ and $(m_1+1)(m_2-2) \geq (m_1-1)(n_1-m_1)$.

By a similar argument, we obtain χ_{12} is regular at ρ_2 if one of the following conditions is satisfied

(r_3) $m_1 \geq 3$ and $m_2 \geq n_2$, or
(r_4) $m_2 < n_2$ and $(m_2+1)(m_1-3) \geq m_2(n_2-m_2)$.

The 1-form χ_{21} is regular at ρ_1 if one of the following conditions is satisfied

(r_5) $m_1 \geq n_1$ and $m_2 \geq 3$, or
(r_6) $n_1 > m_1$ and $(m_1+1)(m_2-3) \geq m_1(n_1-m_1)$,

and χ_{22} is regular at ρ_2 if one of the following conditions is satisfied

(r_7) $m_2 \geq n_2$ and $m_1 \geq 2$, or
(r_8) $m_2 < n_2$ and $(m_2+1)(m_1-2) \geq (m_2-1)(n_2-m_2)$.

Thus, ω_1 is regular on C if one of the following conditions is satisfied

(a) $m_1 \geq n_1,\ m_1 \geq 3,\ m_2 \geq n_2$ and $m_2 \geq 2$,
(b) $m_1 \geq n_1,\ n_2 > m_2 \geq 2$ and $(m_2+1)(m_1-3) \geq m_2(n_2-m_2)$,
(c) $n_1 > m_1 \geq m_2 \geq n_2,\ m_1 \geq 3$ and $(m_1+1)(m_2-2) \geq (m_1-1)(n_1-m_1)$,
(d) $n_1 > m_1,\ n_2 > m_2,\ (m_1+1)(m_2-2) \geq (m_1-1)(n_1-m_1)$
and $(m_2+1)(m_1-3) \geq m_2(n_2-m_2)$.

Similarly, ω_2 is regular on C if one of the following conditions is satisfied

(a') $m_1 \geq n_1,\ m_2 \geq n_2$ and $m_1 \geq m_2 \geq 3$,
(b') $m_1 \geq n_1,\ n_2 > m_2 \geq 3$ and $(m_2+1)(m_1-2) \geq (m_2-1)(n_2-m_2)$,
(c') $n_1 > m_1 \geq m_2 \geq n_2,\ m_1 \geq 2$ and $(m_1+1)(m_2-3) \geq m_1(n_1-m_1)$,
(d') $n_1 > m_1,\ n_2 > m_2,\ (m_1+1)(m_2-3) \geq m_1(n_1-m_1)$
and $(m_2+1)(m_1-2) \geq (m_2-1)(n_2-m_2)$.

Hence, ω_1 and ω_2 are regular on C if one of the following conditions holds

(i) $m_1 \geq m_2 \geq 3, m_1 \geq n_1, m_2 \geq n_2$,
(ii) $m_1 \geq n_1, n_2 > m_2 \geq 3, (m_2+1)(m_1-3) \geq m_2(n_2-m_2)$,
(iii) $n_1 > m_1 \geq m_2 \geq n_2, m_1 \geq 3,\ (m_1+1)(m_2-3) \geq m_1(n_1-m_1)$,
(iv) $n_1 > m_1 \geq m_2, n_2 > m_2,\ (m_1+1)(m_2-3) \geq m_1(n_1-m_1)$
and $(m_2+1)(m_1-3) \geq m_2(n_2-m_2)$.

From (ii), $m_1 > 3$. By (iii), $m_2 > 3$, and by (iv), $m_1 \geq m_2 > 3$. Furthermore, assume that $a\omega_1 + b\omega_2 = 0$, with $a, b \in \mathbb{C}$. Then we obtain $a(Y - \beta_1 Z) + b(Y - \beta_2 Z) = 0$, hence $(a+b)Y - (a\beta_1 + b\beta_2)Z = 0$ with all Y, Z. It follows that $a = b = 0$. Thus, $\omega_1,\ \omega_2$ are linearly independent. Therefore, the curve C is Brody hyperbolic if one of conditions of the proposition is satisfied. ∎

Remark that if $m_1 \geq m_2 \geq 3$ and $m_1 + m_2 - 4 \geq \max\{n_{i_1},\ n_{i_2}\}$, then

$$\mathrm{ord}_{\rho_1,F}\frac{L^{m_1+m_2-3}}{(Y-\beta_1 Z)^{m_1-1}} \geq \begin{cases} (m_2-2)\ \mathrm{ord}_{\rho_1,F}(Y-\beta_1 Z) & \text{if } m_1 \geq n_1 \\ \{(m_1+m_2-3)- \\ -\frac{(m_1-1)(n_1+1)}{m_1+1}\}\mathrm{ord}_{\rho_1,F}(X-\alpha_1 Z) & \text{if } m_1 < n_1, \end{cases}$$

$$\mathrm{ord}_{\rho_2,F}\frac{L^{m_1+m_2-3}}{(Y-\beta_2 Z)^{m_2}} \geq \begin{cases} (m_1-3)\ \mathrm{ord}_{\rho_2,F}(Y-\beta_2 Z) & \text{if } m_2 \geq n_2 \\ \{(m_1+m_2-3)-\frac{m_2(n_2+1)}{m_2+1}\}\mathrm{ord}_{\rho_2,F}(X-\alpha_2 Z) & \text{if } m_2 < n_2, \end{cases}$$

$$\mathrm{ord}_{\rho_1,F}\frac{L^{m_1+m_2-3}}{(Y-\beta_1 Z)^{m_1}} \geq \begin{cases} (m_2-3)\ \mathrm{ord}_{\rho_1,F}(Y-\beta_1 Z) & \text{if } m_1 \geq n_1 \\ \{(m_1+m_2-3)-\frac{m_1(n_1+1)}{m_1+1}\}\mathrm{ord}_{\rho_1,F}(X-\alpha_1 Z) & \text{if } m_1 < n_1, \end{cases}$$

$$\mathrm{ord}_{\rho_2,F}\frac{L^{m_1+m_2-3}}{(Y-\beta_2 Z)^{m_2-1}} \geq \begin{cases} (m_1-2)\ \mathrm{ord}_{\rho_2,F}(Y-\beta_2 Z) & \text{if } m_2 \geq n_2 \\ \{(m_1+m_2-3)- \\ -\frac{(m_2-1)(n_2+1)}{m_2+1}\}\mathrm{ord}_{\rho_2,F}(X-\alpha_2 Z) & \text{if } m_2 < n_2. \end{cases}$$

We obtain

$$\mathrm{ord}_{\rho_1,F}\frac{L^{m_1+m_2-3}}{(Y-\beta_1 Z)^{m_1-1}}\geq 0,\ \mathrm{ord}_{\rho_2,F}\frac{L^{m_1+m_2-3}}{(Y-\beta_2 Z)^{m_2}}\geq 0,$$
$$\mathrm{ord}_{\rho_1,F}\frac{L^{m_1+m_2-3}}{(Y-\beta_1 Z)^{m_1}}\geq 0,\ \mathrm{ord}_{\rho_2,F}\frac{L^{m_1+m_2-3}}{(Y-\beta_2 Z)^{m_2-1}}\geq 0.$$

Thus, we have ω_1 and ω_2 are regular on C. Therefore, we obtain the following corollary.

Corollary 3.7. *If the hypotheses of Proposition 3.5 are satisfied, then the curve C is Brody hyperbolic if $m_1 \geq m_2 \geq 3$ and $m_1 + m_2 - 4 \geq \max\{n_{i_1},\ n_{i_2}\}$.*

In the case $J = \#\Lambda = 1$, we obtain the following result.

Lemma 3.8. *If $k = I = J = l = 1$, then there exist non-constant meromorphic functions f, g such that $P(f) = Q(g)$.*

Proof. If $k = I = J = l = 1$, then we can rewrite the equation $P(f) = Q(g)$ in the form $(f-\alpha)^n = (bg-\beta)^m$, where $b \neq 0$. Assume that h is a non-constant meromorphic function, set

$$f = \alpha + h^m,\ g = \frac{1}{b}h^n + \frac{\beta}{b}.$$

Then f and g are non-constant meromorphic solutions of equation $P(f) = Q(g)$. ∎

Remark. Assume that the equation $P(f) = Q(g)$ has a solution (f, g), when f, g are non-constant meromorphic functions. Then the mapping

$$(f, g, 1) : \mathbb{C} \longrightarrow \mathbb{P}^2(\mathbb{C})$$

has its image contained in C defined by (3). If C is Brody hyperbolic, then $f = g$. From this it follows that $P = Q$, contrary to the fact that $P(x) - Q(y)$ has no linear factors of the form $ax + by + c$. Hence, we prove that under the assumptions of the theorems, the curve C is Brody hyperbolic. ∎

Proof of Theorem 2.1. Theorem 2.1 immediately follows from Lemmas 3.3, 3.4, Proposition 3.5 and Remark 3.9. ∎

Proof of Corollary 2.2. From Theorem 2.1, if $\sum_{j|\beta_j\notin\Lambda} m_j \geq 3$, then the functional equation $P(f) = Q(g)$ has no solution in the set of non-constant meromorphic functions. Since $m_j \geq 1$, we conclude that if $l - J \geq 3$, then $p = \sum_{j|\beta_j\notin\Lambda} m_j \geq 3$. If $l - J = 2$, then there only exist two zeros β_1, β_2 of Q' such that $P(\alpha) \neq Q(\beta_t)$ with all zeros α of P', $t = 1, 2$. This implies that if $m_1 + m_2 \geq 3$ then $p \geq 1$. If $l - J = 1$, then there only exists a unique zero β_1 with multiplicity m_1 of Q' such that $P(\alpha) \neq Q(\beta_1)$ with all zeros α of P'. Since $m_1 \geq 3$ shows that $p \geq 1$, from Remark 3.9, we obtain (ii), (iv) and (vi).

Since $\sum_{j|\alpha_j \notin \Delta} n_j \geq k - I$, therefore, if $k - I \geq n - m + 3$ then the curve C is Brody hyperbolic. If $k - I = 2$ and $n_1 + n_2 \geq n - m + 3$ then $\sum_{j|\alpha_j \notin \Delta} n_j = n_1 + n_2 \geq n - m + 3$. If $k - I = 1$ and $n_1 \geq n - m + 3$ then $\sum_{j|\alpha_j \notin \Delta} n_j = n_1 \geq n - m + 3$. Thus, we obtain (i), (iii) and (v). ■

Proof of Corollary 2.3. If the hypotheses of Corollary 2.3 are satisfied then $J = 0$ and $\sum_{j=1}^{l} m_j = m - 1 \geq 3,\ l \geq 1$, hence $l - J = l$. Using Theorem 2.1 and Corollary 2.2 in the cases (ii), (iv) and (vi), we obtain Corollary 2.3. ■

Note that Corollary 2.3 is Theorem A of Khoai-Yang in [1] and from Theorem 2.1, we can imply the Theorem B of Yang-Li in [2].

Proof of Theorem 2.4 and Corollary 2.5. Theorem 2.4 immediately follows from Proposition 3.6, Lemma 3.8 and Remark 3.9. Similarly, from Corollary 3.7 and Remark 3.9, we can obtain the proof of Corollary 2.5. ■

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