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Strongly Almost Summable Difference Sequences

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Abstract. The idea of difference sequence space was introduced by Kızmaz [12] and was generalized by Et and Çolak [6]. In this paper we introduce and examine some properties of three sequence spaces defined by using a modulus function and give various properties and inclusion relations on these spaces.

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1. Introduction

Let w be the set of all sequences of real numbers and ℓ_∞, c and c_0 be respectively the Banach spaces of bounded, convergent and null sequences $x = (x_k)$ with the usual norm $\|x\| = \sup |x_k|$, where $k \in \mathbb{N} = \{1, 2, \dots\}$, the set of positive integers.

A sequence $x \in \ell_\infty$ is said to be almost convergent [14] if all Banach limits of x coincide. Lorentz [14] defined that

$$\hat{c} = \Big\{x : \ \lim_n \frac{1}{n} \sum_{k=1}^{n} x_{k+m} \quad \text{exists, uniformly in } m\Big\}.$$

Several authors including Lorentz [14], Duran [2] and King [11] have studied almost convergent sequences. Maddox ([16, 17]) has defined x to be strongly almost convergent to a number L if

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} |x_{k+m} - L| = 0, \quad \text{uniformly in} \quad m.$$

By $[\hat{c}]$ we denote the space of all strongly almost convergent sequences. It is easy to see that $c \subset [\hat{c}] \subset \hat{c} \subset \ell_\infty$.

The space of strongly almost convergent sequences was generalized by Nanda ([20, 21]).

Let $p = (p_k)$ be a sequence of strictly positive real numbers. Nanda [20] defined

$$[\hat{c}, p] = \Big\{ x = (x_k) : \lim_n \frac{1}{n} \sum_{k=1}^{n} |x_{k+m} - L|^{p_k} = 0, \quad \text{uniformly in } m \Big\},$$

$$[\hat{c}, p]_0 = \Big\{ x = (x_k) : \lim_n \frac{1}{n} \sum_{k=1}^{n} |x_{k+m}|^{p_k} = 0, \quad \text{uniformly in } m \Big\},$$

$$[\hat{c}, p]_\infty = \Big\{ x = (x_k) : \sup_{n,m} \frac{1}{n} \sum_{k=1}^{n} |x_{k+m}|^{p_k} < \infty \Big\}.$$

Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to ∞ such that $\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1$.

The generalized de la Vallée-Pousin mean is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k,$$

where $I_n = [n - \lambda_n + 1, n]$ for $n = 1, 2, ...$

A sequence $x = (x_k)$ is said to be $(V, \lambda)-$summable to a number L [13] if $t_n(x) \to L$ as $n \to \infty$.

If $\lambda_n = n$, then $(V, \lambda)-$summability and strongly $(V, \lambda)-$summability are reduced to $(C, 1)-$summability and $[C, 1]-$summability, respectively.

The idea of difference sequence spaces was introduced by Kızmaz [12]. In 1981, Kızmaz[12] defined the sequence spaces

$$X(\Delta) = \{x = (x_k) : \Delta x \in X\}$$

for $X = \ell_\infty$, c and c_0, where $\Delta x = (x_k - x_{k+1})$.

Then Et and Çolak [6] generalized the above sequence spaces to the sequence spaces

$$X(\Delta^r) = \big\{x = (x_k) : \Delta^r x \in X\big\}$$

for $X = \ell_\infty$, c and c_0, where $r \in \mathbb{N}$, $\Delta^0 x = (x_k)$, $\Delta x = (x_k - x_{k+1})$, $\Delta^r x = \left(\Delta^{r-1} x_k - \Delta^{r-1} x_{k+1}\right)$, and so $\Delta^r x_k = \sum_{v=0}^{r} (-1)^v \binom{r}{v} x_{k+v}$.

Recently Et and Başarır [5] extended the above sequence spaces to the sequence spaces $X(\Delta^r)$ for $X = \ell_\infty(p)$, $c(p)$, $c_0(p)$, $[\hat{c}, p]$, $[\hat{c}, p]_0$ and $[\hat{c}, p]_\infty$.

We recall that a modulus f is a function from $[0,\infty)$ to $[0,\infty)$ such that

i) $f(x) = 0$ if and only if $x = 0$,

ii) $f(x + y) \leq f(x) + f(y)$ for $x, y \geq 0$,

iii) f is increasing,

iv) f is continuous from the right at 0.

It follows that f must be continuous everywhere on $[0,\infty)$. A modulus may be unbounded or bounded. Ruckle [23] and Maddox [15] used a modulus f to construct some sequence spaces.

Subsequently modulus function has been discussed in ([3, 4, 19, 22, 26]).

Let $X, Y \subset w$. Then we shall write

$$M(X,Y) = \bigcap_{x\in X} x^{-1} * Y = \left\{a \in w : ax \in Y \quad \text{for all } x \in X\right\} \text{ [27]}.$$

The set $X^\alpha = M(X, \ell_1)$ is called the Köthe-Toeplitz dual space or $\alpha-$dual of X.

Let X be a sequence space. Then X is called

i) *Solid* (or *normal*) if $(\alpha_k x_k) \in X$ whenever, $(x_k) \in X$ for all sequences (α_k) of scalars with $|\alpha_k| \le 1$ for all $k \in \mathbb{N}$.

ii) *Symmetric* if $(x_k) \in X$ implies $(x_{\pi(k)}) \in X$, where $\pi(k)$ is a permutation of $\mathbb{N}$.

iii) *Perfect* if $X = X^{\alpha\alpha}$.

iv) *A sequence algebra* if $x.y \in X$, whenever $x, y \in X$.

It is well known that if X is perfect then X is normal [10].

The following inequality will be used throughout this paper.

$$|a_k + b_k|^{p_k} \le C\left\{|a_k|^{p_k} + |b_k|^{p_k}\right\}, \tag{1}$$

where $a_k, b_k \in \mathbb{C}, 0 < p_k \le \sup_k p_k = H$, $C = \max\left(1, 2^{H-1}\right)$ [18].

2. Main Results

In this section we prove some results involving the sequence spaces $\left[\hat{V}, \Delta^r, \lambda, f, p\right]_0$, $\left[\hat{V}, \Delta^r, \lambda, f, p\right]_1$ and $\left[\hat{V}, \Delta^r, \lambda, f, p\right]_\infty$.

Definition 1. *Let f be a modulus function and $p = (p_k)$ be any sequence of strictly positive real numbers. We define the following sequence sets*

$$\left[\hat{V}, \Delta^r, \lambda, f, p\right]_1 = \Big\{x = (x_k) : \lim_n \frac{1}{\lambda_n} \sum_{k\in I_n} \left[f\left(|\Delta^r x_{k+m} - L|\right)\right]^{p_k} = 0, \text{ uniformly in } m, \text{ for some } L > 0\Big\},$$

$$\left[\hat{V}, \Delta^r, \lambda, f, p\right]_0 = \Big\{x = (x_k) : \lim_n \frac{1}{\lambda_n} \sum_{k\in I_n} \left[f\left(|\Delta^r x_{k+m}|\right)\right]^{p_k} = 0, \text{ uniformly in } m\Big\},$$

$$\left[\hat{V}, \Delta^r, \lambda, f, p\right]_\infty = \Big\{x = (x_k) : \sup_{n,m} \frac{1}{\lambda_n} \sum_{k\in I_n} \left[f\left(|\Delta^r x_{k+m}|\right)\right]^{p_k} < \infty\Big\}.$$

If $x \in \left[\hat{V}, \Delta^r, \lambda, f, p\right]_1$ then we shall write $x_k \to L\left[\hat{V}, \Delta^r, \lambda, f, p\right]_1$ and L will be called $\lambda-$strongly almost difference limit of x with respect to the modulus f.

Throughout the paper Z will denote any one of the notation 0, 1, or ∞.

In the case $f(x) = x$ and $p_k = 1$ for all $k \in \mathbb{N}$, we shall write $\left[\hat{V}, \Delta^r, \lambda\right]_Z$ and $\left[\hat{V}, \Delta^r, \lambda, f\right]_Z$ instead of $\left[\hat{V}, \Delta^r, \lambda, f, p\right]_Z$. If $x \in \left[\hat{V}, \Delta^r, \lambda\right]_1$ then we say that x is $\Delta^r_\lambda-$strongly almost convergent to L.

The proofs of the following theorems are obtained by using the known standard techniques, therefore we give them without proofs (For detail see [3, 22]).

Theorem 2.1. *Let (p_k) be bounded. Then the spaces $\left[\hat{V}, \Delta^r, \lambda, f, p\right]_Z$ are linear spaces over the set of complex numbers $\mathbb{C}$.*

Theorem 2.2. *Let the sequence $p = (p_k)$ be bounded and f be a modulus function , then*

$$\left[\hat{V}, \Delta^r, \lambda, f, p\right]_0 \subset \left[\hat{V}, \Delta^r, \lambda, f, p\right]_1 \subset \left[\hat{V}, \Delta^r, \lambda, f, p\right]_\infty .$$

Theorem 2.3. *If $r \geq 1$, then the inclusion $\left[\hat{V}, \Delta^{r-1}, \lambda, f\right]_Z \subset \left[\hat{V}, \Delta^r, \lambda, f\right]_Z$ is strict. In general $\left[\hat{V}, \Delta^i, \lambda, f\right]_Z \subset \left[\hat{V}, \Delta^r, \lambda, f\right]_Z$ for all $i = 1, 2, \ldots, r-1$ and the inclusion is strict.*

Proof. We give the proof for $Z = \infty$ only. It can be proved in a similar way for $Z = 0, 1$. Let $x \in \left[\hat{V}, \Delta^{r-1}, \lambda, f\right]_\infty$. Then we have

$$\sup_{m,n} \frac{1}{\lambda_n} \sum_{k \in I_n} f\left(\left|\Delta^{r-1} x_{k+m}\right|\right) < \infty.$$

By definition of f, we have

$$\frac{1}{\lambda_n} \sum_{k \in I_n} f\left(\left|\Delta^r x_{k+m}\right|\right) \leq \frac{1}{\lambda_n} \sum_{k \in I_n} f\left(\left|\Delta^{r-1} x_{k+m}\right|\right) + \frac{1}{\lambda_n} \sum_{k \in I_n} f\left(\left|\Delta^{r-1} x_{k+m+1}\right|\right) < \infty.$$

Thus $\left[\hat{V}, \Delta^{r-1}, \lambda, f\right]_\infty \subset \left[\hat{V}, \Delta^r, \lambda, f\right]_\infty$. Proceeding in this way one will have $\left[\hat{V}, \Delta^i, \lambda, f\right]_\infty \subset \left[\hat{V}, \Delta^r, \lambda, f\right]_\infty$ for $i = 1, 2, \ldots, r-1$. Let $\lambda_n = n$ for all $n \in \mathbb{N}$, then the sequence $x = (k^r)$, for example, belongs to $\left[\hat{V}, \Delta^r, \lambda, f\right]_\infty$, but does not belong to $\left[\hat{V}, \Delta^{r-1}, \lambda, f\right]_\infty$ for $f(x) = x$. (If $x = (k^r)$, then $\Delta^r x_k = (-1)^r r!$ and $\Delta^{r-1} x_k = (-1)^{r+1} r!(k + \frac{(r-1)}{2})$ for all $k \in \mathbb{N}$). ∎

The proof of the following result is a routine work.

Proposition 2.4. $\left[\hat{V}, \Delta^{r-1}, \lambda, f\right]_1 \subset \left[\hat{V}, \Delta^r, \lambda, f\right]_0$.

Theorem 2.5. *Let f_1, f_2 be modulus functions. Then we have*

i) $\left[\hat{V}, \Delta^r, \lambda, f_1\right]_Z \subset \left[\hat{V}, \Delta^r, \lambda, f_1 \circ f_2\right]_Z$,

ii) $\left[\hat{V}, \Delta^r, \lambda, f_1, p\right]_Z \cap \left[\hat{V}, \Delta^r, \lambda, f_2, p\right]_Z \subset \left[\hat{V}, \Delta^r, \lambda, f_1 + f_2, p\right]_Z$.

Proof. Omitted. ∎

The following result is a consequence of Theorem 2.5 (i).

Proposition 2.6. *Let f be a modulus function. Then* $[\hat{V}, \Delta^r, \lambda]_Z \subset [\hat{V}, \Delta^r, \lambda, f]_Z$.

Theorem 2.7. *The sequence spaces* $[\hat{V}, \Delta^r, \lambda, f, p]_0$, $[\hat{V}, \Delta^r, \lambda, f, p]_1$ *and* $\left[\hat{V}, \Delta^r, \lambda, f, p\right]_\infty$ *are not solid for* $r \geq 1$.

Proof. Let $p_k = 1$ for all k, $f(x) = x$ and $\lambda_n = n$ for all $n \in \mathbb{N}$. Then $(x_k) = (k^r) \in \left[\hat{V}, \Delta^r, \lambda, f, p\right]_\infty$ but $(\alpha_k x_k) \notin \left[\hat{V}, \Delta^r, \lambda, f, p\right]_\infty$ when $\alpha_k = (-1)^k$ for all $k \in \mathbb{N}$. Hence $\left[\hat{V}, \Delta^r, \lambda, f, p\right]_\infty$ is not solid. The other cases can be proved by considering similar examples. ∎

From the above theorem we may give the following corollary.

Corollary 2.8. *The sequence spaces* $\left[\hat{V}, \Delta^r, \lambda, f, p\right]_0$, $\left[\hat{V}, \Delta^r, \lambda, f, p\right]_1$ *and* $\left[\hat{V}, \Delta^r, \lambda, f, p\right]_\infty$ *are not perfect for* $r \geq 1$.

Theorem 2.9. *The sequence spaces* $\left[\hat{V}, \Delta^r, \lambda, f, p\right]_1$ *and* $\left[\hat{V}, \Delta^r, \lambda, f, p\right]_\infty$ *are not symmetric for* $r \geq 1$.

Proof. Let $p_k = 1$ for all k, $f(x) = x$ and $\lambda_n = n$ for all $n \in \mathbb{N}$. Then $(x_k) = (k^r) \in [\hat{V}, \Delta^r, \lambda, f, p]_\infty$. Let (y_k) be a rearrangement of (x_k), which is defined as follows:

$(y_k) = \{x_1, x_2, x_4, x_3, x_9, x_5, x_{16}, x_6, x_{25}, x_7, x_{36}, x_8, x_{49}, x_{10}, ...\}$. Then $(y_k) \notin [\hat{V}, \Delta^r, \lambda, f, p]_\infty$. ∎

Remark. The space $[\hat{V}, \Delta^r, \lambda, f, p]_0$ is not symmetric for $r \geq 2$.

Theorem 2.10. *The sequence spaces* $[\hat{V}, \Delta^r, \lambda, f, p]_Z$ *are not sequence algebras.*

Proof. Let $p_k = 1$ for all $k \in \mathbb{N}$, $f(x) = x$ and $\lambda_n = n$ for all $n \in \mathbb{N}$. Then $x = (k^{r-2})$, $y = (k^{r-2}) \in [\hat{V}, \Delta^r, \lambda, f, p]_Z$, but $x.y \in [\hat{V}, \Delta^r, \lambda, f, p]_Z$. ∎

3. Statistical Convergence

The notion of statistical convergence was introduced by Fast [7] and studied by various authors ([1, 9, 24, 25]).

In this section we define Δ^r_λ−almost statistically convergent sequences and give some inclusion relations between $\hat{s}(\Delta^r_\lambda)$ and $\left[\hat{V}, \Delta^r, \lambda, f, p\right]_1$.

Definition 2. *A sequence $x=(x_k)$ is said to be Δ_λ^r−almost statistically convergent to the number L if for every $\varepsilon>0$,*

$$\lim_n \frac{1}{\lambda_n}\left|\{k\in I_n : |\Delta^r x_{k+m}-L|\geq\varepsilon\}\right|=0, \text{ uniformly in } m.$$

In this case we write $\hat{s}(\Delta_\lambda^r)-\lim x=L$ or $x_k\to L\hat{s}(\Delta_\lambda^r)$.

In the case $\lambda_n=n$ we shall write $\hat{s}(\Delta^r)$ instead of $\hat{s}(\Delta_\lambda^r)$.

The proof of the following theorem is easily obtained by using the same techniques of Theorem 2 in Savas [25], therefore we give it without proof.

Theorem 3.1. *Let $\lambda=(\lambda_n)$ be the same as in Sec. 1, then*

i) *If* $x_k\to L\left[\hat{V},\Delta^r,\lambda\right]_1 \Rightarrow x_k\to L\hat{s}(\Delta_\lambda^r)$,

ii) *If* $x\in\ell_\infty(\Delta^r)$ *and* $x_k\to L\hat{s}(\Delta_\lambda^r)$, *then* $x_k\to L\left[\hat{V},\Delta^r,\lambda\right]_1$,

iii) $\hat{s}(\Delta_\lambda^r)\cap\ell_\infty(\Delta^r)=\left[\hat{V},\Delta^r,\lambda\right]_1\cap\ell_\infty(\Delta^r)$.

Theorem 3.2. $\hat{s}(\Delta^r)\subseteq\hat{s}(\Delta_\lambda^r)$ *if and only if* $\liminf_n\frac{\lambda_n}{n}>0$.

Proof. The sufficiency part of the proof can be obtained using the same technique as the sufficiency part of the proof of Theorem 3 in Savas [25].

For the necessity suppose that $\liminf_n\frac{\lambda_n}{n}=0$. As in ([8], p.510) we can choose a subsequence $(n(j))$ such that $\frac{\lambda_{n(j)}}{n(j)}<\frac{1}{j}$. Define $x=(x_i)$ such that

$$\Delta^r x_i=\begin{cases}1, & \text{if } i\in I_n(j),\ j=1,2,...\\ 0, & \text{otherwise.}\end{cases}$$

Then $x\in[\hat{c}](\Delta^r)$ and by [4, Theorem 3.1 (i)], $x\in\hat{s}(\Delta^r)$. But $x\notin\left[\hat{V},\Delta^r,\lambda\right]_1$ and Theorem 3.1 (ii) implies that $x\notin\hat{s}(\Delta_\lambda^r)$. This completes the proof. ■

Theorem 3.3. *Let f be a modulus function and $\sup_k p_k=H$. Then $[\hat{V},\Delta^r,\lambda,f,p]_1\subset\hat{s}(\Delta_\lambda^r)$.*

Proof. Let $x\in[\hat{V},\Delta^r,\lambda,f,p]_1$ and $\varepsilon>0$ be given. Let Σ_1 denote the sum over $k\leq n$ such that $|\Delta^r x_{k+m}-L|\geq\varepsilon$ and Σ_2 denote the sum over $k\leq n$ such that $|\Delta^r x_{k+m}-L|<\varepsilon$. Then

$$\begin{aligned}
&\frac{1}{\lambda_n}\sum_{k\in I_n}\left[f\left(\left|\Delta^r x_{k+m}-L\right|\right)\right]^{p_k}\\
&=\frac{1}{\lambda_n}\sum_{1}\left[f\left(\left|\Delta^r x_{k+m}-L\right|\right)\right]^{p_k}+\frac{1}{\lambda_n}\sum_{2}\left[f\left(\left|\Delta^r x_{k+m}-L\right|\right)\right]^{p_k}\\
&\geq\frac{1}{\lambda_n}\sum_{1}\left[f\left(\left|\Delta^r x_{k+m}-L\right|\right)\right]^{p_k}\\
&\geq\frac{1}{\lambda_n}\sum_{1}\left[f\left(\varepsilon\right)\right]^{p_k}\\
&\geq\frac{1}{\lambda_n}\sum_{1}\min\left(\left[f\left(\varepsilon\right)\right]^{\inf p_k},\left[f\left(\varepsilon\right)\right]^{H}\right)\\
&\geq\frac{1}{\lambda_n}\left|\left\{k\in I_n:\left|\Delta^r x_{k+m}-L\right|\geq\varepsilon\right\}\right|\min\left(\left[f\left(\varepsilon\right)\right]^{\inf p_k},\left[f\left(\varepsilon\right)\right]^{H}\right).
\end{aligned}$$

Hence $x\in\hat{s}(\Delta^r_\lambda)$. ■

Theorem 3.4. *Let f be bounded and $0<h=\inf_k p_k\leq p_k\leq\sup_k p_k=H<\infty$. Then $\hat{s}(\Delta^r_\lambda)\subset\left[\hat{V},\Delta^r,\lambda,f,p\right]_1$.*

Proof. Suppose that f is bounded. Let $\varepsilon>0$ and Σ_1 and Σ_2 be denoted in the previous theorem. Since f is bounded there exists an integer K such that $f(x)<K$, for all $x\geq 0$. Then

$$\begin{aligned}
&\frac{1}{\lambda_n}\sum_{k\in I_n}\left[f\left(\left|\Delta^r x_{k+m}-L\right|\right)\right]^{p_k}\\
&=\frac{1}{\lambda_n}\sum_{1}\left[f\left(\left|\Delta^r x_{k+m}-L\right|\right)\right]^{p_k}+\frac{1}{\lambda_n}\sum_{2}\left[f\left(\left|\Delta^r x_{k+m}-L\right|\right)\right]^{p_k}\\
&\leq\frac{1}{\lambda_n}\sum_{1}\max\left(K^h,K^H\right)+\frac{1}{\lambda_n}\sum_{2}\left[f\left(\varepsilon\right)\right]^{p_k}\\
&\leq\max\left(K^h,K^H\right)\frac{1}{\lambda_n}\left|\left\{k\in I_n:\left|\Delta^r x_{k+m}-L\right|\geq\varepsilon\right\}\right|\\
&\quad+\max\left(f(\varepsilon)^h,\ f(\varepsilon)^H\right).
\end{aligned}$$

Hence $x\in\left[\hat{V},\Delta^r,\lambda,f,p\right]_1$. ■

Theorem 3.5. *Let f be bounded and $0<h=\inf_k p_k\leq p_k\leq\sup_k p_k=H<\infty$. We have $\hat{s}(\Delta^r_\lambda)=[\hat{V},\Delta^r,\lambda,f,p]_1$ if and only if f is bounded .*

Proof. Let f be bounded. By the Theorem 3.3 and Theorem 3.4 we have $\hat{s}(\Delta^r_\lambda)=[\hat{V},\Delta^r,\lambda,f,p]_1$.

Conversely, suppose that f is unbounded. Then there exists a positive sequence (t_k) with $f(t_k)=k^2$, for $k=1,2,\ldots.$ If we choose

$$\Delta^r x_i = \begin{cases} t_k, & i = k^2, \quad i = 1, 2, \dots \\ 0, & \text{otherwise} \end{cases}.$$

Then we have

$$\frac{1}{\lambda_n} \left|\{k \in I_n : |\Delta^r x_{k+m}| \geq \varepsilon\}\right| \leq \frac{\sqrt{\lambda_{n-1}}}{\lambda_n} \text{ for all } n \text{ and } m$$

and so $x \in \hat{s}(\Delta^r_\lambda)$, but $x \notin \left[\hat{V}, \Delta^r, \lambda, f, p\right]_1$. This contradicts to $\hat{s}(\Delta^r_\lambda) = \left[\hat{V}, \Delta^r, \lambda, f, p\right]$. ■

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