Vietnam Journal of Mathematics **34**:4 (2006) 495–499

Vietnam Journal of MATHEMATICS

Short Communications

Controllability Radii and Stabilizability Radii of Time-Invariant Linear Systems

D. C. Khanh and D. D. X. Thanh

Dept. of Info. Tech. and App. Math., Ton Duc Thang University
98 Ngo Tat To St., Binh Thanh Dist., Ho Chi Minh City, Vietnam

Dedicated to Professor Do Long Van on the occasion of his 65^{th} birthday

Received August 15, 2006

2000 Mathematics Subject Classification: 34K06, 93C73, 93D09.
Keywords: Controllability radii, stabilizability radii.

1. Introduction

Consider the system

$$\dot{x} = Ax + Bu, \tag{1.1}$$

where $A \in \mathbb{C}^{n\times m}$, $B \in \mathbb{C}^{m\times n}$. Some researchers, such as in [2 - ,4], did research on the system when both matrices A and B are subjected to perturbation:

$$\dot{x} = (A + \Delta_A)x + (B + \Delta_B)u. \tag{1.2}$$

In this paper, we get the formulas of controllability radii in Sec. 2, and stabilizability radii in Sec. 3 *for arbitrary operator norm* when both A and B as well as only A or B is perturbed. This means we also concern the perturbed systems:

$$\dot{x} = (A + \Delta_A)x + Bu, \tag{1.3}$$

or

$$\dot{x} = Ax + (B + \Delta_B)u. \tag{1.4}$$

The stabilizability radii when the system (1.1) is already stabilized by a given feedback $u = Fx$ is studied in the end of Sec. 3. And we also answer for the question *whether the system* (1.1) *is also stabilized by perturbed feedback* $u = (F + \Delta_F)x$ *for some* Δ_F.

Let M be a matrix in $\mathbb{C}^{k\times n}$, we denote the smallest singular value of M by $\sigma_{\min}(M)$, the spectrum by $\sigma(M)$. The following lemma is the key to obtain the results of this paper.

Lemma 1.1. *Given* $A \in \mathbb{C}^{m\times n}$ *and* $B \in \mathbb{C}^{k\times n}$ *satisfying* $\operatorname{rank}\begin{pmatrix} A \\ B \end{pmatrix} = n$, *we have*

$$\inf_{\Delta\in\mathbb{C}^{k\times n}} \left\{ \|\Delta\| : \operatorname{rank}\begin{pmatrix} A \\ B+\Delta \end{pmatrix} < n \right\} = \min_{\substack{x\in\operatorname{Ker}A \\ \|x\|=1}} \|Bx\|.$$

A matrix $K \in \mathbb{C}^{k\times n}$ is said to represent a subspace V of $\mathbb{C}^{k\times n}$ if the following conditions is satisfied:

- $V = \operatorname{Im}(K)$,
- $\|y\| = 1 \Leftrightarrow \|Ky\| = 1$.

For example, with spectral norm, K is the matrix the columns of which are the normal orthogonal basis of V.

Remark 1. For convenience on computing with spectral norm, Lemma 1.1 can be rewritten as

$$\inf_{\Delta\in\mathbb{C}^{k\times n}} \left\{ \|\Delta\|_2 : \operatorname{rank}\begin{pmatrix} A \\ B+\Delta \end{pmatrix} < n \right\} = \sigma_{\min}(BK^A),$$

where K^A is the matrix representing $\operatorname{Ker}A$

2. Controllability Radii

The controllability radii of system (1.1) with the perturbation on:

- both A and B are defined by

$$r_{AB} = \inf_{(\Delta_A\ \ \Delta_B)\in\mathbb{C}^{n\times(n+m)}} \{\|(\Delta_A\ \ \Delta_B)\| : \text{the system (1.2) is uncontrollable}\},$$

- only A is defined by

$$r_A = \inf_{\Delta_A\in\mathbb{C}^{n\times n}} \{\|\Delta_A\| : \text{the system (1.3) is uncontrollable}\},$$

- only B is defined by

$$r_B = \inf_{\Delta_B\in\mathbb{C}^{n\times m}} \{\|\Delta_B\| : \text{the system (1.4) is uncontrollable}\}.$$

Theorem 2.1. *The formulas of controllability radii of system* (1.1) *are*

$$r_{AB} = \min_{\lambda\in\mathbb{C}} \min_{\|x\|=1} \|(A-\lambda I\ \ B)x\|,$$

$$r_A = \min_{\lambda\in\mathbb{C}} \min_{\substack{x\in\operatorname{Ker}B^* \\ \|x\|=1}} \|(A^*-\lambda I)x\|,$$

$$r_B = \min_{\lambda\in\mathbb{C}} \min_{\substack{x\in\operatorname{Ker}(A^*-\lambda I) \\ \|x\|=1}} \|B^*x\|.$$

Remark 2. The spectral norm version of Theorem 2.1 is

$$r_{AB} = \min_{\lambda\in\mathbb{C}} \sigma_{\min}(A - \lambda I \ \ B),$$
$$r_{A} = \min_{\lambda\in\mathbb{C}} \sigma_{\min}\left[(A^* - \lambda I)K^B\right],$$
$$r_{B} = \min_{\lambda\in\sigma(A)} \sigma_{\min}(B^* K^\lambda),$$

where K^B and K^λ are the matrices representing Ker$B$ and Ker$(A^* - \lambda I)$, and the formula of r_{AB} is the result obtained in [4].

By the definitions, it is clear that $r_{AB} \le \min\{r_A, r_B\}$, and the strict inequality may happen as in the case of following system:

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} u, \tag{2.1}$$

Applying Remark 2, we obtain

$$r_{AB} = \sqrt{2},\ r_A = +\infty,\ r_B = \sqrt{\frac{5}{2}}.$$

3. StabilizabilityRadii

By the same definitions and proofs as the controllability radii, we get:

Theorem 3.1. *The formalas of stabilizability radii of system* (1.1) *are*

$$r_{AB} = \min_{\lambda\in\overline{C}_+} \min_{\|x\|=1} \|(A - \lambda I \ \ B)x\|,$$
$$r_{A} = \min_{\lambda\in\overline{C}_+} \min_{\substack{x\in \mathrm{Ker}B^* \\ \|x\|=1}} \|(A^* - \lambda I)x\|,$$
$$r_{B} = \min_{\lambda\in\overline{C}_+} \min_{\substack{x\in \mathrm{Ker}(A^*-\lambda I) \\ \|x\|=1}} \|B^* x\|,$$

where $\overline{C}_+$ is the closed right haft complex plane.

Remark 3. The spectral norm version of Theorem 3.1 can be constructed as in Remark 2 and the inequality $r_{AB} \le \min\{r_A, r_B\}$ may also happen strictly.

Now, we assume the system (1.1) is really stabilizable by matrix $F \in \mathbb{C}^{m\times n}$. That means the system

$$\dot{x} = (A + BF)x \tag{3.1}$$

is stable, and we concern following pertubed systems:

$$\dot{x} = [(A + \Delta_A) + (B + \Delta_B)F]x, \tag{3.2}$$

$$\dot{x} = [(A + \Delta_A) + BF]x, \tag{3.3}$$

$$\dot{x} = [A + (B + \Delta_B)F]x, \tag{3.4}$$

$$\dot{x} = [A + B(F + \Delta_F)]x, \tag{3.5}$$

The stabilizability radii of system (3.1) of the feedback matrix F with the pertubation on

- both A and B are defined by

$$r_{AB} = \inf_{(\Delta_A \ \Delta_B) \in \mathbb{C}^{n\times(n+m)}} \{\|(\Delta_A \ \Delta_B)\| : \text{the system (3.2) is unstable}\},$$

- only A is defined by

$$r_A = \inf_{\Delta_A \in \mathbb{C}^{n\times n}} \{\|\Delta_A\| : \text{the system (3.3) is unstable}\},$$

- only B is defined by

$$r_B = \inf_{\Delta_B \in \mathbb{C}^{n\times m}} \{\|\Delta_B\| : \text{the system (3.4) is unstable}\},$$

- only F is defined by

$$r_F = \inf_{\Delta_F \in \mathbb{C}^{m\times n}} \{\|\Delta_F\| : \text{the system (3.5) is unstable}\}.$$

Theorem 3.2. *The formulas of stabilizability radii of system* (3.1) *of the feedback matrix F are*

$$\begin{aligned} r_{AB} &= \min_{\lambda\in\overline{C}_+} \left\| \begin{bmatrix} I \\ F \end{bmatrix} [\lambda I - A - BF]^{-1} \right\|^{-1}, \\ r_A &= \min_{\lambda\in\overline{C}_+} \|(\lambda I - A - BF)^{-1}\|^{-1}, \\ r_B &= \min_{\lambda\in\overline{C}_+} \|F[\lambda I - A - BF]^{-1}\|^{-1}, \\ r_F &= \min_{\lambda\in\overline{C}_+} \|[\lambda I - A - BF]^{-1}B\|^{-1}. \end{aligned}$$

From the r_F, it is clear to see that there is so much matrix F making the system (1.1) stabilizable. And a open question appear: "Which F makes r_{AB}, r_A, or r_B maximum?". For apart result of this question, see [5].

Remark 4. The spectral norm vestion of Theorem 3.2 is

$$\begin{aligned} r_{AB} &= \min_{\lambda\in\overline{C}_+} \sigma_{\min}\left(\begin{bmatrix} I \\ F \end{bmatrix} [\lambda I - A - BF]^{-1} \right), \\ r_A &= \min_{\lambda\in\overline{C}_+} \sigma_{\min}\big((\lambda I - A - BF)^{-1}\big), \\ r_B &= \min_{\lambda\in\overline{C}_+} \sigma_{\min}\big(F[\lambda I - A - BF]^{-1}\big), \\ r_F &= \min_{\lambda\in\overline{C}_+} \sigma_{\min}\big([\lambda I - A - BF]^{-1}B\big). \end{aligned}$$

The inequality $r_{AB} \le \min\{r_A, r_B\}$ may happen strictly as in the case of following system:

$$\dot{x} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} u. \tag{3.6}$$

It easy to see that the system (3.6) is not stable, but stabilized by $F = Id_2$. Applying Remark 4 we obtain

$$r_{AB} = \sqrt{2},\ r_A = 2,\ r_B = 2,\ r_F = 1.$$

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