A Note on Maximal Nonhamiltonian Burkard–Hammer Graphs

Ngo Dac Tan

Institute of Mathematics, 18 Hoang Quoc Viet Road, 10307 Hanoi, Vietnam

Dedicated to Professor Do Long Van on the occasion of his 65th birthday

Received February 22, 2006

Abstract. A graph $G = (V, E)$ is called a split graph if there exists a partition $V = I \cup K$ such that the subgraphs $G[I]$ and $G[K]$ of $G$ induced by $I$ and $K$ are empty and complete graphs, respectively. In 1980, Burkard and Hammer gave a necessary condition for a split graph $G$ with $|I| < |K|$ to be hamiltonian. We will call a split graph $G$ with $|I| < |K|$ satisfying this condition a Burkard–Hammer graph. Further, a split graph $G$ is called a maximal nonhamiltonian split graph if $G$ is nonhamiltonian but $G + uv$ is hamiltonian for every $uv \not\in E$ where $u \in I$ and $v \in K$. In an earlier work, the author and Iamjaroen have asked whether every maximal nonhamiltonian Burkard–Hammer graph $G$ with the minimum degree $\delta(G) \geq |I| - k$ where $k \geq 3$ possesses a vertex adjacent to all vertices of $G$ and whether every maximal nonhamiltonian Burkard–Hammer graph $G$ with $\delta(G) = |I| - k$ where $k > 3$ and $|I| > k + 2$ possesses a vertex with exactly $k - 1$ neighbors in $I$. The first question and the second one have been proved earlier to have a positive answer for $k = 3$ and $k = 4$, respectively. In this paper, we give a negative answer both to the first question for all $k \geq 4$ and to the second question for all $k \geq 5$.

2000 Mathematics Subject Classification: 05C45.

Keywords: Split graph, Burkard–Hammer condition, Burkard–Hammer graph, hamiltonian graph, maximal nonhamiltonian split graph.

1. Introduction

All graphs considered in this paper are finite undirected graphs without loops or multiple edges. If $G$ is a graph, then $V(G)$ and $E(G)$ (or $V$ and $E$ in short)
will denote its vertex-set and its edge-set, respectively. For a subset \( W \subseteq V(G) \), the set of all neighbors of \( W \) is denoted by \( N_G(W) \) or \( N(W) \) in short. For a vertex \( v \in V(G) \), the degree of \( v \), denoted by \( \deg(v) \), is the number \( |N(v)| \). The minimum degree of \( G \), denoted by \( \delta(G) \), is the number \( \min\{\deg(v) \mid v \in V(G)\} \). By \( N_{G,W}(v) \) or \( N_W(v) \) in short we denote the set \( W \cap N_G(v) \). The subgraph of \( G \) induced by \( W \) is denoted by \( G[W] \). Unless otherwise indicated, our graph-theoretic terminology will follow [1].

A graph \( G = (V, E) \) is called a \textit{split graph} if there exists a partition \( V = I \cup K \) such that the subgraphs \( G[I] \) and \( G[K] \) of \( G \) induced by \( I \) and \( K \) are empty and complete graphs, respectively. We will denote such a graph by \( S(I \cup K, E(G)) \) or \( S(I \cup K, E) \) in short. Further, a split graph \( G = S(I \cup K, E) \) is called a \textit{complete split graph} if every \( u \in I \) is adjacent to every \( v \in K \). The notion of split graphs was introduced in 1977 by Földes and Hammer [4]. These graphs are interesting because they are related to many problems in combinatorics (see [3, 5, 10]) and in computer science (see [6, 7]).

In 1980, Burkard and Hammer gave a necessary condition for a split graph \( G = S(I \cup K, E) \) with \( |I| < |K| \) to be hamiltonian [2] (see Sec. 2 for more detail). We will call this condition the \textit{Burkard–Hammer condition}. Also, we will call a split graph \( G = S(I \cup K, E) \) with \( |I| < |K| \), which satisfies the Burkard–Hammer condition, a \textit{Burkard–Hammer graph}.

Thus, by [2] any hamiltonian split graph \( G = S(I \cup K, E) \) with \( |I| < |K| \) is a Burkard–Hammer graph. In general, the converse is not true. The first nonhamiltonian Burkard–Hammer graph has been indicated in [2]. Further infinite families of nonhamiltonian Burkard–Hammer graphs have been constructed recently in [13].

A split graph \( G = S(I \cup K, E) \) is called a \textit{maximal nonhamiltonian split graph} if \( G \) is nonhamiltonian but the graph \( G + uv \) is hamiltonian for every \( uv \notin E \) where \( u \in I \) and \( v \in K \). It is known from a result in [12] that any nonhamiltonian Burkard–Hammer graph is contained in a maximal nonhamiltonian Burkard–Hammer graph. So knowledge about maximal nonhamiltonian Burkard–Hammer graphs provides us certain information about nonhamiltonian Burkard–Hammer graphs.

It has been shown in [12] (see Theorem 2 in Sec. 2) that there are no nonhamiltonian Burkard–Hammer graphs \( G = S(I \cup K, E) \) with \( \delta(G) \geq |I| - 2 \) and no nonhamiltonian Burkard–Hammer graphs \( G = S(I \cup K, E) \) with \( \delta(G) = |I| - 3 \) and \( |I| > 5 \). Therefore, without loss of generality we may assume that all considered in this paper maximal nonhamiltonian Burkard–Hammer graphs \( G = S(I \cup K, E) \) have \( \delta(G) = |I| - k \) where \( |I| \geq 3 \) and all considered maximal nonhamiltonian Burkard–Hammer graphs \( G = S(I \cup K, E) \) with \( \delta(G) = |I| - k \) and \( |I| > k + 2 \) have \( k > 3 \).

It has been proved recently in [14] that a maximal nonhamiltonian Burkard–Hammer graph \( G = S(I \cup K, E) \) with \( \delta(G) = |I| - k \) where \( |I| \geq 3 \) must have \( |I| \geq k + 2 \) and no vertices with exactly \( k + 1, \ldots, |I| - 1 \) neighbors in \( I \). Moreover, if \( G = S(I \cup K, E) \) has \( \delta(G) = |I| - k \) where \( k > 3 \) and \( |I| > k + 2 \), then \( G \) also has no vertices with exactly \( k \) neighbors in \( I \). However, it is shown in [14] that for every integer \( k > 3 \) and every integer \( m > k + 2 \) there
exists a maximal nonhamiltonian Burkard–Hammer graph $G = S(I \cup K, E)$ with $|I| = m$ and $\delta(G) = |I| - k$ which possesses a vertex with exactly $k - 1$ neighbors in $I$. Ngo Dac Tan and Iamjaroen have asked in [14] whether all maximal nonhamiltonian Burkard–Hammer graphs $G = S(I \cup K, E)$ with $\delta(G) = |I| - k$ where $k \geq 3$ possess a vertex adjacent to all vertices of $G$ and whether all maximal nonhamiltonian Burkard–Hammer graphs $G = S(I \cup K, E)$ with $\delta(G) = |I| - k$ where $k > 3$ and $|I| > k + 2$ possess a vertex with exactly $k - 1$ neighbors in $I$. The first question has been proved in [12] to have a positive answer for $k = 3$. Recently, Ngo Dac Tan and Iamjaroen have proved in [14] that the second question also has a positive answer for $k = 4$. In this paper, however, we will give a negative answer both to the first question for all $k \geq 4$ and to the second question for all $k \geq 5$.

We would like to note that there is an interesting discussion about the Burkard–Hammer condition in [9]. Concerning the hamiltonian problem for split graphs, the readers can see also [8] and [11].

2. Preliminaries

Let $G = S(I \cup K, E)$ be a split graph and $I' \subseteq I$, $K' \subseteq K$. Denote by $B_G(I' \cup K', E')$ the graph $G[I' \cup K'] - E(G[K'])$. It is clear that $G' = B_G(I' \cup K', E')$ is a bipartite graph with the bipartition subsets $I'$ and $K'$. So we will call $B_G(I' \cup K', E')$ the bipartite subgraph of $G$ induced by $I'$ and $K'$. For a component $G'_j = B_G(I'_j \cup K'_j, E'_j)$ of $G' = B_G(I' \cup K', E')$ we define

$$k_G(G'_j) = k_G(I'_j, K'_j) = \begin{cases} |I'_j| - |K'_j| & \text{if } |I'_j| > |K'_j|, \\ 0 & \text{otherwise}. \end{cases}$$

If $G' = B_G(I' \cup K', E')$ has $r$ components $G'_1 = B_G(I'_1 \cup K'_1, E'_1), \ldots, G'_r = B_G(I'_r \cup K'_r, E'_r)$ then we define

$$k_G(G') = k_G(I', K') = \sum_{j=1}^r k_G(G'_j).$$

A component $G'_j = B_G(I'_j \cup K'_j, E'_j)$ of $G' = B_G(I' \cup K', E')$ is called a T-component (resp., H-component, L-component) if $|I'_j| > |K'_j|$ (resp., $|I'_j| = |K'_j|$). Let $h_G(G') = h_G(I', K')$ denote the number of H-components of $G'$.

In 1980, Burkard and Hammer proved the following necessary but not sufficient condition for hamiltonian split graphs [2].

**Theorem 1.** [2] Let $G = S(I \cup K, E)$ be a split graph with $|I| < |K|$. If $G$ is hamiltonian, then

$$k_G(I', K') + \max\left\{1, \frac{h_G(I', K')}{2}\right\} \leq |N_G(I')| - |K'|$$

holds for all $\emptyset \neq I' \subseteq I$, $K' \subseteq N_G(I')$ with $(k_G(I', K'), h_G(I', K')) \neq (0, 0)$.
We will shortly call the condition in Theorem 1 the Burkard–Hammer condition. We also call a split graph \( G = S(I \cup K, E) \) with \( |I| < |K| \), which satisfies the Burkard–Hammer condition, a Burkard–Hammer graph. Thus, by Theorem 1 any hamiltonian split graph \( G = S(I \cup K, E) \) with \( |I| < |K| \) is a Burkard–Hammer graph. For split graphs \( G = S(I \cup K, E) \) with \( |I| < |K| \) and \( \delta(G) \geq |I| - 2 \) the converse is true [12]. But it is not true in general. The first example of a nonhamiltonian Burkard–Hammer graph has been indicated in [2]. Recently, Tan and Hung [12] have classified nonhamiltonian Burkard–Hammer graphs \( G = S(I \cup K, E) \) with \( \delta(G) = |I| - 3 \). Namely, they have proved the following result.

**Theorem 2.** [12] Let \( G = S(I \cup K, E) \) be a split graph with \( |I| < |K| \) and the minimum degree \( \delta(G) \geq |I| - 3 \). Then

(i) If \( |I| \neq 5 \) then \( G \) has a Hamilton cycle if and only if \( G \) satisfies the Burkard–Hammer condition;

(ii) If \( |I| = 5 \) and \( G \) satisfies the Burkard–Hammer condition, then \( G \) has no Hamilton cycles if and only if \( G \) is isomorphic to one of the graphs \( H^{1,n} \), \( H^{2,n} \), \( H^{3,n} \) or \( H^{4,n} \) listed in Table 1.

<table>
<thead>
<tr>
<th>The graph ( G )</th>
<th>The vertex-set ( V(G) = I^* \cup K^* )</th>
<th>The edge-set ( E(G) = E_1^* \cup \cdots \cup E_5^* \cup E_{K^*} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H^{1,n} ) ((n &gt; 5))</td>
<td>( I^* = {u_1^<em>, u_2^</em>, u_3^<em>, v_1^</em>, v_2^*} )</td>
<td>( E_1^* = {u_1^<em>v_1^</em>, u_1^<em>v_2^</em>, u_2^<em>v_2^</em>} )</td>
</tr>
<tr>
<td>( H^{2,n} )</td>
<td>( V(H^{2,n}) = V(H^{1,n}) )</td>
<td>( E(H^{2,n}) = E(H^{1,n}) \cup {u_1^<em>v_2^</em>} )</td>
</tr>
<tr>
<td>( H^{3,n} )</td>
<td>( V(H^{3,n}) = V(H^{1,n}) )</td>
<td>( E(H^{3,n}) = E(H^{1,n}) \cup {u_5^<em>v_2^</em>} )</td>
</tr>
<tr>
<td>( H^{4,n} )</td>
<td>( V(H^{4,n}) = V(H^{1,n}) )</td>
<td>( E(H^{4,n}) = E(H^{1,n}) \cup {u_1^<em>v_2^</em>, u_5^<em>v_2^</em>} )</td>
</tr>
</tbody>
</table>

Table 1. The graphs \( H^{1,n}, H^{2,n}, H^{3,n} \) and \( H^{4,n} \)

Theorem 2 shows that there are only four nonhamiltonian Burkard–Hammer graphs \( G = S(I \cup K, E) \) with \( K = N(I) \) and \( \delta(G) = |I| - 3 \), namely, the graphs \( H^{1,6}, H^{2,6}, H^{3,6} \) and \( H^{4,6} \). In contrast with this result, the number of nonhamiltonian Burkard–Hammer graphs \( G = S(I \cup K, E) \) with \( K = N(I) \) and \( \delta(G) = |I| - 4 \) is infinite. This is a recent result of Tan and Iamjaroen [13]. We remind now one of the constructions in this work, which is needed for the next sections.

Let \( G_1 = S(I_1 \cup K_1, E_1) \) and \( G_2 = S(I_2 \cup K_2, E_2) \) be split graphs with \( V(G_1) \cap V(G_2) = \emptyset \).
and \( v \) be a vertex of \( K_1 \). We say that a graph \( G \) is an expansion of \( G_1 \) by \( G_2 \) at \( v \) if \( G \) is the graph obtained from \((G_1 - v) \cup G_2\) by adding the set of edges
\[
E_0 = \{x,v_j \mid x \in V(G_1) \setminus \{v\}, v_j \in K_2 \text{ and } x,v \in E_1\}.
\]
It is clear that such a graph \( G \) is a split graph \( S(I \cup K,E) \) with \( I = I_1 \cup I_2 \), \( K = (K_1 \setminus \{v\}) \cup K_2 \) and is uniquely determined by \( G_1, G_2 \) and \( v \in K_1 \). Because of this, we will denote this graph \( G \) by \( G_1[G_2,v] \). Further, a graph \( G \) is called an expansion of \( G_1 \) by \( G_2 \) if it is an expansion of \( G_1 \) by \( G_2 \) at some vertex \( v \in K_1 \).

The following results which have been proved in [12-14] are needed later.

Lemma 1. [12] Let \( G = S(I \cup K,E) \) be a Burkard–Hammer graph. Then for any \( uv \notin E \) where \( u \in I \) and \( v \in K \), the graph \( G+uv \) is also a Burkard–Hammer graph.

Theorem 3. [13] Let \( G_1 = S(I_1 \cup K_1,E_1) \) be a Burkard–Hammer graph and \( G_2 = S(I_2 \cup K_2,E_2) \) be a complete split graph with \( |I_2| < |K_2| \). Then an expansion of \( G_1 \) by \( G_2 \) is a Burkard–Hammer graph.

Theorem 4. [13] Let \( G_1 = S(I_1 \cup K_1,E_1) \) be an arbitrary split graph and \( G_2 = S(I_2 \cup K_2,E_2) \) be a split graph with \( |K_2| = |I_2| + 1 \). Then an expansion of \( G_1 \) by \( G_2 \) is a hamiltonian graph if and only if both \( G_1 \) and \( G_2 \) are hamiltonian graphs.

Let \( G = S(I \cup K,E) \) be a split graph. Set
\[
B_i(G) = \{v \in K \mid |N_I(v)| = i\}.
\]
If the graph \( G \) is clear from the context then we also write \( B_i \) instead of \( B_i(G) \).

Theorem 5. [14] Let \( G_1 = S(I_1 \cup K_1,E_1) \) be a complete split graph with \( |I_1| < |K_1| \) and \( G_2 = S(I_2 \cup K_2,E_2) \) be a maximal nonhamiltonian Burkard–Hammer graph with \( \delta(G_2) = |I_2| - k_2 \) such that every vertex \( u \in I_2 \) has \( N_{G_2}(u) \neq K_2 \). Then any expansion \( G = S(I \cup K,E) = G_1[G_2,v_1] \) where \( v_1 \in K_1 \) is a maximal nonhamiltonian Burkard–Hammer graph with \( \delta(G) = \delta(G_2) = |I| - (k_2 + |I_1|) \). Moreover, for any \( x \in K_1 \setminus \{v_1\}, |N_{G,x}(x)| = |I_1| \) and for any \( y \in K_2, |N_{G,y}(y)| = |N_{G_2,y}(y)| + |I_1| \).

3. Formulations of the Main Results and Discussions

By Theorem 2 in the previous section there are no nonhamiltonian Burkard–Hammer graphs \( G = S(I \cup K,E) \) with \( \delta(G) \geq |I| - 2 \) and no nonhamiltonian Burkard–Hammer graphs \( G = S(I \cup K,E) \) with \( \delta(G) = |I| - 3 \) and \( |I| > 5 \). Therefore, in further discussions without loss of generality we may assume that all considered maximal nonhamiltonian Burkard–Hammer graphs \( G = S(I \cup K,E) \) with \( \delta(G) = |I| - k \) have \( |I| \geq k \geq 3 \) and all considered maximal nonhamiltonian Burkard–Hammer graphs \( G = S(I \cup K,E) \) with \( \delta(G) = |I| - k \) and \( |I| > k + 2 \) have \( k > 3 \). We start our discussions with the following result proved in [14].
Theorem 6. [14] Let \( G = S(I \cup K, E) \) be a maximal nonhamiltonian Burkard–Hammer graph with the minimum degree \( \delta(G) = |I| - k \) where \(|I| \geq k \geq 3\). Then \(|I| \geq k + 2\) and \( B_{k+1} = \cdots = B_{|I|-1} = \emptyset \). Furthermore, if \( k > 3 \) and \(|I| > k + 2\) then \( B_k \) is also empty.

Two questions raised from Theorem 6 are whether a maximal nonhamiltonian Burkard–Hammer graph \( G = S(I \cup K, E) \) with \( \delta(G) = |I| - k \) where \( k \geq 3 \) must have \( B_{|I|} = \emptyset \) and whether a maximal nonhamiltonian Burkard–Hammer graph \( G = S(I \cup K, E) \) with \( \delta(G) = |I| - k \) where \( k > 3 \) and \(|I| > k + 2\) also must have \( B_{k-1} = \emptyset \). The following results proved in [14] show that both these questions have negative answers.

Theorem 7. [14]
(a) For every integer \( k \geq 3 \) there exists a maximal nonhamiltonian Burkard–Hammer graph \( G = S(I \cup K, E) \) with \(|I| = k + 2\) and \( \delta(G) = |I| - k \), which has \( B_k \neq \emptyset \) and \( B_{|I|} \neq \emptyset \).
(b) For every integer \( k > 3 \) and every integer \( m > k + 2 \) there exists a maximal nonhamiltonian Burkard–Hammer graph \( G = S(I \cup K, E) \) with \(|I| = m\) and \( \delta(G) = |I| - k \), which has \( B_{k-1}(G) \neq \emptyset \) and \( B_{|I|} \neq \emptyset \).

Two natural questions raised from the results in Theorem 7 are whether every maximal nonhamiltonian Burkard–Hammer graph \( G = S(I \cup K, E) \) with \( \delta(G) = |I| - k \) where \( k \geq 3 \) has \( B_{|I|} \neq \emptyset \) and whether every maximal nonhamiltonian Burkard–Hammer graph \( G = S(I \cup K, E) \) with \( \delta(G) = |I| - k \) where \( k > 3 \) and \(|I| > k + 2\) has \( B_{k-1} \neq \emptyset \). These questions have been posed in [14]. Theorem 2 shows that the first question has a positive answer for \( k = 3 \) and Theorem 8 below proved in [14] shows that the second question has a positive answer for \( k = 4 \). These make the questions more attractive for investigation.

Theorem 8. [14] Let \( G = S(I \cup K, E) \) be a maximal nonhamiltonian Burkard–Hammer graph with \(|I| \geq 7\) and the minimum degree \( \delta(G) = |I| - 4 \). Then \( B_4 = B_5 = \cdots = B_{|I|-1} = \emptyset \) but \( B_3 \neq \emptyset \).

In this paper, we get complete answers to the above two questions. Namely, we will prove the following results.

Theorem 9.
(a) For every integer \( k \geq 4 \) there exists a maximal nonhamiltonian Burkard–Hammer graph \( G = S(I \cup K, E) \) with \( \delta(G) = |I| - k \), which has \( B_{|I|} = \emptyset \).
(b) For every integer \( k \geq 5 \) and every integer \( m > k + 2 \) there exists a maximal nonhamiltonian Burkard–Hammer graph \( G = S(I \cup K, E) \) with \(|I| = m\) and \( \delta(G) = |I| - k \), which has \( B_{k-1} = \emptyset \) but \( B_{k-2} \neq \emptyset, B_{k-3} \neq \emptyset \) and \( B_{k-4} \neq \emptyset \).

Thus, by Theorem 9 both the first question for all \( k \geq 4 \) and the second question for all \( k \geq 5 \) have negative answers, although the former question has a positive answer for \( k = 3 \) and the latter one has a positive answer for \( k = 4 \).
4. Proof of Theorem 9

First of all we prove the following lemmas.

**Lemma 2.** Let $L = S(I(L) \cup K(L), E(L))$ be the split graph with

$$I(L) = \{u_1^*, u_2^*, \ldots, u_6^*\},$$

$$K(L) = \{v_1^*, v_2^*, \ldots, v_7^*\},$$

$$E(L) = E_1^* \cup E_2^* \cup \cdots \cup E_6^* \cup E_K^*,$$

where

$$E_1^* = \{u_1^*v_1^*, u_1^*v_2^*, u_1^*v_3^*\},$$

$$E_2^* = \{u_2^*v_2^*, u_2^*v_4^*\},$$

$$E_3^* = \{u_3^*v_3^*, u_3^*v_4^*, u_3^*v_6^*\},$$

$$E_4^* = \{u_4^*v_4^*, u_4^*v_5^*, u_4^*v_7^*\},$$

$$E_5^* = \{u_5^*v_5^*, u_5^*v_6^*, u_5^*v_7^*\},$$

$$E_6^* = \{u_6^*v_6^*, u_6^*v_7^*\},$$

$$E_K^* = \{v_i^*v_j^* \mid i \neq j; i, j \in \{1, \ldots, 7\}\}$$

(see Fig. 1). Then $L$ is a maximal nonhamiltonian Burkard–Hammer graph with $B_{|I(L)|} = \emptyset$.

![Graph L](image)

_Fig. 1. The graph L_

<table>
<thead>
<tr>
<th>Graph $L - u_i^*$</th>
<th>Hamilton cycle $C_{u_i^<em>}$ for $L - u_i^</em>$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L - u_1^*$</td>
<td>$C_{u_1^*} = u_2^*v_2^*u_5^*v_6^*u_6^*v_7^*u_4^*v_4^*u_5^*v_5^*u_3^<em>u_2^</em>$</td>
</tr>
<tr>
<td>$L - u_2^*$</td>
<td>$C_{u_2^*} = u_1^*v_1^*u_1^*v_2^*u_3^*v_3^*u_4^*v_4^*u_5^*v_5^*u_6^*v_6^*u_3^<em>u_1^</em>$</td>
</tr>
<tr>
<td>$L - u_3^*$</td>
<td>$C_{u_3^*} = u_1^*v_2^*u_2^*v_1^*u_4^*v_4^*u_5^*v_5^*u_6^*v_6^*u_2^*v_2^<em>u_1^</em>$</td>
</tr>
<tr>
<td>$L - u_4^*$</td>
<td>$C_{u_4^*} = u_1^*v_1^*u_1^*v_2^*u_3^*v_3^*u_5^*v_5^*u_6^*v_6^*u_3^*v_1^*u_2^*v_2^<em>u_1^</em>$</td>
</tr>
<tr>
<td>$L - u_5^*$</td>
<td>$C_{u_5^*} = u_1^*v_1^*u_1^*v_2^*u_3^*v_3^*u_5^*v_5^*u_6^*v_6^*u_3^*v_1^*u_2^*v_2^<em>u_1^</em>$</td>
</tr>
<tr>
<td>$L - u_6^*$</td>
<td>$C_{u_6^*} = u_1^*v_1^*u_1^*v_2^*u_3^*v_3^*u_5^*v_5^*u_6^*v_6^*u_3^*v_1^*u_2^*v_2^<em>u_1^</em>$</td>
</tr>
</tbody>
</table>
Proof. For any vertex \( u^*_i \in I(L) \), the graph \( L - u^*_i \) has a Hamilton cycle \( C_{u^*_i} \), which is shown in Table 2. Therefore, by Theorem 1 the Burkard–Hammer condition holds for any \( \emptyset \neq I' \subseteq I(L) \) and \( K' \subseteq N_L(I') \) with \( |I'| \leq 5 \) and \((k(I', K'), h(I', K')) \neq (0, 0)\). For \( I' = I(L) \) and \( K' \subseteq N_L(I(L)) \), by direct computations we can verify that the Burkard–Hammer condition also holds. (It is tedious to do this, but we don’t know other ways to verify the last assertion.) Thus, \( L \) satisfies the Burkard–Hammer condition.

Now suppose that \( L \) has a Hamilton cycle \( C \). Since \( \deg(u^*_i) = \deg(u^*_j) = 2 \), \( C \) must contain the paths \( v^*_2 u^*_1 v^*_4 \) and \( v^*_3 u^*_6 v^*_7 \). We consider separately the following possibilities for \( C \):

(i) \( v^*_2 u^*_1 v^*_3 \) is in \( C \).

In this case \( C \) must contain the path \( v^*_1 u^*_2 v^*_2 v^*_1 v^*_3 u^*_6 v^*_7 \). So both \( v^*_2 u^*_1 \) and \( v^*_3 u^*_3 \) cannot be in \( C \). Therefore, \( v^*_2 u^*_5 v^*_2 \) and \( v^*_3 u^*_5 v^*_3 \) must be in \( C \) because \( \deg(u^*_3) = \deg(u^*_5) = 3 \). It follows that both \( u^*_4 v^*_4 \) and \( u^*_4 v^*_6 \) cannot be in \( C \). Hence, \( u^*_4 \) is not in \( C \) because \( \deg(u^*_4) = 3 \), contradicting our assumption that \( C \) is a Hamilton cycle of \( L \). Thus, this case cannot occur.

(ii) \( v^*_1 u^*_1 v^*_4 \) is in \( C \).

In this case, \( C \) must contain the path \( v^*_1 u^*_2 v^*_2 v^*_1 v^*_4 \). Therefore, \( v^*_1 u^*_5 \) cannot be in \( C \). Since \( \deg(u^*_5) = 3 \), \( v^*_1 u^*_5 v^*_2 \) must be in \( C \). It follows that \( v^*_1 u^*_4 \) cannot be in \( C \) because \( v^*_2 u^*_5 \) and \( v^*_1 u^*_5 \) are already in \( C \). So, \( v^*_1 u^*_4 v^*_4 \) must be in \( C \) because \( \deg(u^*_4) = 3 \). Thus, \( v^*_1 u^*_1 v^*_2 v^*_4 u^*_6 v^*_7 \) is a proper subcycle of \( C \), which is impossible. This means that this case also cannot occur.

(iii) \( v^*_1 u^*_1 v^*_3 \) is in \( C \).

By arguments similar to those of Case (ii), we can get a contradiction for this case. Hence, this case also cannot occur.

Thus, the assumption that \( L \) has a Hamilton cycle is false. So \( L \) must be nonhamiltonian.

Now we prove that \( L \) is a maximal nonhamiltonian split graph. Since \( L \) is nonhamiltonian as we have proved above, it remains to prove that \( L + u^*_i v^*_j \) is hamiltonian for any \( u^*_i \neq v^*_j \in E(L) \) and \( u^*_i \in I(L) \) and \( v^*_j \in K(L) \). This is done in Table 3.

Finally, the fact that \( B_{|I(L)|} = \emptyset \) is trivial. The proof of Lemma 2 is complete. □

Lemma 3. Let \( H^{4,6} \) be a graph defined in Table 1 and \( X = \text{S}(I(X) \cup K(X), E(X)) \) be the complete split graph with \( I(X) = \{u_{x,1}\} \) and \( K(X) = \{v_{x,1}, v_{x,2}\} \). Then the graph \( T = \text{S}(I(T) \cup K(T), E(T)) = H^{4,6}[X, v^*_1] + u_{x,1}v^*_2 \) (see Fig. 2) is a maximal nonhamiltonian Burkard–Hammer graph with \( B_4(T) = \emptyset \) but \( B_3(T) \neq \emptyset, B_2(T) \neq \emptyset \) and \( B_1(T) \neq \emptyset \).

Proof. The following assertions (a) and (b) are true for \( T \).
Table 3. The Hamilton cycle for $L + u_i^j v_j^i$

<table>
<thead>
<tr>
<th>Graph $L + u_i^j v_j^i$</th>
<th>Hamilton cycle $C_{u_i^j v_j^i}$ for $L + u_i^j v_j^i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L + u_1^1 v_1^1$</td>
<td>$C_{u_1^1 v_1^1} = u_1^1 v_1^1 u_4^1 v_2^1 u_6^1 v_3^1 u_3^1 v_5^1 u_5^1 v_6^1 u_2^1 v_4^1 u_1^1$</td>
</tr>
<tr>
<td>$L + u_1^2 v_2^1$</td>
<td>$C_{u_1^2 v_2^1} = u_1^2 v_2^1 u_4^1 v_1^1 u_2^1 v_3^1 u_5^1 v_5^1 u_6^1 v_3^1 u_3^1 v_2^1 u_1^2$</td>
</tr>
<tr>
<td>$L + u_1^3 v_3^1$</td>
<td>$C_{u_1^3 v_3^1} = u_1^3 v_3^1 u_4^1 v_1^1 u_2^1 v_5^1 u_6^1 v_6^1 u_5^1 v_2^1 u_3^1 v_1^1 u_1^3$</td>
</tr>
<tr>
<td>$L + u_1^4 v_4^1$</td>
<td>$C_{u_1^4 v_4^1} = u_1^4 v_4^1 u_4^1 v_1^1 u_2^1 v_5^1 u_6^1 v_6^1 u_5^1 v_2^1 u_3^1 v_1^1 u_1^4$</td>
</tr>
<tr>
<td>$L + u_1^5 v_5^1$</td>
<td>$C_{u_1^5 v_5^1} = u_1^5 v_5^1 u_4^1 v_1^1 u_2^1 v_3^1 u_5^1 v_5^1 u_6^1 v_3^1 u_3^1 v_2^1 u_1^5$</td>
</tr>
<tr>
<td>$L + u_1^6 v_6^1$</td>
<td>$C_{u_1^6 v_6^1} = u_1^6 v_6^1 u_4^1 v_1^1 u_2^1 v_3^1 u_5^1 v_5^1 u_6^1 v_3^1 u_3^1 v_2^1 u_1^6$</td>
</tr>
<tr>
<td>$L + u_1^7 v_1^2$</td>
<td>$C_{u_1^7 v_1^2} = u_1^7 v_1^2 u_4^1 v_1^1 u_2^1 v_3^1 u_5^1 v_5^1 u_6^1 v_3^1 u_3^1 v_2^1 u_1^7$</td>
</tr>
<tr>
<td>$L + u_1^8 v_2^2$</td>
<td>$C_{u_1^8 v_2^2} = u_1^8 v_2^2 u_4^1 v_1^1 u_2^1 v_3^1 u_5^1 v_5^1 u_6^1 v_3^1 u_3^1 v_2^1 u_1^8$</td>
</tr>
<tr>
<td>$L + u_1^9 v_3^2$</td>
<td>$C_{u_1^9 v_3^2} = u_1^9 v_3^2 u_4^1 v_1^1 u_2^1 v_3^1 u_5^1 v_5^1 u_6^1 v_3^1 u_3^1 v_2^1 u_1^9$</td>
</tr>
<tr>
<td>$L + u_1^{10} v_4^2$</td>
<td>$C_{u_1^{10} v_4^2} = u_1^{10} v_4^2 u_4^1 v_1^1 u_2^1 v_3^1 u_5^1 v_5^1 u_6^1 v_3^1 u_3^1 v_2^1 u_1^{10}$</td>
</tr>
<tr>
<td>$L + u_1^{11} v_5^2$</td>
<td>$C_{u_1^{11} v_5^2} = u_1^{11} v_5^2 u_4^1 v_1^1 u_2^1 v_3^1 u_5^1 v_5^1 u_6^1 v_3^1 u_3^1 v_2^1 u_1^{11}$</td>
</tr>
<tr>
<td>$L + u_1^{12} v_6^2$</td>
<td>$C_{u_1^{12} v_6^2} = u_1^{12} v_6^2 u_4^1 v_1^1 u_2^1 v_3^1 u_5^1 v_5^1 u_6^1 v_3^1 u_3^1 v_2^1 u_1^{12}$</td>
</tr>
<tr>
<td>$L + u_2^1 v_1^1$</td>
<td>$C_{u_2^1 v_1^1} = u_2^1 v_1^1 u_4^1 v_2^1 u_6^1 v_3^1 u_3^1 v_5^1 u_5^1 v_6^1 u_2^1 v_4^1 u_2^1$</td>
</tr>
<tr>
<td>$L + u_2^2 v_2^1$</td>
<td>$C_{u_2^2 v_2^1} = u_2^2 v_2^1 u_4^1 v_1^1 u_2^1 v_3^1 u_5^1 v_5^1 u_6^1 v_3^1 u_3^1 v_2^1 u_2^2$</td>
</tr>
<tr>
<td>$L + u_2^3 v_3^1$</td>
<td>$C_{u_2^3 v_3^1} = u_2^3 v_3^1 u_4^1 v_1^1 u_2^1 v_5^1 u_6^1 v_6^1 u_5^1 v_2^1 u_3^1 v_1^1 u_2^3$</td>
</tr>
<tr>
<td>$L + u_2^4 v_4^1$</td>
<td>$C_{u_2^4 v_4^1} = u_2^4 v_4^1 u_4^1 v_1^1 u_2^1 v_5^1 u_6^1 v_6^1 u_5^1 v_2^1 u_3^1 v_1^1 u_2^4$</td>
</tr>
<tr>
<td>$L + u_2^5 v_5^1$</td>
<td>$C_{u_2^5 v_5^1} = u_2^5 v_5^1 u_4^1 v_1^1 u_2^1 v_3^1 u_5^1 v_5^1 u_6^1 v_3^1 u_3^1 v_2^1 u_2^5$</td>
</tr>
<tr>
<td>$L + u_2^6 v_6^1$</td>
<td>$C_{u_2^6 v_6^1} = u_2^6 v_6^1 u_4^1 v_1^1 u_2^1 v_3^1 u_5^1 v_5^1 u_6^1 v_3^1 u_3^1 v_2^1 u_2^6$</td>
</tr>
<tr>
<td>$L + u_2^7 v_1^2$</td>
<td>$C_{u_2^7 v_1^2} = u_2^7 v_1^2 u_4^1 v_2^1 u_6^1 v_3^1 u_3^1 v_5^1 u_5^1 v_6^1 u_2^1 v_4^1 u_2^7$</td>
</tr>
<tr>
<td>$L + u_2^8 v_2^2$</td>
<td>$C_{u_2^8 v_2^2} = u_2^8 v_2^2 u_4^1 v_1^1 u_2^1 v_3^1 u_5^1 v_5^1 u_6^1 v_3^1 u_3^1 v_2^1 u_2^8$</td>
</tr>
<tr>
<td>$L + u_2^9 v_3^2$</td>
<td>$C_{u_2^9 v_3^2} = u_2^9 v_3^2 u_4^1 v_2^1 u_6^1 v_3^1 u_3^1 v_5^1 u_5^1 v_6^1 u_2^1 v_4^1 u_2^9$</td>
</tr>
<tr>
<td>$L + u_2^{10} v_4^2$</td>
<td>$C_{u_2^{10} v_4^2} = u_2^{10} v_4^2 u_4^1 v_2^1 u_6^1 v_3^1 u_3^1 v_5^1 u_5^1 v_6^1 u_2^1 v_4^1 u_2^{10}$</td>
</tr>
<tr>
<td>$L + u_2^{11} v_5^2$</td>
<td>$C_{u_2^{11} v_5^2} = u_2^{11} v_5^2 u_4^1 v_2^1 u_6^1 v_3^1 u_3^1 v_5^1 u_5^1 v_6^1 u_2^1 v_4^1 u_2^{11}$</td>
</tr>
<tr>
<td>$L + u_2^{12} v_6^2$</td>
<td>$C_{u_2^{12} v_6^2} = u_2^{12} v_6^2 u_4^1 v_2^1 u_6^1 v_3^1 u_3^1 v_5^1 u_5^1 v_6^1 u_2^1 v_4^1 u_2^{12}$</td>
</tr>
</tbody>
</table>

Note on Maximal Nonhamiltonian Burkard–Hammer Graphs
(a) $T$ is a Burkard–Hammer graph.

In fact, since $H^{4.6}$ is a Burkard–Hammer graph, by Theorem 3 the graph $H^{4.6}[X, v_i]$ is a Burkard–Hammer graph. Therefore, by Lemma 1 the graph $T$ is a Burkard–Hammer graph.

(b) $T$ is a maximal nonhamiltonian split graph.

Since $H^{4.6}$ is nonhamiltonian, by Theorem 4 the graph $H^{4.6}[X, v_i]$ is nonhamiltonian. Therefore, if $T$ has a Hamilton cycle $C$ then $C$ must contain the edge $u_1v_2$. So $C$ must contain the path $u_1v_2u_2v_4$ because $N_T(u_2^*) = \{v_2^*, v_4^*\}$. It follows that the edges $u_1^*v_2^*, u_3^*v_2^*, u_5^*v_2^*$ are not in $C$. Hence, $C$ must contain the paths $v_1u_3v_3$ and $v_3u_5v_5v_5u_5$ because $u_1^*, u_3^*$ and $u_5^*$ have degree 3 in $T$. From these facts we see that both $u_3^*v_2^*$ and $u_5^*v_5^*$ cannot be in $C$. Now if $u_1v_2$ is in $C$ then $u_1^*v_1^*$ also cannot be in $C$ because the edges $u_1v_2$ and $u_1^*v_1^*$ are already in $C$. Therefore $C_1 = u_1v_2^*u_2v_4^*u_4v_2v_1v_1u_1v_1u_1v_1$ is a proper subcycle of $C$, a contradiction. Similarly, if $u_2v_2$ is in $C$ then $u_2^*v_2^*$ cannot be in $C$ and therefore $C_2 = u_1v_2u_2^*v_4^*u_4v_2v_1v_1u_2v_2u_1v_1$ is a proper subcycle of $C$, a contradiction again. Thus, $T$ must be nonhamiltonian.

To prove Assertion (b) it remains to prove that $T + uv$ is hamiltonian for every $uv \notin E(T)$ where $u \in I(T)$ and $v \in K(T)$.

First suppose that $u \in I^*$ and $v \in K^* \setminus \{v^*_1\}$. Then $uv$ also is not an edge of $H^{4.6}$. Since $H^{4.6}$ is a maximal nonhamiltonian split graph by Theorem 2, the graph $H^{4.6} + uv$ is hamiltonian. Therefore, $(H^{4.6} + uv)[X, v_i]$ is hamiltonian by Theorem 4 because the graph $X$ trivially has a Hamilton cycle. It is clear that in this case $T + uv = (H^{4.6} + uv)[X, v_i] + u_1v_1$. Hence, $T + uv$ is hamiltonian if $u \in I^*$ and $v \in K^* \setminus \{v^*_1\}$.

Next suppose that $u \in I^*$ and $v \in \{v_{x,1}, v_{x,2}\}$. Then $u$ is not adjacent to $v^*_1$ in $H^{4.6}$. Since $H^{4.6}$ is a maximal nonhamiltonian split graph, $H^{4.6} + uv^*_1$ has a Hamilton cycle $C$ containing the edge $uv^*_1$. Now it is not difficult to see that if $v = v_{x,1}$ (resp., $v = v_{x,2}$) then we can get a Hamilton cycle for $T + uv$ by replacing the vertex $v^*_1$ in $C$ with the path $v_{x,1}u_{x,1}v_{x,2}$ (resp., $v_{x,2}u_{x,1}v_{x,1}$).

Finally suppose that $u = u_{x,1}$ and $v$ is one of the vertices $v^*_2, v^*_4$ or $v^*_6$.

Then

$$C_3 = u_{x,1}v^*_1u^*_3u^*_6v^*_2v^*_2u^*_2u^*_4v^*_4v^*_2u^*_2u^*_4v^*_4v^*_1v^*_1u^*_1u_{x,1},$$

$$C_4 = u_{x,1}v^*_1u^*_4u^*_2v^*_2u^*_3u^*_3u^*_5u^*_5v^*_4v^*_2u^*_2v^*_2u^*_2v^*_1u^*_1u_{x,1}.$$
Burkard–Hammer graph constructed in Lemma

we have

are Hamilton cycles of $T$ and $T + u_{x,1}v_x^6$, respectively.

Thus, $T$ is a maximal nonhamiltonian split graph.

By Assertions (a) and (b) the graph $T = S(I(T) \cup K(T), E(T)) = H^{4,6}[X, v_t]$ + $u_{x,1}v_x^5$ is a maximal nonhamiltonian Burkard–Hammer graph. Furthermore, it is clear that $B_4(T) = \emptyset$ but $B_3(T) \neq \emptyset$, $B_2(T) \neq \emptyset$ and $B_1(T) \neq \emptyset$.

The proof of Lemma 12 is complete.

---

**Lemma 4.** Let $T = S(I(T) \cup K(T), E(T))$ be the maximal nonhamiltonian Burkard–Hammer graph constructed in Lemma 3 and $Y_t = S(I(Y_t) \cup K(Y_t), E(Y_t))$ be a complete split graph with $I(Y_t) = \{u_{y,1}, u_{y,2}, ..., u_{y,t}\}$ and $K(Y_t) = \{v_{y,1}, v_{y,2}, ..., v_{y,t}, v_{y,t+1}\}$ where $t \geq 1$ is an integer. Then the graph $H_t = S(I(H_t) \cup K(H_t), E(H_t))$ is a maximal nonhamiltonian Burkard–Hammer graph with $|I(H_t)| = 6 + t, \delta(H_t) = t + 1 = |I(H_t)| - 5$. Moreover, $B_4(H_t) = \emptyset$ but $B_3(H_t) \neq \emptyset, B_2(H_t) \neq \emptyset$ and $B_1(H_t) \neq \emptyset$.

**Proof.** By Lemma 3, graph $T$ is a nonhamiltonian Burkard–Hammer graph. Therefore, by Theorems 3 and 4, the graph $H_t$ is a nonhamiltonian Burkard–Hammer graph. We prove now that $H_t + uv$ is hamiltonian for every $uv \notin E(H_t)$ where $u \in I(H_t)$ and $v \in K(H_t)$. There are two separate cases to consider.

**Case 1:** $u \in I(T), v \in K(T) \setminus \{v_2^2\}$.

In this case, $uw \notin E(T)$ and $H_t + uv = (T + uv)[Y_t, v_2^2]$. Since $T$ is a maximal nonhamiltonian Burkard–Hammer graph by Lemma 3, the graph $T + uv$ is hamiltonian. The graph $Y_t = S(I(Y_t) \cup K(Y_t), E(Y_t))$ is also hamiltonian because it is a complete split graph with $|K(Y_t)| = |I(Y_t)| + 1$. By Theorem 4, the graph $(T + uv)[Y_t, v_2^2]$ has a Hamilton cycle. Hence, the graph $H_t + uv$ is hamiltonian.

**Case 2:** $u \in I(Y_t), v \in K(T) \setminus \{v_2^2\}$.

Since $v \in K(T) \setminus \{v_2^2\}$, we have $|N_{I(T)}(v)| \leq 3$. Therefore, there exists a vertex $w \in I(T)$ such that $uw \notin E(T)$. By Case 1, the graph $H_t + uv$ has a Hamilton cycle $C$ which must contain the edge $uv$ because $H_t$ is nonhamiltonian.

Let $C'$ be the cycle $C$ with an orientation. By $C'$ we denote the cycle $C$ with the reverse orientation. If $x, y \in V(C)$, then $xC'y$ denotes the consecutive vertices of $C$ from $x$ to $y$ in the direction specified by $C'$. The same vertices in the reverse order are given by $yC'x$. If $x \in V(C)$ then $x^+$ denotes the successor of $x$ on $C$, and $x^-$ denotes its predecessor. Without loss of generality, we may assume that $w^+ = v$ in $C$. By the definitions of $T$ and $T[Y_t, v_2^2]$, vertex $w$ is adjacent to both $u^+$ and $w^–$. Therefore, $C' = vC'u^-wCuv$ is a Hamilton cycle in $H_t + uv$.

Thus, $H_t + uv$ is hamiltonian for every $uv \notin E(H_t)$ where $u \in I(H_t)$ and $v \in K(H_t)$. Therefore, $H_t$ is a maximal nonhamiltonian split graph. Further, we have

$$|I(H_t)| = |I(T)| + |I(Y_t)| = 6 + t,$$
\[ \delta(H_t) = |K(Y_t)| = t + 1 = |I(H_t)| - 5. \]

It is also clear that \( B_4(H_t) = \emptyset \) but \( B_3(H_t) \neq \emptyset, B_2(H_t) \neq \emptyset \) and \( B_1(H_t) \neq \emptyset \).

The proof of Lemma 4 is complete. \( \blacksquare \)

**Proof of Theorem 9.**

(a) Let \( k = 4 \). Then the graph \( L = S(I(L) \cup K(L), E(L)) \) of Lemma 2 is a maximal nonhamiltonian Burkard–Hammer graph with \( \delta(L) = 2 = |I(L)| - 4 \) and \( B_{\mid I(L)\mid} = \emptyset \). Thus, Assertion (a) is true for \( k = 4 \).

Now suppose that \( k > 4 \). Let \( G_1 = S(I_1 \cup K_1, E_1) \) be a complete split graph with \( |K_1| > |I_1| = k-4 \) and \( v \) be a vertex of \( K_1 \). Since the graph \( L \) of Lemma 2 is a maximal nonhamiltonian Burkard–Hammer graph which has \( N_{L}(u) \neq K(L) \) for every \( u \in I(L) \), by Theorem 6 the graph \( G = S(I \cup K, E) = G_1[l, v] \) is a maximal nonhamiltonian Burkard–Hammer graph with \( \delta(G) = \delta(L) = |I| - (4 + |I_1|) = |I| - k \). Moreover, by Theorem 5 and Lemma 2, \( B_{\mid I\mid} = \emptyset \). Thus, Assertion (a) is also true for \( k > 4 \).

(b) Let \( k = 5 \) and \( m \) be an integer with \( m > 7 \). Further, let \( H_t = T[Y_t, v_t^2] \) be a graph constructed from \( T \) and \( Y_t \) with \( |I(Y_t)| = t = m - 6 \) as in Lemma 4. Then by this lemma, the graph \( H_t \) is a maximal nonhamiltonian Burkard–Hammer graph with \( |I(H_t)| = |I(T)| + |I(Y_t)| = 6 + (m - 6) = m \) and \( \delta(H_t) = |I(H_t)| - 5 \).

Also by Lemma 4, \( B_4(H_t) = \emptyset \) but \( B_3(H_t) \neq \emptyset, B_2(H_t) \neq \emptyset \) and \( B_1(H_t) \neq \emptyset \). Thus, Assertion (b) is true for \( k = 5 \) and any integer \( m > 7 \).

Now suppose that \( k \) and \( m \) are integers with \( k \geq 6 \) and \( m > k + 2 \). Let \( G_1 = S(I_1 \cup K_1, E_1) \) be a complete split graph with \( |K_1| > |I_1| = k-5 \) and \( v \) be a vertex of \( K_1 \). Further, let \( G_2 = S(I_2 \cup K_2, E_2) \) be the graph \( H_t = T[Y_t, v_t^2] \) defined in Lemma 4 where \( l = m - k - 1 \). Then by Lemma 4, the graph \( G_2 \) is a maximal nonhamiltonian Burkard–Hammer graph with \( |I_2| = |I(H_t)| = m - k + 5, \delta(G_2) = \delta(H_t) = |I(G_2)| - 5 \) and \( B_4(G_2) = \emptyset \) but \( B_3(G_2) \neq \emptyset, B_2(G_2) \neq \emptyset \) and \( B_1(G_2) \neq \emptyset \). Moreover, it is clear that for every vertex \( u \in I_2 \), \( N_{G_2}(u) \neq K_2 \).

Therefore, by Theorem 6 the graph \( G = S(I \cup K, E) = G_1[G_2, v] \) is a maximal nonhamiltonian Burkard–Hammer graph. Further, we have \( |I| = |I_1| + |I_2| = (k - 5) + (m - k + 5) = m \) and by Theorem 5 and Lemma 4

\[ \delta(G) = \delta(G_2) = |I| - (5 + |I_1|) = |I| - k, \]

\[ B_{k-1}(G) = B_{k+|I_1|}(G) = \emptyset, \]

\[ B_{k-2}(G) = B_{3+|I_1|}(G) \neq \emptyset, \]

\[ B_{k-3}(G) = B_{2+|I_1|}(G) \neq \emptyset \] and

\[ B_{k-4}(G) = B_{1+|I_1|}(G) \neq \emptyset. \]

Thus, Assertion (b) is also true for any \( k \geq 6 \) and \( m > k + 2 \).

The proof of Theorem 10 is complete. \( \blacksquare \)

**References**