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Lower Semicontinuity of the KKT Point Set in Quadratic Programs Under Linear Perturbations*

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Abstract. We establish necessary and sufficient conditions for the lower semicontinuity of the Karush-Kuhn-Tucker point set in indefinite quadratic programs under linear perturbations. The obtained results are illustrated by examples.

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1. Introduction

The problem of minimizing or maximizing a linear-quadratic function on a convex polyhedral set is called a quadratic program. Since the appearance of the paper by Daniel [4] in 1973, continuity and differentiability properties of the solution map, the local solution map, the Karush-Kuhn-Tucker (KKT, for brevity) point set mapping and the optimal value function in parametric quadratic programming have been studied intensively in the literature. In particular, upper

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semicontinuity and also lower semicontinuity of the KKT point set mapping in indefinite quadratic programs under perturbations were investigated in [11 - 13] where it was assumed that every component of the data is subject to perturbation. If only the linear part of the data is subject to perturbation, then the upper semicontinuity of the KKT point set mapping can be studied via a theorem of Robinson [9] on the upper Lipschitz continuity of polyhedral multifunctions.

The aim of this paper is to derive necessary and sufficient conditions for the lower semicontinuity of the Karush-Kuhn-Tucker point set in indefinite quadratic programs under linear perturbations. The necessary conditions are relatively simple. But the sufficient conditions are rather sophisticated. A series of examples is designed to show how each set of the sufficient conditions can be realized in practice.

We consider the quadratic program

$$\text{Minimize} \quad f(x) := \frac{1}{2}x^T Dx + c^T x \quad \text{subject to} \;\; x \in \Delta(A, b), \tag{1}$$

where $\Delta(A, b) = \{x \in R^n : Ax \geq b\}$, D is a symmetric $(n \times n)-$matrix, A is an $(m \times n)-$matrix, $b \in R^m$ and $c \in R^n$ are some given vectors. Here the superscript ^{T} denotes transposition. In what follows, matrices D and A will be fixed, while vectors c and b are subject to change. Since D is not assumed to be a positive semidefinite matrix, the function f is not necessarily convex. Thus we will have deal with indefinite quadratic programs under linear perturbations.

We say that $x \in R^n$ is a Karush–Kuhn–Tucker point of (1) if there exists a Lagrange multiplier $\lambda \in R^m$ corresponding to x, that is

$$Dx - A^T\lambda + c = 0, \quad Ax \geq b, \quad \lambda \geq 0, \quad \lambda^T(Ax - b) = 0. \tag{2}$$

The KKT point set of (1) is denoted by $S(c, b)$. The solution set and the local solution set of (1) are denoted, respectively, by $\mathrm{Sol}(c, b)$ and $\mathrm{loc}(c, b)$. It is well-known (see [3, p. 115]) that $S(c, b) \supset \mathrm{loc}(c, b) \supset \mathrm{Sol}(c, b)$. We are interested in studying the lower semicontinuity of the multifunction

$$S(\cdot) : R^n \times R^m \to 2^{R^n}, \; (c', b') \mapsto S(c', b').$$

Note that lower semicontinuity properties of the multifunctions $\mathrm{Sol}(\cdot)$ and $\mathrm{loc}(\cdot)$, have been studied in [5] and [7].

Recall [14, p. 451] that a multifunction $F : R^k \to 2^{R^n}$ is said to be lower semicontinuous (l.s.c.) at $\omega \in R^k$ if $F(\omega) \neq \emptyset$ and, for each open set $V \subset R^n$ satisfying $F(\omega) \cap V \neq \emptyset$, there exists $\delta > 0$ such that $F(\omega') \cap V \neq \emptyset$ for every $\omega' \in R^k$ with the property that $\|\omega' - \omega\| < \delta$. This definition differs slightly from the corresponding one given in [1, p. 39], where only the points from the effective domain of F are taken into account.

We obtain the necessary and sufficient conditions for the lower semicontinuity of the multifunction $S(\cdot)$, our main results, in Sec. 2. Then, in Sec. 3, we consider several illustrative examples.

Throughout this paper, the scalar product and the norm in an Euclidean space R^k are denoted by $\langle , \rangle$ and $\| \cdot \|$, respectively. In matrix computations, vectors in R^k are understood as columns of real numbers. In the usual text they are written as rows of real numbers. For two vectors $x = (x_1, \ldots, x_k)$, $y =$

$(y_1, \ldots, y_k) \in R^k$, the inequality $x \geq y$ (resp., $x > y$) means $x_i \geq y_i$ (resp., $x_i > y_i$) for all $i = 1, \ldots, k$. For a matrix $A \in R^{m \times n}$, A_i denotes the i−th row of A. For a subset $I \subset \{1, \ldots, m\}$, A_I is the matrix composed by the rows A_i $(i \in I)$ of A. For a vector $x = (x_1, \ldots, x_k) \in R^k$ and an index set $J \subset \{1, \ldots, k\}$, x_J is the vector with the components x_j $(j \in J)$. The norm in the product space $R^n \times R^m$ is defined by setting $\|(c, b)\| = (\|c\|^2 + \|b\|^2)^{1/2}$ for every $(c, b) \in R^n \times R^m$.

2. Main Results

Necessary and sufficient conditions for the lower semicontinuity of the multifunction $S(\cdot)$ will be established in this section. Recall that the inequality system $Ax \geq b$ is said to be regular if the Slater condition is satisfied, i.e., there exists $\bar{x} \in R^n$ such that $A\bar{x} > b$. It is easily seen that if the system $Ax \geq b$ is irregular then there exists a sequence $\{b^k\} \subset R^m$ converging to b such that, for each k, the system $Ax \geq b^k$ has no solutions.

Theorem 2.1 (Necessary conditions for lower semicontinuity). *If the multifunction $S(\cdot)$ is lower semicontinuous at (c, b), then the system $Ax \geq b$ is regular and the set $S(c, b)$ is nonempty and finite.*

Proof. Suppose that $S(\cdot)$ is l.s.c. at (c, b). By definition, $S(c, b) \neq \emptyset$. If the system $Ax \geq b$ is irregular, then there exists a sequence $\{b^k\} \subset R^m$ converging to b such that $\Delta(A, b^k) = \emptyset$ for all $k \in N$. This implies that $S(c, b^k) = \emptyset$ for all $k \in N$. Then $S(\cdot)$ cannot be l.s.c. at (c, b), a contradiction.

In order to prove that $S(c, b)$ is a finite set, for each subset $I \subset \{1, \cdots, m\}$ we define a matrix $M_I \in R^{(n+|I|)\times(n+|I|)}$, where $|I|$ is the number of elements of I, by setting

$$M_I = \begin{pmatrix} D & -A_I^T \\ A_I & O \end{pmatrix}$$

(If $I = \emptyset$, then we put $M_I = D$). Let

$$Q_I = \left\{(u, v) \in R^n \times R^m : \begin{pmatrix} u \\ v_I \end{pmatrix} = M_I \begin{pmatrix} x \\ \lambda_I \end{pmatrix} \quad \text{for some } (x, \lambda) \in R^n \times R^m \right\},$$

and

$$Q = \bigcup \{Q_I : I \subset \{1, \cdots, m\},\ \det M_I = 0\}.$$

If $\det M_I = 0$, then Q_I is a proper linear subspace of $R^n \times R^m$. By the Baire Lemma, Q is nowhere dense in $R^n \times R^m$. Hence there exists a sequence $\{(c^k, b^k)\} \subset R^n \times R^m$ converging to (c, b) such that $(-c^k, b^k) \notin Q$ for all k. Fix any $\bar{x} \in S(c, b)$. Since $S(\cdot)$ is l.s.c at (c, b), without loss of generality we can assume that there is a sequence $\{x^k\} \subset R^n$ converging to $\bar{x}$ such that $x^k \in S(c^k, b^k)$ for all k. Then for each $k \in N$ there exists $\lambda^k \in R^m$ such that

$$\begin{cases} Dx^k - A^T\lambda^k + c^k = 0, \\ Ax^k \geq b^k, \quad \lambda^k \geq 0, \\ (\lambda^k)^T(Ax^k - b^k) = 0. \end{cases}$$

For every k, let $I_k := \{i \in \{1,\dots,m\} : \lambda_i^k > 0\}$. (It may happen that $I_k = \emptyset$.) Clearly, there exists a subset $I \subset \{1,\cdots,m\}$ such that $I_k = I$ for infinitely many k. Without loss of generality we can assume that $I_k = I$ for all k. Then we have

$$Dx^k - A_I^T\lambda_I^k + c^k = 0, \quad A_I x^k = b_I^k,$$

or, equivalently,

$$M_I \begin{pmatrix} x^k \\ \lambda_I^k \end{pmatrix} = \begin{pmatrix} -c^k \\ b_I^k \end{pmatrix}.$$

We claim that det $M_I \neq 0$. Indeed, if det $M_I = 0$ then by the definitions of Q_I and Q we have

$$(-c^k, b^k) \in Q_I \subset Q,$$

contrary to the fact that $(-c^k, b^k) \notin Q$ for all k. We have proved that det $M_I \neq 0$. So

$$\begin{pmatrix} x^k \\ \lambda_I^k \end{pmatrix} = M_I^{-1}\begin{pmatrix} -c^k \\ b_I^k \end{pmatrix}.$$

Letting $k \to \infty$, we get

$$\lim_{k\to\infty} \begin{pmatrix} x^k \\ \lambda_I^k \end{pmatrix} = M_I^{-1}\begin{pmatrix} -c \\ b_I \end{pmatrix}.$$

(If $I = \emptyset$ then the last formula becomes $\lim_{k\to\infty} x^k = D^{-1}(-c)$.) It follows that the sequence $\{\lambda_I^k\}$ converges to some $\lambda_I \geq 0$ in $R^{|I|}$. Since the sequence $\{x^k\}$ converges to $\bar{x}$, we have

$$\begin{pmatrix} \bar{x} \\ \lambda_I \end{pmatrix} = M_I^{-1}\begin{pmatrix} -c \\ b_I \end{pmatrix}.$$

Set

$$Z = \Big\{ (x,\lambda) \in R^n \times R^m : \exists J \subset \{1,\cdots,m\} \text{ such that } \det M_J \neq 0 \text{ and } \begin{pmatrix} x \\ \lambda_J \end{pmatrix} = M_J^{-1}\begin{pmatrix} -c \\ b_J \end{pmatrix} \Big\}.$$

Let

$$X = \big\{ x \in R^n : \exists \lambda \in R^m \text{ such that } (x,\lambda) \in Z \big\}.$$

It is clear that $\bar{x} \in X$. From the definitions of Z and X it follows that X is a finite set. Since $\bar{x} \in X$ for every $\bar{x} \in S(c,b)$, we conclude that $S(c,b)$ is a finite set. The proof is complete. ■

Example 3.1 in the next section shows that the regularity of the system $Ax \geq b$, the nonemptiness and finiteness of $S(c,b)$, altogether, do not imply that $S(\cdot)$ is l.s.c. at (c,b).

Our next goal is to find sufficient conditions for the lower semicontinuity of the KKT point set mapping $(c',b') \mapsto S(c',b')$ at the given point $(c,b) \in R^n \times R^m$.

Let $x \in S(c,b)$ and let $\lambda \in R^m$ be a Lagrange multiplier corresponding to x. We set $I = \{1, 2, \ldots, m\}$,

$$K = \{i \in I : A_i x = b_i,\ \lambda_i > 0\}, \quad J = \{i \in I : A_i x = b_i,\ \lambda_i = 0\}. \tag{3}$$

It is clear that K and J are two disjoint sets (possibly empty).

Theorem 2.2. (Sufficient conditions for lower semicontinuity). *Suppose that the system $Ax \geq b$ is regular, the set $S(c,b)$ is finite and nonempty. If for every $x \in S(c,b)$ there exists a Lagrange multiplier λ corresponding to x such that at least one of the following conditions holds:*
(c1) $x \in \mathrm{loc}(c,b)$,
(c2) $K = \emptyset$,
(c3) $J = \emptyset$, $K \neq \emptyset$, *and the system $\{A_i : i \in K\}$ is linearly independent,*
(c4) $J \neq \emptyset$, $K = \emptyset$, *D is nonsingular, and $A_J D^{-1} A_J^T$ is a positive definite matrix,*
where K and J are defined via (x, λ) by (3). *Then, the multifunction $S(\cdot)$ is lower semicontinuous at (c,b).*

Proof. Since $S(c,b)$ is nonempty, in order to prove that $S(\cdot)$ is l.s.c. at (c,b) we only need to show that, for any $x \in S(c,b)$ and for any open neighborhood V_x of x, there exists $\delta > 0$ such that

$$S(c', b') \cap V_x \neq \emptyset \tag{4}$$

for every $(c', b') \in R^n \times R^m$ satisfying $\|(c', b') - (c, b)\| < \delta$.

Let $x \in S(c,b)$ and let V_x be an open neighborhood of x. By our assumptions, there exists a Lagrange multiplier λ corresponding to x such that at least one of the four conditions (c1)–(c4) holds.

We first examine the case where (c1) holds, that is $x \in \mathrm{loc}(c,b)$. Since $S(c,b)$ is finite, $\mathrm{loc}(c,b)$ is finite. So x is an isolated local solution of (1). It can be shown that the second-order sufficient condition [10, Def. 2.1] holds at (x, λ). Since the system $Ax \geq b$ is regular, we can apply Theorem 3.1 from [10] to find a $\delta > 0$ such that

$$\mathrm{loc}(D, A, c', b') \cap V_x \neq \emptyset$$

for every $(c', b') \in R^n \times R^n$ with $\|(c', b') - (c, b)\| < \delta$. Since $\mathrm{loc}(c,b) \subset S(c', b')$, we conclude that (4) is valid for every (c', b') satisfying $\|(c', b') - (c, b)\| < \delta$.

Consider the case where (c2) holds, that is $A_i x > b_i$ for every $i \in I$. Since λ is a Lagrange multiplier corresponding to x, system (2) is satisfied. Because $Ax > b$, from (2) we deduce that $\lambda = 0$. Then the first equality in (2) implies that $Dx = -c$. Thus x is a solution of the linear system

$$Dz = -c \quad (z \in R^n). \tag{5}$$

Since $S(c,b)$ is finite, x is a locally unique KKT point of (1). Combining this with the fact that x is an interior point of $\Delta(A,b)$, we can assert that x is a unique solution of (5). Hence the matrix D is nonsingular and we have

$$x = -D^{-1}c. \tag{6}$$

Since $Ax > b$, there exist $\delta_1 > 0$ and an open neighborhood $U_x \subset V_x$ of x such that $U_x \subset \Delta(A, b')$ for all $b' \in R^m$ satisfying $\|b' - b\| < \delta_1$. By (6), there exists $\delta_2 > 0$ such that if $\|c' - c\| < \delta_2$ and $x' = -D^{-1}c'$ then $x' \in U_x$. Set $\delta = \min\{\delta_1,\ \delta_2\}$. Let (c', b') be such that $\|(c', b') - (c, b)\| < \delta$. Since $x' := -D^{-1}c'$ belongs to the open set $U_x \subset \Delta(A, b')$, we deduce that

$$Dx' + c' = 0, \quad Ax' > b'.$$

From this it follows that $x' \in S(c', b')$. (Observe that $\lambda' = 0$ is a Lagrange multiplier corresponding to x'.) We have thus shown that (4) is valid for every $(c', b') \in R^n \times R^m$ satisfying $\|(c', b') - (c, b)\| < \delta$.

We now suppose that (c3) holds. First, we prove that the matrix $M_K \in R^{(n+|K|)\times(n+|K|)}$ defined by setting

$$M_K = \begin{bmatrix} D & -A_K^T \\ A_K & 0 \end{bmatrix},$$

where $|K|$ denotes the number of elements in K, is nonsingular. To obtain a contradiction, suppose that M_K is singular. Then there exists a nonzero vector $(v, w) \in R^n \times R^{|K|}$ such that

$$M_K \begin{pmatrix} v \\ w \end{pmatrix} = \begin{bmatrix} D & -A_K^T \\ A_K & 0 \end{bmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = 0.$$

This implies that

$$Dv - A_K^T w = 0, \quad A_K v = 0. \tag{7}$$

Since the system $\{A_i : i \in K\}$ is linearly independent by (c3), from (7) it follows that $v \neq 0$. Because $A_{I\setminus K}x > b_{I\setminus K}$ and $\lambda_K > 0$, there exists $\delta_3 > 0$ such that $A_{I\setminus K}(x + tv) \geq b_{I\setminus K}$ and $\lambda_K + tw \geq 0$ for every $t \in [0, \delta_3]$. By (2) and (7), we have

$$\begin{cases} D(x + tv) - A_K^T(\lambda_K + tw) + c = 0, \\ A_K(x + tv) = b_K, \quad \lambda_K + tw \geq 0, \\ A_{I\setminus K}(x + tv) \geq b_{I\setminus K}, \quad \lambda_{I\setminus K} = 0 \end{cases} \tag{8}$$

for every $t \in [0, \delta_3]$. From (8) we deduce that $x + tv \in S(c, b)$ for all $t \in [0, \delta_3]$. This contradicts the assumption that $S(c, b)$ is finite. We have thus proved that M_K is nonsingular. From (2) and the definition of K it follows that

$$\begin{cases} Dx - A_K^T\lambda_K + c = 0, \\ A_K x = b_K, \quad \lambda_K > 0, \\ A_{I\setminus K}x > b_{I\setminus K}, \quad \lambda_{I\setminus K} = 0. \end{cases}$$

The last system can be rewritten equivalently as follows

$$M_K \begin{pmatrix} x \\ \lambda_K \end{pmatrix} = \begin{pmatrix} -c \\ b_K \end{pmatrix}, \quad \lambda_K > 0, \quad \lambda_{I\setminus K} = 0, \quad A_{I\setminus K}x > b_{I\setminus K}. \tag{9}$$

As M_K is nonsingular, (9) yields

$$\begin{pmatrix} x \\ \lambda_K \end{pmatrix} = M_K^{-1} \begin{pmatrix} -c \\ b_K \end{pmatrix}, \quad \lambda_K > 0, \quad \lambda_{I\setminus K} = 0, \quad A_{I\setminus K}x > b_{I\setminus K}.$$

Hence there exists $\delta > 0$ such that if $(c', b') \in R^n \times R^m$ is such that $\|(c', b') - (c, b)\| < \delta$, then the formula

$$\begin{pmatrix} x' \\ \lambda'_K \end{pmatrix} = M_K^{-1} \begin{pmatrix} c' \\ b'_K \end{pmatrix}$$

defines a vector $(x', \lambda'_K) \in R^n \times R^{|K|}$ satisfying the following conditions

$$x' \in V_x, \quad \lambda'_K > 0, \quad A_{I\setminus K} x' > b'_{I\setminus K}.$$

We see at once that vector x' defined in this way belongs to $S(c', b') \cap V_x$ and $\lambda' := (\lambda'_K, \lambda'_{I\setminus K})$, where $\lambda'_{I\setminus K} = 0$, is a Lagrange multiplier corresponding to x'. We have shown that (4) is valid for every $(c', b') \in R^n \times R^m$ satisfying $\|(c', b') - (c, b)\| < \delta$.

Finally, suppose that (c4) holds. In this case, from (2) we get

$$Dx + c = 0, \quad A_J x = b_J, \quad \lambda_J = 0, \quad A_{I\setminus J} x > b_{I\setminus J}, \quad \lambda_{I\setminus J} = 0. \tag{10}$$

To prove that there exists $\delta > 0$ such that (4) is valid for every $(c', b') \in R^n \times R^m$ satisfying $\|(c', b') - (c, b)\| < \delta$, we consider the following system of equations and inequalities of variables $(z, \mu) \in R^n \times R^m$:

$$\begin{cases} Dz - A_J^T \mu_J + c' = 0, \quad A_J z \geq b'_J, \quad \mu_J \geq 0, \\ A_{I\setminus J} z \geq b'_{I\setminus J}, \quad \mu_{I\setminus J} = 0, \quad \mu_J^T (A_J z - b'_J) = 0. \end{cases} \tag{11}$$

Since D is nonsingular, (11) is equivalent to the following system

$$\begin{cases} z = D^{-1}(-c' + A_J^T \mu_J), \quad A_J z \geq b'_J, \quad \mu_J \geq 0, \\ A_{I\setminus J} z \geq b'_{I\setminus J}, \quad \mu_{I\setminus J} = 0, \quad \mu_J^T (A_J z - b'_J) = 0. \end{cases} \tag{12}$$

By (10), $A_{I\setminus J} x > b_{I\setminus J}$. Hence there exist $\delta_4 > 0$ and an open neighborhood $U_x \subset V_x$ of x such that $A_{I\setminus J} z \geq b'_{I\setminus J}$ for any $z \in U_x$ and $(c', b') \in R^n \times R^m$ satisfying $\|(c', b') - (c, b)\| < \delta_4$. Consequently, for every (c', b') satisfying $\|(c', b') - (c, b)\| < \delta_4$, the verification of (4) is reduced to the problem of finding $z \in U_x$ and $\mu_J \in R^{|J|}$ such that (12) holds. Here $|J|$ denotes the number of elements in J. We substitute z from the first equation of (12) into the first inequality and the last equation of that system to get the following

$$\begin{cases} A_J D^{-1} A_J^T \mu_J \geq b'_J + A_J D^{-1} c', \quad \mu_J \geq 0, \\ \mu_J^T (A_J D^{-1} A_J^T \mu_J - b'_J - A_J D^{-1} c') = 0. \end{cases} \tag{13}$$

Let $S := A_J D^{-1} A_J^T$ and $q' := -b'_J - A_J D^{-1} c'$. We can rewrite (13) as follows

$$S\mu_J + q' \geq 0, \quad \mu_J \geq 0, \quad (\mu_J)^T (S\mu_J + q') = 0. \tag{14}$$

Problem of finding $\mu_J \in R^{|J|}$ satisfying (14) is the linear complementarity problem (see [3]) defined by the matrix $S \in R^{|J|\times|J|}$ and the vector $q' \in R^{|J|}$. By assumption (c4), S is a positive definite matrix, that is $y^T S y > 0$ for every $y \in R^{|J|} \setminus \{0\}$. Then S is a P-matrix. The latter means [3, Def. 3.3.1] that every principal minor of S is positive. According to Theorem 3.3.7 in [3], for each $q' \in R^{|J|}$, problem (14) has a unique solution $\mu_J \in R^{|J|}$. Since D is nonsingular, from (10) it follows that

$$A_J D^{-1}(-c) - b_J = 0.$$

Setting $q = -b_J - A_J D^{-1}c$ we have $q = 0$. Substituting $q' = q = 0$ into (14) we find the unique solution $\bar{\mu}_J = 0 = \lambda_J$. By Theorem 7.2.1 in [3], there exist $\ell > 0$ and $\varepsilon > 0$ such that for every $q' \in R^{|J|}$ satisfying $\|q' - q\| < \varepsilon$ we have

$$\|\mu_J - \lambda_J\| \le \ell \|q' - q\|.$$

Therefore

$$\|\mu_J\| = \|\mu_J - \lambda_J\| \le \ell \|b'_J - b_J + A_J D^{-1}(c' - c)\|.$$

From this we conclude that there exists $\delta \in (0, \delta_4]$ such that if (c', b') satisfies the condition $\|(c', b') - (c, b)\| < \delta$, then the vector z defined by the formula

$$z = D^{-1}(-c' + A_J^T \mu_J),$$

where μ_J is the unique solution of (14), belongs to U_x. From the definition of μ_J and z we see that system (12), where $\mu_{I \setminus J} := 0$, is satisfied. Then $z \in S(c', b')$. We have thus shown that, for any (c', b') satisfying $\|(c', b') - (c, b)\| < \delta$, property (4) is valid.

The proof is complete. ■

To verify condition (c1), we can use the following result, which is due to Majthay [6] and Contesse [2].

Theorem 2.3. (See [3, p. 116]) *The necessary and sufficient condition for $x \in R^n$ to be a local solution of* (1) *is that the next two properties are valid:*

(i) $\nabla f(x)v = (Dx + c)^T v \ge 0$ *for every* $v \in T_\Delta(x) = \{v \in R^n : A_{I_0} v \ge 0\}$, *where* $I_0 = \{i \in I : A_i x = b_i\}$;

(ii) $v^T D v \ge 0$ *for every* $v \in T_\Delta(x) \cap (\nabla f(x))^\perp$, *where* $(\nabla f(x))^\perp = \{v \in R^n : \nabla f(x) v = 0\}$.

The ideas of the proof of Theorem 2.2 are adapted from [8, Theorem 4.1] and [12, Theorem 6]. In [8], some results involving Schur complements were obtained.

Let $x \in S(c, b)$ and let $\lambda \in R^m$ be a Lagrange multiplier corresponding to x. We define K and J by (3). Consider the case where both the sets K and J are nonempty. If the matrix

$$M_K = \begin{bmatrix} D & -A_K^T \\ A_K & 0 \end{bmatrix} \in R^{(n+|K|)\times(n+|K|)}$$

is nonsingular, then we denote by S_J the Schur complement [3, p. 75] of M_K in the matrix

$$\begin{bmatrix} D & -A_K^T & -A_J^T \\ A_K & 0 & 0 \\ A_J & 0 & 0 \end{bmatrix} \in R^{(n+|K|+|J|)\times(n+|K|+|J|)}.$$

That is

$$S_J = [A_J\ 0] M_K^{-1} [A_J\ 0]^T.$$

Note that S_J is a symmetric matrix [8, p. 56]. Consider the following condition:

(c5) $J \neq \emptyset$, $K \neq \emptyset$, the system $\{A_i : i \in K\}$ is linearly independent, $v^T Dv \neq 0$ for every nonzero vector v satisfying $A_K v = 0$, and S_J is positive definite.

Modifying some arguments of the proof of Theorem 2.2 we can show that if $J \neq \emptyset$, $K \neq \emptyset$, the system $\{A_i : i \in K\}$ is linearly independent, and $v^T Dv \neq 0$ for every nonzero vector v satisfying $A_K v = 0$, then M_K is nonsingular.

It can be proved that the assertion of Theorem 2.2 remains valid if instead of (c1)–(c4) we use (c1)–(c5). The method of dealing with (c5) is similar to that of dealing with (c4) in the proof of Theorem 2.2. Up to now we have not found any example of quadratic programs of the form (1) for which there exists a pair (x, λ), $x \in S(c, b)$ and λ is a Langrange multiplier corresponding to x, such that (c1)–(c4) are not satisfied, but (c5) is satisfied. Thus the usefulness of (c5) in characterizing the lower semicontinuity property of the multifunction $S(\cdot)$ is to be investigated furthermore. This is the reason why we omit (c5) in the formulation of Theorem 2.2.

3. Examples

The following example shows that the conditions stated in Theorem 2.1 are not sufficient for having the lower semicontinuity property of $S(\cdot)$ at (c, b).

Example 3.1. (see [12, Example 2]). Consider problem (1) with $n = 2,\ m = 3$,

$$D = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad c = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix}.$$

For every $\varepsilon > 0$, we set $c(\varepsilon) = (1, -\varepsilon)$. Since

$$\Delta(A, b) = \{x = (x_1, x_2) \in R^2 : x_1 \geq 0,\ x_2 \geq 0,\ -x_1 - x_2 \geq -2\},$$

we check at once that the system $Ax \geq b$ is regular. A direct computation shows that if $\varepsilon > 0$ is small enough then

$$S(c, b) = \left\{(0, 0), (1, 0), (2, 0), \left(\frac{5}{3}, \frac{1}{3}\right), (0, 2)\right\},$$
$$S(c(\varepsilon), b) = \left\{(2, 0), \left(\frac{5+\varepsilon}{3}, \frac{1-\varepsilon}{3}\right), (0, 2)\right\}.$$

For the open set $V := \{x \in R^2 : \frac{1}{2} < x_1 < \frac{3}{2},\ -1 < x_2 < 1\}$, we have $S(c, b) \cap V = \{(1, 0)\}$ and $S(c(\varepsilon), b) \cap V = \emptyset$ for every $\varepsilon > 0$ small enough. We thus conclude that $S(\cdot)$ is not l.s.c. at (c, b).

We now consider three examples to see how the conditions (c1)–(c4) can be verified for concrete quadratic programs.

Example 3.2. (see [8, p. 56]). Let

$$f(x) = \frac{1}{2}x_1^2 - \frac{1}{2}x_2^2 - x_1 \quad \text{for all } x = (x_1, x_2) \in R^2. \tag{15}$$

Consider the problem

$$\min\{f(x) : x = (x_1, x_2) \in R^2,\ \ x_1 - 2x_2 \geq 0,\ \ x_1 + 2x_2 \geq 0\}. \tag{16}$$

For this problem, we have

$$D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & -2 \\ 1 & 2 \end{bmatrix}, \quad c = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$S(c,b) = \Big\{(1,0), \Big(\frac{4}{3}, \frac{2}{3}\Big), \Big(\frac{4}{3}, -\frac{2}{3}\Big)\Big\},$$

$$\mathrm{loc}(c,b) = \Big\{\Big(\frac{4}{3}, \frac{2}{3}\Big), \Big(\frac{4}{3}, -\frac{2}{3}\Big)\Big\}.$$

For any feasible vector $x = (x_1, x_2)$ of (16), we have $x_1 \geq 2|x_2|$. Therefore

$$f(x) + \frac{2}{3} = \frac{1}{2}x_1^2 - \frac{1}{2}x_2^2 - x_1 + \frac{2}{3} \geq \frac{3}{8}x_1^2 - x_1 + \frac{2}{3} \geq 0. \tag{17}$$

For $\bar{x} := \left(\frac{4}{3}, \frac{2}{3}\right)$ and $\hat{x} := \left(\frac{4}{3}, -\frac{2}{3}\right)$, we have $f(\bar{x}) = f(\hat{x}) = -\frac{2}{3}$. Hence from (17) it follows that $\bar{x}$ and $\hat{x}$ are the solutions of (16). Actually,

$$\mathrm{Sol}(c,b) = \mathrm{loc}(c,b) = \{\bar{x},\ \hat{x}\}.$$

Setting $\widetilde{x} = (1,0)$ we have $\tilde{x} \in S(c,b) \setminus \mathrm{loc}(c,b)$. Note that $\widetilde{\lambda} := (0,0)$ is a Lagrange multiplier corresponding to $\tilde{x}$. We check at once that the inequality system defining the constraint set of (16) is regular and, for each KKT point $x \in S(c,b)$, either (c1) or (c2) is satisfied. Theorem 2.2 shows that the multifunction $S(\cdot)$ is l.s.c. at (c,b).

Example 3.3. Let $f(\cdot)$ be defined by (15). Consider the problem

$$\min\{f(x) : x = (x_1, x_2) \in R^2,\ \ x_1 - 2x_2 \geq 0,\ \ x_1 + 2x_2 \geq 0,\ \ x_1 \geq 1\}.$$

For this problem, we have

$$D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & -2 \\ 1 & 2 \\ 1 & 0 \end{bmatrix}, \quad c = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Let $\bar{x}$, $\hat{x}$, $\widetilde{x}$ be the same as in the preceding example. Note that $\widetilde{\lambda} := (0,0,0)$ is a Lagrange multiplier corresponding to $\widetilde{x}$. We have

$$S(c,b) = \{\widetilde{x},\ \bar{x},\ \hat{x}\}, \quad \mathrm{Sol}(c,b) = \mathrm{loc}(c,b) = \{\bar{x},\ \hat{x}\}.$$

Clearly, for $x = \bar{x}$ and $x = \hat{x}$, assumption (c1) is satisfied. It is easily seen that, for the pair $(\widetilde{x}, \widetilde{\lambda})$, we have $K = \emptyset$, $J = \{3\}$. Since $A_J = (1\ 0)$ and $D^{-1} = D$, we get $A_J D^{-1} A_J^T = 1$. Thus (c4) is satisfied. By Theorem 2.2, $S(\cdot)$ is l.s.c. at (c,b).

Example 3.4. Let $f(x)$ be as in (15). Consider the problem

$$\min\{f(x) : x = (x_1, x_2) \in R^2,\ \ x_1 - 2x_2 \geq 0,\ \ x_1 + 2x_2 \geq 0,\ \ x_1 \geq 2\}.$$

For this problem, we have

$$D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & -2 \\ 1 & 2 \\ 1 & 0 \end{bmatrix}, \quad c = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix},$$

$$S(c, b) = \big\{(2, 0), (2, 1), (2, -1)\big\},$$

$$\mathrm{Sol}(c, b) = \mathrm{loc}(c, b) = \{(2, 1), (2, -1)\}.$$

Let $\bar{x} = (2, -1)$, $\hat{x} = (2, 1)$, $\widetilde{x} = (2, 0)$. Note that $\widetilde{\lambda} := (0, 0, 1)$ is a Lagrange multiplier corresponding to $\widetilde{x}$. For $x = \bar{x}$ and $x = \hat{x}$, we see at once that (c1) is satisfied. For the pair $(\widetilde{x}, \widetilde{\lambda})$, we have $K = \{3\}$, $J = \emptyset$. Since

$$\{A_i : i \in K\} = \{A_3\} = \{(1\ 0)\},$$

assumption (c3) is satisfied. According to Theorem 2.2, $S(\cdot)$ is l.s.c. at (c, b).

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