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The Translational Hull of a Strongly Right or Left Adequate Semigroup

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Abstract. We prove that the translational hull of a strongly right or left adequate semigroup is still of the same type. Our result amplifies a well known result of Fountain and Lawson on translational hull of an adequate semigroup given in 1985.

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1. Introduction

We call a mapping λ from a semigroup S into itself a left translation of S if $\lambda(ab) = (\lambda a)b$ for all $a, b \in S$. Similarly, we call a mapping ρ from S into itself a right translation of S if $(ab)\rho = a(b\rho)$ for all $a, b \in S$. A left translation λ and a right translation ρ of S are said to be linked if $a(\lambda b) = (a\rho)b$ for all $a, b \in S$. In this case, we call the pair (λ, ρ) a bitranslation of S. The set $\Lambda(S)$ of all left

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translations (and also the set $P(S)$ of all right translations) of the semigroup S forms a semigroup under the composition of mappings. By the translational hull of S, we mean a subsemigroup $\Omega(S)$ consisting of all bitranslations (λ, ρ) of S in the direct product $\Lambda(S) \times P(S)$. The concept of translational hull of semigroups and rings was first introduced by Petrich in 1970 (see [11]). The translational hull of an inverse semigroup was first studied by Ault [1] in 1973. Later on, Fountain and Lawson [2] further studied the translational hulls of adequate semigroups. Recently, Guo and Shum [6] investigated the translational hull of a type-A semigroup, in particular, the result obtained by Ault [1] was substantially generalized and extended. Thus, the translational hull of a semigroup plays an important role in the general theory of semigroups.

Recall that the generalized Green left relation $\mathcal{L}^*$ is defined on a semigroup S by $a\mathcal{L}^*b$ when $ax = ay$ if and only if $bx = by$, for all $x, y \in S^1$ (see, for example, [4]). We now call a semigroup S an rpp semigroup if every $\mathcal{L}^*$-class of S contains an idempotent of S. According to Fountain in [3], an rpp semigroup whose idempotents commute is called a right adequate semigroup. By Guo, Shum and Zhu [7], an rpp semigroup S is called a strongly rpp semigroup if for any $a \in S$, there is a unique idempotent e such that $a\mathcal{L}^*e$ and $a = ea$. Thus, we naturally call a right adequate semigroup S *a strongly right adequate semigroup* if S is a strongly rpp semigroup. Dually, we may define the Green star right relation $\mathcal{R}^*$ on a semigroup S and define similarly *a strongly left adequate semigroup* .

In this paper, we shall show that the translational hull of a strongly right (left) adequate semigroup is still the same type. Thus, the result obtained by Fountain and Lawson in [2] for the translational hull of an adequate semigroup will be amplified. As a consequence, we also prove that the translational hull of a C-rpp semigroup is still a C-rpp semigroup.

2. Preliminaries

Throughout this paper, we will use the notions and terminologies given in [3, 8, 9].

We first call a semigroup S an *idempotent balanced* semigroup if for any $a \in S$, there exist idempotents e and f in S such that $a = ea = af$ holds.

The following lemmas will be useful in studying the translational hull of a strongly right (left) adequate semigroup.

Lemma 2.1. *Let S be an idempotent balanced semigroup. Then the following statements hold:*

(i) *If λ_1 and λ_2 are left translations of S, then $\lambda_1 = \lambda_2$ if and only if $\lambda_1 e = \lambda_2 e$ for all $e \in E$.*

(ii) *If ρ_1 and ρ_2 are right translations of S, then $\rho_1 = \rho_2$ if and only if $e\rho_1 = e\rho_2$ for all $e \in E$.*

Proof. We only need to show that (i) holds because (ii) can be proved similarly. The necessity part of (i) is immediate. For the sufficiency part of (i), we first note that for any $a \in S$, there is an idempotent e such that $a = ea$. Hence, we have

$$\lambda_1 a = \lambda_1 ea = (\lambda_1 e)a = (\lambda_2 e)a = \lambda_2 ea = \lambda_2 a.$$

This implies that $\lambda_1 = \lambda_2$. ∎

Lemma 2.2. *Let S be an idempotent balanced semigroup. If $(\lambda_i, \rho_i) \in \Omega(S)$, for $i = 1, 2$, then the following statements are equivalent:*
(i) $(\lambda_1, \rho_1) = (\lambda_2, \rho_2)$*;*
(ii) $\rho_1 = \rho_2$*;*
(iii) $\lambda_1 = \lambda_2$*.*

Proof. We note that (i)⇔(ii) is the dual of (i) ⇔(iii) and (i)⇒ (ii) is trivial. We only need to show that (ii)⇒(i). Suppose that $\rho_1 = \rho_2$. Then by our hypothesis, for any $e \in E$ there exists an idempotent f such that

$$\lambda_1 e = f(\lambda_1 e) = (f\rho_1)e = (f\rho_2)e = f(\lambda_2 e).$$

Similarly, there exists an idempotent h such that $\lambda_2 e = h(\lambda_1 e)$. Hence,we have $\lambda_1 e \,\mathcal{L}\, \lambda_2 e$. Since S is an idempotent balanced semigroup, there exists an idempotent g such that $f(\lambda_2 e) = (\lambda_2 e)g$. Thus, we have $\lambda_1 e = (\lambda_2 e)g$ and consequently, $\lambda_1 e = (\lambda_2 e)g \cdot g = (\lambda_1 e)g$. Since $\mathcal{L} \subseteq \mathcal{L}^*$, we have $\lambda_2 e = (\lambda_2 e)g$ and so $\lambda_1 e = \lambda_2 e$. By Lemma 2.1, $\lambda_1 = \lambda_2$ and hence, $(\lambda_1, \rho_1) = (\lambda_2, \rho_2)$. ∎

By definition, we can easily obtain the following result.

Lemma 2.3. *If S is a strongly right (left) adequate semigroup, then every $\mathcal{L}^*$-class ($\mathcal{R}^*$-class) of S contains a unique idempotent of S.*

Consequently, for a strongly right adequate semigroup S we always denote the unique idempotent in the $\mathcal{L}^*$-class of a in S by a^+. Now, we have the following lemma.

Lemma 2.4. *Let a, b be elements of a strongly right adequate semigroup S. Then the following conditions hold in S:*
(i) $a^+a = a = aa^+$*;*
(ii) $(ab)^+ = (a^+b)^+$*;*
(iii) $(ae)^+ = a^+e$*, for all $e \in E$.*

Proof. Clearly, (i) holds by definition. For (ii), since $\mathcal{L}^*$ is a right congruence on S, we have $ab \,\mathcal{L}^*\, a^+b$. Now, by Lemma 2.3, we have $(ab)^+ = (a^+b)^+$. Part (iii) follows immediately from (ii).

3. Strongly Right Adequate Semigroups

Throughout this section, we always use S to denote a strongly right adequate semigroup with a semilattice of idempotents E. Let $(\lambda, \rho) \in \Omega(S)$. Then we define the mappings λ^+ and ρ^+ which map S into itself by

$$a\rho^+ = a(\lambda a^+)^+ \quad \text{and} \quad \lambda^+ a = (\lambda a^+)^+ a,$$

for all $a \in S$.

For the mappings λ^+ and ρ^+, we have the following lemma.

Lemma 3.1. *For any $e \in E$, we have*
(i) $\lambda^+ e = e\rho^+$, *and* $e\rho^+ \in E$;
(ii) $\lambda^+ e = (\lambda e)^+$.

Proof.
(i) Since we assume that the set of all idempotents E of the semigroup S forms a semilattice, all idempotents of S commute. Hence, $\lambda^+ e = (\lambda e)^+ e = e(\lambda e)^+ = e\rho^+$. Also, the element $e\rho^+$ is clearly an idempotent.
(ii) Since $\mathcal{L}^*$ is a right congruence on S, we see that $\lambda^+ e = (\lambda e)^+ e \; \mathcal{L}^* \; \lambda e \cdot e = \lambda e$. Now, by Lemma 2.3, we have $\lambda^+ e = (\lambda e)^+$, as required. ∎

Lemma 3.2. *The pair (λ^+, ρ^+) is an element of the translational hull $\Omega(S)$ of S.*

Proof. We first show that λ^+ is a left translation of S. For any $a, b \in S$, by Lemma 2.4, we have

$$\begin{aligned}
\lambda^+(ab) &= [\lambda(ab)^+]^+ \cdot ab = [\lambda(ab)^+]^+ \cdot a^+ \cdot ab \\
&= [\lambda(ab)^+ \cdot a^+]^+ \cdot ab = \{\lambda[(ab)^+ a^+]\}^+ \cdot ab \\
&= \{\lambda[a^+(ab)^+]\}^+ \cdot ab = [(\lambda a^+) \cdot (ab)^+]^+ \cdot ab \\
&= (\lambda a^+)^+ \cdot (ab)^+ \cdot ab = (\lambda a^+)^+ a \cdot b \\
&= (\lambda^+ a)b.
\end{aligned}$$

We now proceed to show that ρ^+ is a right translation of S. For all $a, b \in S$, we first observe that $ab = (ab) \cdot b^+$ and so $(ab)^+ = (ab)^+ b^+$, by Lemma 2.4. Now, we have

$$\begin{aligned}
(ab)\rho^+ &= ab \cdot [\lambda(ab)^+]^+ = ab \cdot \{\lambda[(ab)^+ b^+]\}^+ \\
&= ab \cdot \{\lambda[b^+ \cdot (ab)^+]\}^+ = ab \cdot [(\lambda b^+) \cdot (ab)^+]^+ \\
&= ab \cdot (\lambda b^+)^+ \cdot (ab)^+ = (ab)(ab)^+ \cdot (\lambda b^+)^+ \\
&= a \cdot b(\lambda b^+)^+ = a(b\rho^+).
\end{aligned}$$

In fact, the pair (λ^+, ρ^+) is clearly linked because for all $a, b \in S$, we have

$$\begin{aligned}
a(\lambda^+ b) &= a \cdot (\lambda b^+)^+ b = a \cdot a^+ \cdot (\lambda b^+)^+ \cdot b \\
&= a \cdot (\lambda b^+)^+ a^+ \cdot b = a \cdot [\lambda b^+ \cdot a^+]^+ \cdot b \\
&= a \cdot [\lambda(b^+ a^+)]^+ \cdot b = a \cdot [\lambda(a^+ b^+)]^+ \cdot b \\
&= a \cdot [\lambda a^+ \cdot b^+]^+ \cdot b = a \cdot (\lambda a^+)^+ \cdot b^+ \cdot b \\
&= a(\lambda a^+)^+ \cdot b = (a\rho^+)b.
\end{aligned}$$

Consequently, the pair (λ^+, ρ^+) is an element of the translational hull $\Omega(S)$ of S. ∎

Note. By Lemma 2.4, it can be easily seen that a strongly right (left) adequate semigroup is an idempotent balanced semigroup. This is an useful property of the strongly right (left) adequate semigroups and we shall use this property in proving our main result later on.

Lemma 3.3. *Let S be a strongly right adequate semigroup and (λ, ρ) be an element of $\Omega(S)$. Then $(\lambda, \rho) = (\lambda, \rho)(\lambda^+, \rho^+) = (\lambda^+, \rho^+)(\lambda, \rho)$.*

Proof. For all $e \in E$, we have $\lambda\lambda^+ e = \lambda[(\lambda e)^+ e] = \lambda[e(\lambda e)^+] = \lambda e$. This implies that $\lambda\lambda^+ = \lambda$ by Lemma 2.2. Since $(\lambda, \rho) \in \Omega(S)$, by Lemma 3.2, we have $(\lambda^+, \rho^+) \in \Omega(S)$. Hence, $(\lambda, \rho)(\lambda^+, \rho^+) = (\lambda\lambda^+, \rho\rho^+) \in \Omega(S)$. Since $\lambda\lambda^+ = \lambda$ as we have shown above, by Lemma 2.2, we have $\rho\rho^+ = \rho$. This shows that the first equality above holds. Furthermore, we have, by Lemma 3.1, that $\lambda^+\lambda e = [\lambda(\lambda e)^+]^+(\lambda e) = [\lambda\lambda^+ e]^+(\lambda e) = \lambda e$. Consequently, we obtain $\lambda^+\lambda = \lambda$ and again by Lemma 2.2 as before, we have $(\lambda, \rho) = (\lambda^+, \rho^+)(\lambda, \rho)$. ■

Lemma 3.4. *Let S be a strongly right adequate semigroup and $(\lambda, \rho) \in \Omega(S)$. Then (λ, ρ) is $\mathcal{L}^*$-related to (λ^+, ρ^+).*

Proof. Let $(\lambda_1, \rho_1), (\lambda_2, \rho_2)$ be elements of $\Omega(S)$. In order to prove $(\lambda, \rho)\mathcal{L}^*(\lambda^+, \rho^+)$, we only need to show that

$$(\lambda, \rho)(\lambda_1, \rho_1) = (\lambda, \rho)(\lambda_2, \rho_2) \Longleftrightarrow (\lambda^+, \rho^+)(\lambda_1, \rho_1) = (\lambda^+, \rho^+)(\lambda_2, \rho_2).$$

That is,

$$(\lambda\lambda_1, \rho\rho_1) = (\lambda\lambda_2, \rho\rho_2) \Longleftrightarrow (\lambda^+\lambda_1, \rho^+\rho_1) = (\lambda^+\lambda_2, \rho^+\rho_2). \tag{3.1}$$

By Lemma 2.2, it suffices to show that

$$\rho\rho_1 = \rho\rho_2 \Longleftrightarrow \rho^+\rho_1 = \rho^+\rho_2. \tag{3.2}$$

In proving the necessity part of (3.2), we first note that for any $e \in E$, we have $[(\lambda e)^+\rho]e = (\lambda e)^+(\lambda e) = \lambda e$ and hence, by Lemma 2.3, we have

$$(\lambda e)^+ = [(\lambda e)^+\rho]^+ e = e[(\lambda e)^+\rho]^+. \tag{3.3}$$

Now suppose that $\rho\rho_1 = \rho\rho_2$. Then, it is clear that $(\lambda e)^+\rho\rho_1 = (\lambda e)^+\rho\rho_2$. Since $((\lambda e)^+\rho) \cdot [(\lambda e)^+\rho]^+ = (\lambda e)^+\rho$, we have

$$((\lambda e)^+\rho)[(\lambda e)^+\rho]^+\rho_1 = ((\lambda e)^+\rho)[(\lambda e)^+\rho]^+\rho_2.$$

Again since $(\lambda e)^+\rho\mathcal{L}^*[(\lambda e)^+\rho]^+$ and by the definition of $\mathcal{L}^*$, we can deduce that

$$[(\lambda e)^+\rho]^+\rho_1 = [(\lambda e)^+\rho]^+\rho_2.$$

Combining the above equality with the equality (3.3), we can easily deduce that $(\lambda e)^+\rho_1 = (\lambda e)^+\rho_2$. By using Lemma 3.1, we immediately have

$$e\rho^+\rho_1 = (\lambda e)^+\rho_1 = (\lambda e)^+\rho_2 = e\rho^+\rho_2.$$

This leads to $\rho^+\rho_1 = \rho^+\rho_2$, by Lemma 2.1.

For the proof of the sufficiency part of (3.2), we only need to note that $\rho\rho^+ = \rho$ by Lemma 3.3. Hence, it can be easily seen that (λ, ρ) and (λ^+, ρ^+) are indeed $\mathcal{L}^*$-related. ■

Lemma 3.5. *Let $\Phi(S) = \{(\lambda, \rho) \in \Omega(S) \mid \lambda E \cup E\rho \subseteq E\}$. Then $\Phi(S)$ is the set of all idempotents of $\Omega(S)$.*

Proof. Suppose that $(\lambda, \rho) \in \Omega(S)$ and $e \in E$. Then, $\lambda e \in E$ and $e\rho \in E$. Hence, we have

$$e\rho^2 = (e\rho)\rho = (e(e\rho))\rho = ((e\rho)e)\rho = (e\rho)(e\rho) = e\rho.$$

Similarly, $\lambda^2 e = \lambda e$. By Lemma 2.1, we obtain immediately that $(\lambda, \rho)^2 = (\lambda, \rho)$. Conversely, suppose that $(\lambda, \rho) \in E(\Omega(S))$. Then by Lemma 3.4, we see that $(\lambda, \rho)\mathcal{L}^*(\lambda^+, \rho^+)$. This leads to $(\lambda^+, \rho^+) = (\lambda^+, \rho^+)(\lambda, \rho)$. However,we always have $(\lambda, \rho) = (\lambda^+, \rho^+)(\lambda, \rho)$, by Lemma 3.3 and so $(\lambda, \rho) = (\lambda^+, \rho^+)$. Again, by Lemma 3.1, we have $\lambda E \cup E\rho \subseteq E$ and hence $(\lambda, \rho) \in \Phi(S)$. ■

Corollary 3.6. *The element (λ^+, ρ^+) is an idempotent of $\Omega(S)$.*

Lemma 3.7. *The elements of $\Phi(S)$ commute with each other.*

Proof. Let $(\lambda_i, \rho_i) \in \Phi(S), i = 1, 2$. Then, by Lemma 3.5, we have $\lambda_i E \cup E\rho_i \subseteq E$. Thus, for any $e \in E$, we have

$$\begin{aligned} e\rho_1\rho_2 &= [e(e\rho_1)]\rho_2 = [(e\rho_1)e]\rho_2 = (e\rho_1)(e\rho_2) \\ &= (e\rho_2)(e\rho_1) = e\rho_2\rho_1. \end{aligned}$$

This fact implies that $\rho_1\rho_2 = \rho_2\rho_1$. Similarly, we have $\lambda_1\lambda_2 = \lambda_2\lambda_1$. Thus, we have $(\lambda_1, \rho_1)(\lambda_2, \rho_2) = (\lambda_2, \rho_2)(\lambda_1, \rho_1)$, as required. ■

By using the above Lemmas 3.2 - 3.5, Corollary 3.6 and Lemma 3.7, we can easily verify that for any $(\lambda, \rho) \in \Omega(S)$ there exists a unique idempotent (λ^+, ρ^+) such that $(\lambda, \rho)\mathcal{L}^*(\lambda^+, \rho^+)$ and $(\lambda, \rho) = (\lambda^+, \rho^+)(\lambda, \rho)$. Thus, $\Omega(S)$ is indeed a strongly rpp semigroup. Again by Lemma 3.7 and the definition of a strongly right (left) adequate semigroup, we can formulate our main theorem.

Theorem 3.8.
(i) *The translational hull of a strongly right adequate semigroup is still a strongly right adequate semigroup.*
(ii) *The translational hull of a strongly left adequate semigroup is still a strongly left adequate semigroup.*

Note. Since the set E of all idempotents in a C-rpp semigroup S lies in the center of S, we see immediately that a C-rpp semigroup is a strongly right adequate semigroup (see [4]). As a direct consequence of Theorem 3.8, we deduce the following corollary.

Corollary 3.9. *The translational hull of a C-rpp semigroup is still a C-rpp semigroup.*

In closing this paper, We remark that the Fundamental Ehresmann semigroups were first initiated and studied by Gomes and Gould in [5]. As a generalization of the Fundamental C- Ehreshmann semigroups, the quasi-C Ehresmann semigroups have been investigated by Li, Guo and Shum in [10]. These kinds of Ehreshmann semigroups are in fact the generalized C-rpp semigroups. Since Guo and Shum have shown that the translational hull of a type-A semigroup is still of the same type [6], it is natural to ask whether the translational hull of C-Ehresmann semigroups and their generalized classes are still of the same type?

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