Vietnam Journal of Mathematics **34**:4 (2006) 449–458

Vietnam Journal
of
MATHEMATICS

A Blowing-up Characterization of Pseudo Buchsbaum Modules

Nguyen Tu Cuong[1] **and Nguyen Thi Hong Loan**[2]

[1]*Institute of Mathematics,18 Hoang Quoc Viet, 10307 Hanoi, Vietnam*
[2]*Department of Mathematics, Vinh University*
182 Le Duan Street, Vinh City, Vietnam

Dedicated to Professor Do Long Van on the occasion of his 65th birthday

Received June 22, 2005

Abstract. Let $(A, \mathfrak{m})$ be a commutative Noetherian local ring and M a finitely generated A-module. The aim of this paper is to give a blow-up characterization of pseudo Buchsbaum modules defined in [2], which says that M is a pseudo Buchsbaum module if and only if the Rees module $R_{\mathfrak{q}}(M)$ is pseudo Buchsbaum for all parameter ideals $\mathfrak{q}$ of M. We also show that the associated graded module $G_{\mathfrak{q}}(M)$ is pseudo Cohen Macaulay (resp. pseudo Buchsbaum) provided M is pseudo Cohen Macaulay (resp. pseudo Buchsbaum).

2000 Mathematics Subject Classification: 13H10, 13A30.
Keywords: Pseudo Cohen-Macaulay module, pseudo Buchsbaum module, Rees module, associate graded module.

1. Introduction

Let A be a commutative Noetherian local ring with the maximal ideal $\mathfrak{m}$, M a finitely generated A-module with $\dim M = d > 0$. Let $\underline{x} = (x_1, \ldots, x_d)$ be a system of parameters of A-module M. We consider the difference between the multiplicity and the length

$$J_M(\underline{x}) = e(\underline{x}; M) - \ell(M/Q_M(\underline{x})),$$

where $Q_M(\underline{x}) = \bigcup_{t>0}((x_1^{t+1}, \ldots, x_d^{t+1})M : x_1^t \ldots x_d^t)$ is a submodule of M. It should be mentioned that $J_M(\underline{x})$ gives a lot of informations on the structure of M.

For example, if M is a Cohen–Macaulay module then $Q_M(\underline{x}) = (x_1, \dots, x_d)M$ by [7]. Therefore $J_M(\underline{x}) = 0$ for all system of parameters $\underline{x}$ of M. Further, we have known that $\ell(M/Q_M(\underline{x}))$ is just the length of generalized fraction (see [10]). Therefore by [10], $\sup\limits_{\underline{x}} J_M(\underline{x}) < \infty$ if M is a generalized Cohen-Macaulay module. In [1] we also showed that if M is a Buchsbaum module then, $J_M(\underline{x})$ takes a constant value for every system of parameters $\underline{x}$ of M. Unfortunately, the converses of all above statements are not true in general. The structure of modules M satisfying $J_M(\underline{x}) = 0$ or $\sup\limits_{\underline{x}} J_M(\underline{x}) < \infty$ was studied in [5] and such modules were called *pseudo Cohen-Macaulay* modules or *pseudo generalized Cohen-Macaulay* modules, respectively. In [2] we studied the structure of modules M having $J_M(\underline{x})$ a constant value for all systems of parameters. We called it *pseudo Buchsbaum* modules. Note that pseudo Cohen Macaulay (resp. pseudo Buchsbaum, pseudo generalized Cohen Macaulay) modules still have many nice properties and they are relatively closed to Cohen Macaulay (resp. Buchsbaum, generalized Cohen Macaulay) modules.

For a parameter ideal $\mathfrak{q}$ of M we set $R_{\mathfrak{q}}(M) = \bigoplus\limits_{i \geq 0} \mathfrak{q}^i M T^i$ the Rees module and $G_{\mathfrak{q}}(M) = \bigoplus\limits_{i \geq 0} \mathfrak{q}^i M / \mathfrak{q}^{i+1} M$ the associated graded module of M with respect to $\mathfrak{q}$. Let $\mathfrak{M} = \mathfrak{m} \oplus \bigoplus\limits_{i \geq 1} \mathfrak{q}^i T^i$ be the unique homogeneous maximal ideal of $R_{\mathfrak{q}}(A)$. Then $R_{\mathfrak{q}}(M)$ or $G_{\mathfrak{q}}(M)$ is called a *pseudo Cohen Macaulay* (resp. *pseudo Buchsbaum*) *module* if and only if $R_{\mathfrak{q}}(M)_{\mathfrak{M}}$ or $G_{\mathfrak{q}}(M)_{\mathfrak{M}}$ is a pseudo Cohen Macaulay (resp. pseudo Buchsbaum) module. The purpose of this paper is to prove the following result.

Theorem 1. *Let A be a commutative Noetherian local ring and M a finitely generated A-module. Then the following statements are true.*

(i) *M is a pseudo Buchsbaum module if and only if $R_{\mathfrak{q}}(M)$ is a pseudo Buchsbaum module for all parameter ideals $\mathfrak{q}$ of M.*

(ii) *Let M be a pseudo Cohen Macaulay (resp. pseudo Buchsbaum) module. Then $G_{\mathfrak{q}}(M)$ is a pseudo Cohen Macaulay (resp. pseudo Buchsbaum) module for all parameter ideals $\mathfrak{q}$ of M.*

It should be noted that an analogous result of the first statement in the above theorem for Buchsbaum modules was only proved under the assumption that depth $M > 0$ (see [11, Theorem 3.3, Chap. IV]).

The paper is divided into 4 sections. In Sec. 2, we outline some properties of pseudo Cohen Macaulay (resp. pseudo Buchsbaum) modules over local ring which will be needed later. The proof of Theorem 1 is given in Sec. 3. As consequences of Theorem 1 we will show in the last section that the Rees module $R_{\mathfrak{q}}(M)$ and the associated graded module $G_{\mathfrak{q}}(M)$ are always locally pseudo Cohen-Macaylay if M is a pseudo Buchsbaum module.

2. Preliminaries

Let $(A, \mathfrak{m})$ be a commutative Noetherian local ring and M a finitely generated module with $\dim M = d > 0$. Let $\underline{x} = (x_1, \ldots, x_d)$ be a system of parameters of M and $\underline{n} = (n_1, \ldots, n_d)$ a d-tuple of positive integers. Set $\underline{x}(\underline{n}) = (x_1^{n_1}, \ldots, x_d^{n_d})$. Then the difference between multiplicities and lengths

$$J_M(\underline{x}(\underline{n})) = n_1 \ldots n_d e(\underline{x}; M) - \ell(M/Q_M(\underline{x}(\underline{n})))$$

can be considered as a function in $\underline{n}$. Note that this function is non-negative ([1, Lemma 3.1]) and ascending, i.e., for $\underline{n} = (n_1, \ldots, n_d)$, $\underline{m} = (m_1, \ldots, m_d)$ with $n_i \geq m_i$, $i = 1, \ldots, d$, $J_M(\underline{x}(\underline{n})) \geq J_M(\underline{x}(\underline{m}))$ ([1, Corollary 4.3]). Moreover, we know that $\ell(M/Q_M(\underline{x}(\underline{n})))$ is just the length of generalized fraction $M(1/(x_1^{n_1}, \ldots, x_d^{n_d}, 1))$ defined by Sharp and Hamieh [10]. Therefore, we can describe Question 1.2 of [10] as follows: is $J_M(\underline{x}(\underline{n}))$ a polynomial for large enough $\underline{n}$ ($\underline{n} \gg 0$ for short)? A negative answer for this question is given in [4]. But, the function $J_M(\underline{x}(\underline{n}))$ is bounded above by the polynomial $n_1 \ldots n_d J_M(\underline{x})$, and more general, we have the following result.

Theorem 2. [3, Theorem 3.2] *The least degree of all polynomials in $\underline{n}$ bounding above the function $J_M(\underline{x}(\underline{n}))$ is independent of the choice of a system of parameters $\underline{x}$.*

The numerical invariant of M given in the above theorem is called the *polynomial type of fractions of M* and denoted by $pf(M)$ [3, Definition 3.3]. For convenience, we stipulate that the degree of the zero-polynomial is equal to $-\infty$.

Definition 1.

(i) [5, Definition 2.2] *M is said to be a pseudo Cohen Macaulay module if $pf(M) = -\infty$.*

(ii) [2, Definition 3.1] *An A-module M is called a pseudo Buchsbaum module if there exists a constant K such that $J_M(\underline{x}) = K$ for every system of parameters $\underline{x}$ of M.*

A is called a pseudo Cohen Macaulay (resp. pseudo Buchsbaum) ring if it is a pseudo Cohen Macaulay (resp. pseudo Buchsbaum) module as a module over itself.

It should be mentioned that every Cohen Macaulay module is pseudo Cohen Macaulay and the class of pseudo Buchsbaum modules contains the class of pseudo Cohen Macaulay modules. In [1] and [2], we showed that the class of pseudo Buchsbaum modules strictly contains the class of Buchsbaum modules, but it does not contain the class of generalized Cohen Macaulay modules.

Next, we recall characterizations of these modules from [5] and [2].

Proposition 1. *M is a pseudo Cohen Macaulay (resp. pseudo Buchsbaum) A-module if and only if $\widehat{M}$ is a pseudo Cohen Macaulay (resp. pseudo Buchsbaum) $\widehat{A}$-module.*

Note that for an A-module M (A is not necessarily a local ring) we usually use in this paper the following notations

$$\operatorname{Assh} M = \{\mathfrak{p} \in \operatorname{Ass} M \mid \dim A/\mathfrak{p} = \dim M\}.$$

Let $0 = \underset{\mathfrak{p}_i \in \mathrm{Ass} M}{\cap} N(\mathfrak{p}_i)$ be a reduced primary decomposition of the submodule 0 of M. We put

$$U_M(0) = \underset{\mathfrak{p}_j \in \mathrm{Assh} M}{\cap} N(\mathfrak{p}_j) \text{ and } \overline{M} = M/U_M(0).$$

Then $U_M(0)$ does not depend on the choice of a primary decomposition of the zero-submodule of M. Notice that $U_M(0)$ is the largest submodule of M of dimension less than $\dim M$ and $\operatorname{Ass} \overline{M} = \operatorname{Assh} M$, $\dim \overline{M} = \dim M$.

Theorem 3. ([5, Theorem 3.1], [2, Lemma 4.4]) *Suppose that A admits a dualizing complex. Then the following statements are true.*

(i) *M is a pseudo Cohen Macaulay module if and only if $\overline{M}$ is a Cohen Macaulay module.*

(ii) *M is a pseudo Buchsbaum A-module if and only if $\overline{M}$ is a Buchsbaum A-module. Moreover, in this case we have*

$$J_M(\underline{x}) = \sum_{i=1}^{d-1} \binom{d-1}{i-1} \ell(H^i_{\mathfrak{m}}(\overline{M})),$$

for every system of parameters $\underline{x} = (x_1, \ldots, x_d)$ of M, where $H^i_{\mathfrak{m}}(\overline{M})$ stands for the i^{th} local cohomology module of $\overline{M}$ with respect to the maximal ideal $\mathfrak{m}$.

3. Proof of Theorem 1

Let

$$\varphi : R_{\mathfrak{q}}(M) \to R_{\mathfrak{q}}(\overline{M}) \quad \text{and} \quad \pi : G_{\mathfrak{q}}(M) \to G_{\mathfrak{q}}(\overline{M})$$

be the canonical epimorphisms, where $\overline{M} = M/U_M(0)$. Then we have

$$\operatorname{Ker} \ \varphi = \underset{i \geq 0}{\oplus} (U_M(0) \cap \mathfrak{q}^i M) T^i \text{ and } \operatorname{Ker} \pi = \underset{i \geq 0}{\oplus} \frac{\mathfrak{q}^i M \cap (\mathfrak{q}^{i+1} M + U_M(0))}{\mathfrak{q}^{i+1} M}.$$

To prove Theorem 1 we need some auxiliary lemmata.

Lemma 1. *With the same notations as above, then we have Ker $\varphi = U_{R_{\mathfrak{q}}(M)}(0)$.*

Proof. It is clear that $\operatorname{Ass} \operatorname{Ker} \varphi \subseteq \operatorname{Ass} R_{\mathfrak{q}}(M)$. For each $\mathfrak{p} \in \operatorname{Spec} A$ we denote $\widetilde{\mathfrak{p}} := \underset{i \geq 0}{\oplus} (\mathfrak{p} \cap \mathfrak{q}^i) T^i$. Take any $\mathfrak{P} \in \operatorname{Assh} R_{\mathfrak{q}}(M)$. Then there exists $\mathfrak{p} \in \operatorname{Assh} M$ such that $\mathfrak{P} = \widetilde{\mathfrak{p}}$ (see [11, Lemma 1.7 and Lemma 3.1, chap IV]). Since $\dim U_M(0) < \dim M$,$(U_M(0))_{\mathfrak{p}} = 0$. Therefore we have

$$(\operatorname{Ker} \varphi)_{(\widetilde{\mathfrak{p}})} = (\underset{i \geq 0}{\oplus} (U_M(0) \cap \mathfrak{q}^i M) T^i)_{(\widetilde{\mathfrak{p}})} = 0.$$

Thus $(\mathrm{Ker}\ \varphi)_{(\mathfrak{P})} = 0$. It follows that $(\mathrm{Ker}\ \varphi)_{\mathfrak{P}} = 0$. Therefore $\dim \mathrm{Ker}\,\varphi < \dim R_{\mathfrak{q}}(M)$.

Let $K = \underset{i\geq 0}{\oplus} K_i T^i$ be a homogeneous submodule of $R_{\mathfrak{q}}(M)$ with $K \supset \mathrm{Ker}\,\varphi$. Then we have $K_i T^i \supseteq (U_M(0) \cap \mathfrak{q}^i M)T^i$ for all $i \geq 0$ and there exists $j \geq 0$ such that $K_j \supset (U_M(0) \cap \mathfrak{q}^j M)$. Since $K \subseteq R_{\mathfrak{q}}(M)$, $K_j \subseteq \mathfrak{q}^j M$. Hence $K_j \not\subseteq U_M(0)$. Set $V = K_j + U_M(0)$. We have $V \supset U_M(0)$. Thus $\dim V = \dim M$. Therefore there exists $\mathfrak{p} \in \mathrm{Assh}\, V \cap \mathrm{Assh}\, M$. Hence $0 \neq V_{\mathfrak{p}} = (K_j)_{\mathfrak{p}} \subseteq K_{\widetilde{\mathfrak{p}},h} \subseteq K_{\widetilde{\mathfrak{p}}}$. Thus we get $K_{\widetilde{\mathfrak{p}}} \neq 0$, i.e, $\widetilde{\mathfrak{p}} \in \mathrm{Supp}\, K \subseteq \mathrm{Supp}\, R_{\mathfrak{q}}(M)$. On the other hand $\widetilde{\mathfrak{P}} \in \mathrm{Assh}\, R_{\mathfrak{q}}(M)$ (see [11, Lemma 1.7 and Lemma 3.1, chap IV]). Combining these facts, we get $\dim K = \dim R_{\mathfrak{q}}(M)$ and therefore $\mathrm{Ker}\,\varphi$ is the largest homogeneous submodule of $R_{\mathfrak{q}}(M)$ of dimension less than $\dim R_{\mathfrak{q}}(M)$.

Moreover, we can choose a reduced primary decomposition of the submodule 0 in $R_{\mathfrak{q}}(M)$ such that $0_{R_{\mathfrak{q}}(M)} = \overset{l}{\underset{i=1}{\cap}} Q_i$ with Q_i is the homogeneous primary submodule of $R_{\mathfrak{q}}(M)$ belonging to homogeneous prime $\mathfrak{P}_i$ (see [9, Proposition 10 B]). Then $U_{R_{\mathfrak{q}}(M)}(0)$ is a homogeneous submodule of $R_{\mathfrak{q}}(M)$. On the other hand, $U_{R_{\mathfrak{q}}(M)}(0)$ is the largest submodule of $R_{\mathfrak{q}}(M)$ of dimension less than $\dim R_{\mathfrak{q}}(M)$. Therefore $\mathrm{Ker}\ \varphi = U_{R_{\mathfrak{q}}(M)}(0)$. ■

Let N be a submodule of M such that $\dim N < \dim M$. Set $M' = M/N$. If $\mathfrak{q}$ is a parameter ideal of M, then it is clear that $\mathfrak{q}$ is a parameter ideal of M'. But the converse is not true. It means that there exists a parameter ideal $\mathfrak{q}'$ of M' but $\mathfrak{q}'$ is not a parameter ideal of M. However, we have the following result.

Lemma 2. *Let $\mathfrak{q}'$ be a parameter ideal of M'. Then there exists a parameter ideal $\mathfrak{q}$ of M such that $\mathfrak{q} + \mathrm{Ann}\, M' = \mathfrak{q}' + \mathrm{Ann}\, M'$. In particular we have $R_{\mathfrak{q}}(M') = R_{\mathfrak{q}'}(M')$ and $G_{\mathfrak{q}}(M') = G_{\mathfrak{q}'}(M')$.*

Proof. Let $\mathfrak{q}'$ be a parameter ideal of M' and let $(x_1, \dots, x_d)$ be a system of parameters of M' such that $\mathfrak{q}' = (x_1, \dots, x_d)A$. Then the lemma is proved if we can show the existence of a system of parameters $(y_1, \dots, y_d)$ of M such that

$$(y_1, \dots, y_d)R + \mathrm{Ann}\, M' = (x_1, \dots, x_d)R + \mathrm{Ann}\, M'.$$

To prove this we first claim by introduction on i that there exists a system of parameters $(y_1, \dots, y_d)$ of M such that $y_i = x_i + a_i$ with $a_i \in (x_{i+1}, \dots, x_d)A + \mathrm{Ann}\, M'$ for all $i = 1, \dots, d$. In fact, since x_1 is a parameter element of M' and $\mathrm{Assh}\, M = \mathrm{Assh}\, M'$, we have x_1 is a parameter element of M. We choose $y_1 = x_1$. Suppose that we already have for $1 \leqslant k < d$ a part of the system of parameters $(y_1, \dots, y_k)$ of M as required. We have to show that there exists a parameter element y_{k+1} of $M/(y_1, \dots, y_k)M$ such that $y_{k+1} = x_{k+1} + a_{k+1}$ with $a_{k+1} \in (x_{k+2}, \dots, x_d)A + \mathrm{Ann}\, M'$. Let $\mathfrak{q}_1 = (x_1, \dots, x_d)A + \mathrm{Ann}\, M'$. Since $(x_1, \dots, x_d)$ is a system of parameters of M', we have $\mathfrak{q}_1$ is a $\mathfrak{m}$-primary ideal. Therefore $\mathfrak{q}_1 \not\subseteq \mathfrak{p}$ for all prime ideals $\mathfrak{p}$ with $\dim A/\mathfrak{p} > 0$. It then follows that

$$(x_{k+1}, \dots, x_d)A + \mathrm{Ann}\, M' \not\subseteq \mathfrak{p}$$

for all $\mathfrak{p} \in \operatorname{Assh}(M/(y_1, \dots, y_k)M$. Indeed, if $(x_{k+1}, \dots, x_d)A + \operatorname{Ann} M' \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Assh}(M/(y_1, \dots, y_d)M)$, then

$$\begin{aligned}\mathfrak{q}_1 &= (x_1, \dots, x_d)A + \operatorname{Ann} M' \\ &= (y_1, \dots, y_k, x_{k+1}, \dots, x_d)A + \operatorname{Ann} M' \subseteq (y_1, \dots, y_k)A + \mathfrak{p} = \mathfrak{p}\end{aligned}$$

as the choice of $y_1, \dots, y_k$. This gives a contradiction since $\dim A/\mathfrak{p} > 0$. Therefore we can choose by [8, Theorem 124] an element $a_{k+1} \in (x_{k+2}, \dots, x_d)A + \operatorname{Ann} M'$ such that $x_{k+1} + a_{k+1} \notin \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Assh}(M/(y_1, \dots, y_k)M)$. Let $y_{k+1} = x_{k+1} + a_{k+1}$. Then y_{k+1} is a parameter element of $M/(y_1, \dots, y_k)M$ and the claim is therefore proved.

Now, let $(y_1, \dots, y_d)$ be a system of parameters of M as required. Then we can check that

$$(y_1, \dots, y_d)R + \operatorname{Ann} M' = (x_1, \dots, x_d)R + \operatorname{Ann} M'$$

by the choice of $y_1, \dots, y_d$. We set $\mathfrak{q} = (y_1, \dots, y_d)A$. Then we have $\mathfrak{q} + \operatorname{Ann} M' = \mathfrak{q}' + \operatorname{Ann} M'$; $R_{\mathfrak{q}}(M') = R_{\mathfrak{q}'}(M')$ and $G_{\mathfrak{q}}(M') = G_{\mathfrak{q}'}(M')$. ■

Now we are able to prove the first statement of Theorem 1.

Proof of Statement (i) *of Theorem* 1. Let $\mathfrak{q}$ be a parameter ideal of M. We have known that $R_{\mathfrak{q}\widehat{A}}(\widehat{A}) \cong R_{\mathfrak{q}}(A) \otimes_A \widehat{A}$ and $R_{\mathfrak{q}\widehat{A}}(\widehat{M}) \cong R_{\mathfrak{q}}(M) \otimes_A \widehat{A}$. Moreover, let $\widehat{\mathfrak{q}}$ denote a parameter ideal of $\widehat{M}$. Then there is a parameter ideal $\mathfrak{q}$ of M with $\widehat{\mathfrak{q}}\widehat{M} = (\mathfrak{q}\widehat{A})\widehat{M}$. Hence $R_{\widehat{\mathfrak{q}}}(\widehat{M}) = R_{\mathfrak{q}\widehat{A}}(\widehat{M})$. Therefore $R_{\widehat{\mathfrak{q}}}(\widehat{M})$ is a pseudo Buchsbaum (resp. pseudo Cohen Macaulay) module for all parameter ideals $\widehat{\mathfrak{q}}$ of $\widehat{M}$ if and only if $R_{\mathfrak{q}}(M)$ is a pseudo Buchsbaum (resp. pseudo Cohen Macaulay) module for all parameter ideals $\mathfrak{q}$ of M. On the other hand, $\widehat{M}$ is a pseudo Buchsbaum (resp. pseudo Cohen Macaulay) module if and only if M is a pseudo Buchsbaum (resp. pseudo Cohen Macaulay) by Proposition 1. Therefore without any loss of generality, we may assume that $A = \widehat{A}$.

Let M be a pseudo Buchsbaum module and $\mathfrak{q}$ any parameter ideal of M. Then $\overline{M}$ is Buchsbaum by Theorem 3. Hence $R_{\mathfrak{q}}(\overline{M})$ is Buchsbaum (see [11, Theorem 2.10, Chap. IV]). Thus $R_{\mathfrak{q}}(M)_{\mathfrak{M}}/U_{R_{\mathfrak{q}}(M)}(0)_{\mathfrak{M}}$ is Buchsbaum by Lemma 1. Since A is complete, $R_{\mathfrak{q}}(A)$ is catenary. Then we can check that $U_{R_{\mathfrak{q}}(M)}(0)_{\mathfrak{M}} = U_{R_{\mathfrak{q}}}(M)_{\mathfrak{M}}(0)$. Therefore $R_{\mathfrak{q}}(M)_{\mathfrak{M}}$ is a pseudo Buchsbaum by Theorem 5.

Conversely, let $R_{\mathfrak{q}}(M)$ be a pseudo Buchsbaum module for all parameter ideals $\mathfrak{q}$ of M. Let $\mathfrak{q}$ be any parameter ideal of M. Then we have $R_{\mathfrak{q}}(\overline{M}) \cong R_{\mathfrak{q}}(M)/U_{R_{\mathfrak{q}}(M)}(0)$ by Lemma 1. Hence $R_{\mathfrak{q}}(\overline{M})_{\mathfrak{M}} \cong R_{\mathfrak{q}}(M)_{\mathfrak{M}}/U_{R_{\mathfrak{q}}(M)}(0)_{\mathfrak{M}} = R_{\mathfrak{q}}(M)_{\mathfrak{M}}/U_{R_{\mathfrak{q}}(M)_{\mathfrak{M}}}(0)$. Therefore $R_{\mathfrak{q}}(\overline{M})_{\mathfrak{M}}$ is a Buchsbaum module by Theorem 3. Take any parameter ideal $\overline{\mathfrak{q}}$ of $\overline{M}$, there exists by Lemma 2 a parameter ideal $\mathfrak{q}$ of M such that $R_{\overline{\mathfrak{q}}}(\overline{M}) = R_{\mathfrak{q}}(\overline{M})$.

Combining these facts we get that $R_{\overline{\mathfrak{q}}}(\overline{M})$ is a Buchsbaum module for all parameter ideals $\overline{\mathfrak{q}}$ of $\overline{M}$. On the other hand, depth $\overline{M} > 0$. Therefore, $\overline{M}$ is a Buchsbaum module by [11, Theorem 3.3, Chap IV]. Thus M is a pseudo Buchsbaum module by Theorem 3. Statement (i) of Theorem 1 is proved. ∎

In order to prove the second statement of Theorem 1 we need some more lemmas.

Lemma 3. $\mathfrak{P} \notin \operatorname{Supp}(\operatorname{Ker}\varphi)$, *for all* $\mathfrak{P} \in \operatorname{Assh} G_{\mathfrak{q}}(M)$.

Proof. Let $\mathfrak{P} \in \operatorname{Assh} G_{\mathfrak{q}}(M)$. Suppose that $\mathfrak{P} \in \operatorname{Supp}(\operatorname{Ker}\varphi)$. Then we have

$$\dim M = \dim R_{\mathfrak{q}}(A)/\mathfrak{P} \leqslant \dim R_{\mathfrak{q}}(A)/\operatorname{Ann}\operatorname{Ker}\varphi = \dim(\operatorname{Ker}\varphi) < \dim M + 1$$

by Lemma 1. It follows that $\dim(\operatorname{Ker}\varphi) = \dim M$ and $\mathfrak{P} \in \operatorname{Assh}(\operatorname{Ker}\varphi)$. Thus $\dim(\operatorname{Ker}\varphi)_{\mathfrak{P}} = 0$. Hence

$$\dim(\operatorname{Ker}\varphi)_{(\mathfrak{P})} = 0.$$

On the other hand, $[\mathfrak{P}]_0 = \mathfrak{M} \in \operatorname{Supp} M$ (see [11, Lemma 3.1, Chap. IV]). Further, $[\mathfrak{P}]_1 \subset \mathfrak{q}T$. Because, if $[\mathfrak{P}]_1 = \mathfrak{q}T$ then $\mathfrak{P} \supseteq \mathfrak{q}T$. It follows that $\mathfrak{P} = [\mathfrak{P}]_0^* = \mathfrak{M}^* = \mathfrak{M} \oplus (\bigoplus_{i>0} \mathfrak{q}^i T^i) = \mathfrak{M}$. However, $\mathfrak{M} \notin \operatorname{Assh} G_{\mathfrak{q}}(M)$. Therefore, by [11, Lemma 1.3 (ii), Chap IV], there exists $x \in \mathfrak{q}$, $xT \notin [\mathfrak{P}]_1$ such that x is a non-zero divisor with respect to $R_{\mathfrak{q}}(M)_{(\mathfrak{P})}$. Since $(\operatorname{Ker}\varphi)_{(\mathfrak{P})} \subset R_{\mathfrak{q}}(M)_{(\mathfrak{P})}$, x is a non-zero divisor with respect to $(\operatorname{Ker}\varphi)_{(\mathfrak{P})}$. This is a contradiction. Therefore the lemma is proved. ∎

Lemma 4. $\dim \operatorname{Ker}\pi < \dim G_{\mathfrak{q}}(M)$.

Proof. We have

$$\begin{aligned}\operatorname{Ker}\pi &= \bigoplus_{i\geq 0} \frac{\mathfrak{q}^i M \cap (\mathfrak{q}^{i+1}M + U_M(0))}{\mathfrak{q}^{i+1}M} = \bigoplus_{i\geq 0} \frac{\mathfrak{q}^{i+1}M + (\mathfrak{q}^i M \cap U_M(0))}{\mathfrak{q}^{i+1}M} \\ &\cong \frac{\mathfrak{q}R_{\mathfrak{q}}(M) + U_{R_{\mathfrak{q}}(M)}(0)}{\mathfrak{q}R_{\mathfrak{q}}(M)} \cong \frac{U_{R_{\mathfrak{q}}(M)}(0)}{\mathfrak{q}R_{\mathfrak{q}}(M) \cap U_{R_{\mathfrak{q}}(M)}(0)}.\end{aligned}$$

Then we get

$$(\operatorname{Ker}\pi)_{\mathfrak{P}} \cong U_{R_{\mathfrak{q}}(M)}(0)_{\mathfrak{P}}/(\mathfrak{q}R_{\mathfrak{q}}(M) \cap U_{R_{\mathfrak{q}}(M)}(0))_{\mathfrak{P}} = 0,$$

for all $\mathfrak{P} \in \operatorname{Assh} G_{\mathfrak{q}}(M)$ by Lemma 3. Thus $\dim \operatorname{Ker}\pi < \dim G_{\mathfrak{q}}(M)$. ∎

Lemma 5. *Let A be a commutative Notherian local ring, M be a finitely generated A-module. Suppose that N is a submodule of M such that* $\dim N < \dim M$. *Then M is a pseudo Buchsbaum module if and only if so is M/N.*

Proof. Recall that $U_{\widehat{M}}(0)$ is a largest submodule of $\widehat{M}$ of dimension less than $\dim \widehat{M}$. Then $\widehat{N} \subseteq U_{\widehat{M}}(0)$ and $U_{\widehat{M}}(0)/\widehat{N}$ is a largest submodule of $\widehat{M}/\widehat{N}$ of dimension less than $\dim \widehat{M}/\widehat{N}$. Further, $(\widehat{M}/\widehat{N})/(U_{\widehat{M}}(0)/\widehat{N}) \cong \widehat{M}/U_{\widehat{M}}(0)$.

Let M be a pseudo Buchsbaum module. Then $\widehat{M}/U_{\widehat{M}}(0)$ is a Buchsbaum $\widehat{A}$-module by Proposition 1 and Theorem 3. Thus $\widehat{M}/\widehat{N}$ is a pseudo Buchsbaum

$\widehat{A}$-module by Theorem 3. It follows that M/N is a pseudo Buchsbaum A-module by Proposition 1.

For the converse, let M/N be a pseudo Buchsbaum A-module. Then $\widehat{M}/\widehat{N}$ is a pseudo Buchsbaum $\widehat{A}$-module by Proposition 1. Therefore $\widehat{M}/U_{\widehat{M}}(0)$ is a Buchsbaum $\widehat{A}$-module by Theorem 3. Thus $\widehat{M}$ is a pseudo Buchsbaum $\widehat{A}$-module by Theorem 3. So M is a pseudo Buchsbaum module by Proposition 1. ∎

Now we prove the second statement of Theorem 1.

Proof of Statement (ii) *of Theorem* 1. By the same argument in the proof of Stament (i) of Theorem 1, we can assume without loss of generality that A is complete.

Assume that M is a pseudo Cohen Macaulay (resp. pseudo Buchsbaum) module. Then $\overline{M}$ is a Cohen Macaulay (resp. pseudo Buchsbaum) module by Theorem 3. Let $\mathfrak{q}$ be any parameter ideal of M. Then $G_{\mathfrak{q}}(\overline{M})$ is a Cohen Macaulay (resp. Buchsbaum) module (see [11, Theorem 2.1, Chap IV]). Hence $G_{\mathfrak{q}}(M)/\mathrm{Ker}\ \pi$ is a Cohen Macaulay (resp. Buchsbaum) module. It means that $G_{\mathfrak{q}}(M)_{\mathfrak{M}}/(\mathrm{Ker}\ \pi)_{\mathfrak{M}}$ is a Cohen Macaulay (resp. Buchsbaum) module. On the other hand, we have $\dim(\mathrm{Ker}\ \pi)_{\mathfrak{M}} \leqslant \dim \mathrm{Ker}\ \pi < \dim G_{\mathfrak{q}}(M) = \dim G_{\mathfrak{q}}(M)_{\mathfrak{M}}$ by Lemma 4. Therefore, if M is a pseudo Cohen-Macaulay module, we can check that $pf(G_{\mathfrak{q}}(M)_{\mathfrak{M}}) = pf(G_{\mathfrak{q}}(M)_{\mathfrak{M}}/(\mathrm{Ker}\ \pi)_{\mathfrak{M}}) = -\infty$. This means that $G_{\mathfrak{q}}(M)$ is a pseudo Cohen Macaulay. Further, if M is a pseudo Buchsbaum module, then by Lemma 5 $G_{\mathfrak{q}}(M)$ is a pseudo Buchsbaum module. ∎

For pseudo Cohen Macaulayness of Rees module, we only have the following result.

Proposition 2. *Let M be a pseudo Cohen Macaulay module. Then $R_{\mathfrak{q}}(M)$ is a pseudo Cohen Macaulay module for all parameter ideals $\mathfrak{q}$ of M.*

Proof. By the same argument in the proof of Statement (i) of Theorem 1, we can assume without loss of generality that A is complete. Since M is pseudo Cohen Macaulay, $\overline{M}$ is Cohen Macaulay by Theorem 3. Thus $R_{\overline{\mathfrak{q}}}(\overline{M})$ is Cohen Macaulay for all parameter ideals $\overline{\mathfrak{q}}$ of $\overline{M}$ (see [11, Theorem 2.11, Chap. IV]). Let $\mathfrak{q}$ be any parameter ideal of M. We have $R_{\mathfrak{q}}(\overline{M}) \cong R_{\mathfrak{q}}(M)/U_{R_{\mathfrak{q}}(M)}(0)$ by Lemma 1. Therefore

$$R_{\mathfrak{q}}(M)_{\mathfrak{M}} \cong R_{\mathfrak{q}}(M)_{\mathfrak{M}}/U_{R_{\mathfrak{q}}(M)}(0)_{\mathfrak{M}} = R_{\mathfrak{q}}(M)_{\mathfrak{M}}/U_{R_{\mathfrak{q}}(M)_{\mathfrak{M}}}(0).$$

It follows that $R_{\mathfrak{q}}(M)_{\mathfrak{M}}$ is a Cohen Macaulay. The statement is proved.

∎

Remark 1. The converse of Proposition 2 is not true. In fact, let k be a field and s, t indeterminates. Take $A = k[[s^4, s^3t, st^3, t^4]]$. Then the Rees algebra $R_{\mathfrak{q}}(A)$ is a Cohen Macaulay ring for every parameter ideal $\mathfrak{q}$ of A by [6, Proposition 4.8]. But it is well-known that A is not a Cohen Macaulay ring. However, A is Buchsbaum with $H^0_{\mathfrak{M}}(A) = 0$ and $H^1_{\mathfrak{M}}(A) = k$. Therefore $\overline{A} = A/U_A(0) = A$ is not pseudo Cohen Macaulay by Theorem 3.

4. Locally Pseudo Cohen–Macaulay Modules

For any module M we set

$$\text{Supph } M = \{\mathfrak{p} \in \text{Supp } M \mid \exists \mathfrak{q} \in \text{Assh } M, \mathfrak{q} \subseteq \mathfrak{p}\}.$$

We start with the following definition.

Definition 2. *$R_{\mathfrak{q}}(M)$ (resp. $G_{\mathfrak{q}}(M)$) is called a locally pseudo Cohen–Macaulay module if $R_{\mathfrak{q}}(M)_{(\mathfrak{P})}$ (resp. $G_{\mathfrak{q}}(M)_{(\mathfrak{P})}$) is a pseudo Cohen–Macaulay module for all homogeneous prime ideals $\mathfrak{P} \in \text{Supph}\, R_{\mathfrak{q}}(M) \setminus \mathfrak{M}$ (resp. $\mathfrak{P} \in \text{Supph}\, G_{\mathfrak{q}}(M) \setminus \mathfrak{M}$) of $R_{\mathfrak{q}}(A)$.*

Lemma 6. *Assume that A has a dualizing complex. Then $U_{R_{\mathfrak{q}}(M)}(0)_{(\mathfrak{P})}$ is the largest submodule of $R_{\mathfrak{q}}(M)_{(\mathfrak{P})}$ of dimension less than $\dim R_{\mathfrak{q}}(M)_{(\mathfrak{P})}$ for all homogeneous prime ideals $\mathfrak{P} \in \text{Supph}\, R_{\mathfrak{q}}(M)$.*

Proof. Let $\mathfrak{P} \in$ Supph $R_{\mathfrak{q}}(M)$. Since A has a dualizing complex, we can check that $U_{R_{\mathfrak{q}}(M)}(0)_{\mathfrak{P}}$ is the largest submodule of $R_{\mathfrak{q}}(M)_{\mathfrak{P}}$ of dimension less than $\dim R_{\mathfrak{q}}(M)_{\mathfrak{P}}$. Furthermore, $\dim U_{R_{\mathfrak{q}}(M)}(0)_{(\mathfrak{P})} = \dim U_{R_{\mathfrak{q}}(M)}(0)_{\mathfrak{P}}$ and $\dim R_{\mathfrak{q}}(M)_{\mathfrak{P}} = \dim R_{\mathfrak{q}}(M)_{(\mathfrak{P})}$ (see [11, Lemma 2.27, Chap IV]). This implies that $\dim U_{R_{\mathfrak{q}}(M)}(0)_{(\mathfrak{P})} < \dim R_{\mathfrak{q}}(M)_{(\mathfrak{P})}$.

On the other hand, let N be a submodule of $R_{\mathfrak{q}}(M)_{(\mathfrak{P})}$ with $\dim N < \dim R_{\mathfrak{q}}(M)_{(\mathfrak{P})}$. Then $N \subset R_{\mathfrak{q}}(M)_{\mathfrak{P}}$ and $\dim N < \dim R_{\mathfrak{q}}(M)_{\mathfrak{P}}$. Thus $N \subseteq U_{R_{\mathfrak{q}}(M)}(0)_{\mathfrak{P}}$. It follows that $N \subseteq U_{R_{\mathfrak{q}}(M)}(0)_{(\mathfrak{P})}$. Therefore the lemma is proved. ■

Proposition 3. *Let M be a pseudo Buchsbaum module. Then $R_{\mathfrak{q}}(M)$ is a locally pseudo Cohen Macaulay module for all parameter ideals $\mathfrak{q}$ of M.*

Proof. Let M be a pseudo Buchsbaum module. Then $\overline{M}$ is a Buchsbaum module by Theorem 3, (ii). Hence $R_{\overline{\mathfrak{q}}}(\overline{M})$ is a locally Cohen Macaulay module for all parameter ideals $\overline{\mathfrak{q}}$ of $\overline{M}$ by [11, Theorem 3.2, Chap. IV].
Let $\mathfrak{q}$ be a parameter ideal of M. Then $\mathfrak{q}$ is also a parameter ideal of $\overline{M}$ and $R_{\mathfrak{q}}(M)/U_{R_{\mathfrak{q}}(M)}(0)$ is a locally Cohen Macaulay module by Lemma 6. It means that $R_{\mathfrak{q}}(M)_{(\mathfrak{P})}/U_{R_{\mathfrak{q}}(M)}(0)_{(\mathfrak{P})}$ is a Cohen Macaulay module for all homogeneous prime ideals $\mathfrak{P} \in \text{Supph}\, R_{\mathfrak{q}}(M) \setminus \mathfrak{M}$. Therefore $R_{\mathfrak{q}}(M)_{(\mathfrak{P})}$ is a pseudo Cohen Macaulay module for all homogeneous prime ideals $\mathfrak{P} \in \text{Supph}\, R_{\mathfrak{q}}(M) \setminus \mathfrak{M}$ by Lemma 6 and Theorem 3, (i), i.e., $R_{\mathfrak{q}}(M)$ is a locally pseudo Cohen Macaulay module. ■

Lemma 7. *Let $R_{\mathfrak{q}}(M)$ be a locally pseudo Cohen Macaulay module. Then $G_{\mathfrak{q}}(M)$ is a locally pseudo Cohen Macaulay module.*

Proof. Suppose that $R_{\mathfrak{q}}(M)$ is a locally pseudo Cohen Macaulay module i.e.,

$R_{\mathfrak{q}}(M)_{(\mathfrak{P})}$ is a pseudo Cohen Macaulay module for all homogeneous prime ideals $\mathfrak{P} \in \operatorname{Supph} R_{\mathfrak{q}}(M) \setminus \mathfrak{M}$ of $R_{\mathfrak{q}}(A)$.

Let $\mathfrak{P} \in \operatorname{Supph} G_{\mathfrak{q}}(M) \setminus \mathfrak{M}$. If $G_{\mathfrak{q}}(M)_{(\mathfrak{P})} = 0$ then $G_{\mathfrak{q}}(M)_{(\mathfrak{P})}$ is a pseudo Cohen Macaulay module. If $G_{\mathfrak{q}}(M)_{(\mathfrak{P})} \neq 0$ and $[\mathfrak{P}]_1 = \mathfrak{q}T$, then $\mathfrak{P} = \mathfrak{M}$. Thus we may assume that $G_{\mathfrak{q}}(M)_{(\mathfrak{P})} \neq 0$ and $[\mathfrak{P}]_1 \neq \mathfrak{q}T$. Then we can choose an element x such that $x \in \mathfrak{q}$, $xT \notin [\mathfrak{P}]_1$. Moreover, x is a non-zero divisor with respect to $R_{\mathfrak{q}}(M)_{(\mathfrak{P})}$ and $G_{\mathfrak{q}}(M)_{(\mathfrak{P})} \cong R_{\mathfrak{q}}(M)_{(\mathfrak{P})}/xR_{\mathfrak{q}}(M)_{(\mathfrak{P})}$ by [11, Lemma 1.3 (ii), Chap. IV]. Therefore $G_{\mathfrak{q}}(M)_{(\mathfrak{P})}$ is a pseudo Cohen Macaulay module by [5, Corollary 3.4]. Therefore $G_{\mathfrak{q}}(M)$ is a locally pseudo Cohen Macaulay module. ■

Proposition 4. *Let M be a pseudo Buchsbaum module. Then $G_{\mathfrak{q}}(M)$ is a locally pseudo Cohen Macaulay module for all parameter ideals $\mathfrak{q}$ of M.*

Proof. Since M is a pseudo Buchsbaum module, $R_{\mathfrak{q}}(M)$ is a locally pseudo Cohen Macaulay module for all parameter ideals $\mathfrak{q}$ of M by Proposition 3. Therefore the statement follows from Lemma 7. ■

Acknowledgments. The authors would like to thank Macel Morales for his useful suggestions and conversations.

References

1. N. T. Cuong, N. T. Hoa, and N. T. H. Loan, On certain length functions associated to a system of parameters in local rings, *Vietnam J. Math.* **27** (1999) 259–272.
2. N. T. Cuong and N. T. H. Loan, A characterization for pseudo Buchsbaum modules, *Japanese J. Math.* **30** (2004) 165–181.
3. N. T. Cuong and N. D. Minh, Lengths of generalized fractions of modules having small polynomial type, *Math. Proc. Camb. Phil. Soc.* **128** (2000) 269–282.
4. N. T. Cuong, M. Morales, and L. T. Nhan, On the length of generalized fractions, *J. Algebra* **265** (2003) 100–113.
5. N. T. Cuong and L. T. Nhan, Pseudo Cohen Macaulay and pseudo generalized Cohen Macaulay modules, *J. Algebra* **267** (2005) 156–177.
6. S. Goto and Y. Shimoda, On Rees algebra over Buchsbaum rings, *J. Math. Kyoto. Univ.* (JMKYAZ), **20** (1980) 691–708.
7. R. Hartshorne, Property of A-sequence, *Bull. Soc. Math. France* **4** (1966) 61–66.
8. I. Kaplansky, *Commutative Rings,* Allyn and Bacon, Boston, 1970.
9. H. Matsumura, *Commutative Algebra*, W. A. Benjamin. Inc., 1970.
10. R. Y. Sharp and M. A. Hamieh, Lengths of certain generalized fractions, *J. Pure Appl. Algebra* **38** (1985) 323–336.
11. J. Stückrad and W. Vogel, *Buchsbaum Rings and Applications,* Spinger–Verlag, Berlin–Heidelberg–New York, 1986.