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Generating of Minimal Unavoidable Sets

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Dedicated to Professor Do Long Van on the occasion of his 65$^{\text{th}}$ birthday

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Abstract. In this paper we investigate several transformations on unavoidable sets which preserve the minimality of such sets. The main result confirms that any minimal unavoidable set over an alphabet A can be obtained from A, regarded as the initial manimal unavoidable set, by finitely many applications of such transformations. As a consequence, a procedure to generate all possible minimal unavoidable sets over A is proposed. This allows in particular to generate easily counter-examples for both the Ehrenfeucht's conjecture and Haussler's one on unavoidable sets.

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1. Introduction

In this section we recall some definitions and notations concerning with unavoidable sets and with two well-known conjectures on these sets: the *Ehrenfeucht's conjecture* and *Haussler's conjecture*. For more background we refer to [2, 5].

All the alphabets considered in this paper are supposed to be finite. Given an alphabet A, we denote by A^* the free monoid generated by A and we put $A^+ = A^* - \{\varepsilon\}$, where ε is the empty word. For any $x, y \in A^*$, we denote by xy^{-1} (resp. $y^{-1}x$) the word z satisfying the condition $zy = x$ (resp. $yz = x$) if it exists). Then, for any $X, Y \subseteq A^*$ the right quotient of X by Y, denoted by XY^{-1}, is the set of all xy^{-1} with $x \in X$ and $y \in Y$. The left quotient of X by Y, denoted by $Y^{-1}X$, is defined similarly.

Given $X \subseteq A^*$ and $u, w \in A^*$. We say that w *w meets* u if w contains u as a factor, i.e. $w = xuy$ for some $x, y \in A^*$. We say that w *avoids* X if there does not exist any u in X such that w meets u.

Definition 1.1. *Given an alphabet A and $X \subseteq A^+$. The set X is called an unavoidable set over A if all words in A^*, except for a finite number of them, have factors in X, or equivalently, if there exist only finitely many words avoiding X, i.e. the set $A^* - A^*XA^*$ is finite. A set X not being an unavoidable set is called an avoidable set.*

Given $X \subseteq A^*$, $u \in X$ and $u' = au$ or $u' = ua$ for some $a \in A$. Then the set $X' = X - \{u\} \cup \{u'\}$ is called an *extention* of X at u by a on the left or on the right according as $u' = au$ or $u' = ua$. Also u' is called an extention of u by a on the left or on the right according to the case.

Definition 1.2. *Let X be an unavoidable set over an alphabet A, and $|X| = n$. We say that*

1) *X is extendible if there exists an extention X' of X, $X' = X - \{u\} \cup \{u'\}$, which is still an unavoidable set.*
2) *X is n-minimal (or simply minimal) if $\nexists u \in X$ such that $X - \{u\}$ is still an unavoidable set.*
3) *X is n-reductive (or simply reductive) if X is n-minimal and $\nexists u \in X$ such that $au' = u$ or $u'a = u$ for some $a \in A$.*

Example 1.1.

(1) $X = \{a^2, ab, b^2\}$ is an unavoidable set over $A = \{a, b\}$ because $A^* - A^*XA^* = \{\varepsilon, a, b, ba\}$ which is a finite set. It is easily verified that X is a reduced unavoidable set.
(2) $X = \{a^3, ab, b^2\}$ is an unavoidable set over $A = \{a, b\}$ because $A^* - A^*XA^* = \{\varepsilon, a, a^2, b, ba, ba^2\}$. It can be verified that X is a minimal but not reduced unavoidable set.
(3) $X = \{a^3, ab^2, b^3\}$ is not an avoidable set because all the words of the form $w = (ab)^n$, the number of which is infinite, avoid X.

Remark 1.1. When replacing in X a word u by an extention u' of it, the possibility of avoiding X for any word w does not decrease, and therefore the possibility for X to be unavoidable does not increase.

Now we recall two well-known conjectures on unavoidable sets [3].

Ehrenfeucht's conjecture

For every unavoidable set over an alphabet A, there exists always an extension X' of X which remains an unavoidable set over A. In other words, every unavoidable set is extendible.

Haussler's conjecture

For any reduced unavoidable set X over an alphabet A, the maximal wordlenth of X must be smaller or equal to the cardinality $|X|$ of X.

The following results are due to Choffrut and Culik II [1].

Proposition 1.1. *Every unavoidable set contains a finite unavoidable set. In particular, every minimal unavoidable sets is finite.*

Theorem 1.1. *The Ehrenfeucht's conjecture is true if and only if it is true for the case of two-letter alphabets.*

In [6] Rosaz has constructed a counter-example for Ehrenfeucht's conjecture. In [4], by anather approach, namely by introducing and studying in deep the sets $S_X(u, v)$ (see Definition 1.4), the authors have obtained some other counter-examples for both Ehrenfeucht's Conjecture. In this paper, we investigate several transformations on unavoidable sets which preserve the minimality of such sets. The main result confirms that any minimal unavoidable set over an alphabet A can be obtained from A – the initial minimal unavoidable set – by finitely many applications of such transformations. As a consequence, a procedure to generate all possible minimal unavoidable sets over A is proposed. This allows in particular to generate easily counter-examples for the Ehrenfeucht's and Hausler's conjectures on unavoidable sets.

Convention: By Theorem 1.1, from now on we may restrict ourselves to consider only the case of two-letter alphabets, namely $A = \{a, b\}$.

For any subset $X \subseteq A^+$ we say that

1. *X is a prefix set if $\nexists u \in X : u = vx$ with $v \in X$, $v \neq u$ and $x \in A^*$.*
2. *X is a suffix set if $\nexists u \in X : u = xv$ with $v \in X$, $v \neq u$ and $x \in A^*$.*
3. *X is an infix set if $\nexists u \in X : u = xv$ with $v \in X$, $v \neq u$ and $x, y \in A^*$.*
4. *X is a bifix set if X is suffix and prefix.*

Definition 1.3.

1) *For u, v, w in A^*, if $w = ux = yv$ for some $x, y \in A^*$, $xy \neq \varepsilon$, then w is called a u-v arrow.*
2) *For any u-u arrow w, we define $w_n = w.(w')^n$, where $w' = u^{-1}w$, $n \geq 1$* (Figure 1).
3) *For any u-v arrow w, we say that w avoids X if $w \notin A^+XA^+$.*
4) *For any u-v arrow w, we say that w is X-atomic if w avoids $X \cup \{u, v\}$.*

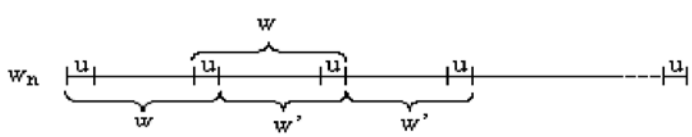

Fig. 1. Definition of u-u arrow w_n basing on u-u arrow w

Lemma 1.1. *Let X be an infix set and $u \in X$. Then, the existence of a X-atomic u-u arrow implies the existence of arbitrarily long u-u arrows avoiding $X - \{u\}$.*

Proof. Let W be a X-atomic u-u arrow. Let us consider a u-u arrow $w_n = w.(w')^n$, where $w' = u^{-1}w$. Because w avoids X and X is infix, w_n avoids $X - \{u\}$. With n large enough, w_n is the u-u arrow required. ∎

Lemma 1.2. *Let X be an unavoidable set. Let $u \in X$ and u is not redundant, i.e. $X - \{u\}$ is not an unavoidable set. Then for all words w, if w is long enough and w avoids $X - \{u\}$ then w meets u.*

Proof. Because u is not redundant, $X - \{u\}$ is an avoidable set. This means that there are infinitely many words avoiding $X - \{u\}$. Among such words there are only finitely many words avoiding u, otherwise X is no more an unavoidable set, that contradicts the hypothesis. Thus, if w is long enough and avoids $X - \{u\}$ then w must meet u. ∎

Consequence 1.1. *For each minimal unavoidable set X, there exists a natural number N_0 such that for all $w \in A^+$ with $|w| \geq N_0$, if $w \notin A^*(X - \{u\})A^*$ then $w \in A^*uA^*$ for any u in X.*

Definition 1.4. *Let X be a subset of A^*. For each pair of words u, v in X, we associate a set $S_X(u, v)$ consisting of all X-atomic u-v arrows:*

$$S_X(u, v) = uA^+ \cap A^+v - A^+XA^+$$

which we write simply $S(u, v)$ when no confusion may arise.

Proposition 1.2. *Let $X \subseteq A^+$, be an unavoidable set. X is minimal if and only if hold the following conditions:*
(i) *X is an infix set.*
(ii) *For all $u \in X$, $S(u, u) \neq \emptyset$.*

Proof. (⇒) Suppose X is minimal, $n = |X|$. We will prove that the conditions (i) and (ii) must be satisfied.
(i) If the condition (i) does not hold then there exist two words $u, v \in X$ such that u is a proper factor of v. Then $X - \{v\}$ is also an unavoidable set, which contradicts the minimality of X. Thus (i) must hold.
(ii) Now assume that the condition (ii) does not hold, i.e. there exists u in X such that $S(u, u) = \emptyset$. Because X is a minimal unavoidable set, by Consequence 1.1, there exists N_0 such that for any $w \in A^+$, if $|w| \geq N_0$ and w avoids $X - \{u\}$ then w meets u. Let w be such a word. Consider the word $w^2 = w.w$. Because w contains at least one u as factor and w avoids $X - \{u\}$, from w^2 we can always extract a u-u arrow that is X-atomic. This means $S(u, u) \neq \emptyset$, a contradiction. Thus (ii) must be hold.

(⇐) Let X be an infix set and $S(u, u) \neq \emptyset$ for all $u \in X$. Since $S(u, u) \neq \emptyset$, there exists a u-u arrow w which is X-atomic. By Lemma 1.1, there exists arbitrarily long u-u one which avoids $X - \{u\}$. This means that, for any u in X, $X - \{u\}$ is not an unavoidable set. Thus X is a minimal unavoidable set. ∎

In fact, Proposition 1.2 is a part of the following result proved in [4] in another way.

Theorem 1.2. [4] *A set X is unavoidable set if and only if the set $S(u, v)$ is finite for any u, v in X, and the unavoidable set X is minimal if and only if X is an infix set and $S(u, u)$ is not empty for all u in X.*

2. Generating Minimal Unavoidable Sets

In this section we introduce and consider transformations on unavoidable sets which preserve the minimality, and show that every minimal unavoidable set over the alphabet $A = \{a, b\}$ can be obtained from A, regarded as the *initial* minimal unavoidable set, by a finite number of applications of such transformations.

First we introduce a label-assigning function, denoted by Asg, which assigns to every word u in an unavoidable set X a pair (l_u, r_u) of labels called the left and the right label of u respectively.

Definition 2.1. *Let X be a minimal unavoidable set on the alphabet $A = \{a, b\}$. For every word u in X, $Asg_X(u) = (l_u, r_u)$ is defined as follows, where S stands for $S(u, u)$.*

+ *If suffix*$(S) \cap Au = \{au, bu\}$ *then* $l_u = Lab$.
+ *If suffix*$(S) \cap Au = \{au\}$ *and* $|S(u, u)| = 1$ *then* $l_u = Lia$.
+ *If suffix*$(S) \cap Au = \{bu\}$ *and* $|S(u, u)| = 1$ *then* $l_u = Lib$.
+ *If suffix*$(S) \cap Au = \{au\}$ *and* $|S(u, u)| > 1$ *then* $l_u = La$.
+ *If suffix*$(S) \cap Au = \{bu\}$ *and* $|S(u, u)| > 1$ *then* $l_u = Lb$.
+ *If prefix*$(S) \cap uA = \{ua, ub\}$ *then* $r_u = Rab$.
+ *If prefix*$(S) \cap uA = \{ua\}$ *and* $|S(u, u)| = 1$ *then* $r_u = Ria$.
+ *If prefix*$(S) \cap uA = \{ub\}$ *and* $|S(u, u)| = 1$ *then* $r_u = Rib$.
+ *If prefix*$(S) \cap uA = \{ua\}$ *and* $|S(u, u)| > 1$ *then* $r_u = Ra$.
+ *If prefix*$(S) \cap uA = \{ub\}$ *and* $|S(u, u)| > 1$ *then* $r_u = Rb$.

The main result of this section is the following:

Theorem 2.1. *Let X be a minimal unavoidable set, where $n = |X|$. Let $u \in X$, and $Asg_X(u) = (l_u, r_u)$.*

(a) *If $l_u = Lab$, then X can not be extended at u on the left to get a new n-minimal unavoidable set, although the set $X' = X - \{u\} \cup \{au, bu\}$ is also an* (n+1)*-minimal unavoidable set.*

(b) *If $l_u = Lia$, then X can be extended consecutively on the left, beginning at u by the letter a and then by appropriate letters, to obtain infinitely many new n-minimal unavoidable sets. Similarly for the case $l_u = Lib$.*

(c) *If $l_u = La$, then X can be extended consecutively on the left only finitely many times, beginning at u by the letter a and then by appropriate letters, to obtain n-minimal unavoidable sets. Similarly for the case $l_u = Lb$.*

(d) *If $r_u = Rab$, then X can not be extended at u on the right to obtain a new n-minimal unavoidable set, although the set $X' = X - \{u\} \cup \{au, bu\}$ is also an* (n+1)*-minimal unavoidable set.*

(e) *If $r_u = Ria$, then X can be extended consecutively on the right, beginning at u by the letter a and then by appropriate letters, to obtain infinitely many new n-minimal unavoidable sets. Similarly for the case $r_u = Rib$.*

(f) *If $r_u = Ra$, then X can be extended consecutively on the right only finitely many times, beginning at u by the letter a and then by appropriate letters, to obtain new n-minimal unavoidable sets. Similarly for the case $r_u = Rb$.*

Proof. (a) A u-u arrows w is said to be of a-form if $w = uy = xau$ with $x \in A^*$, $y \in A^*$, $a \in A$. Arrows of b-form are defined similarly. By Proposition 1.2, $S(u,u) \neq \emptyset$ for all $u \in X$. Because $l_u = Lab$, by Definition 2.1, we have $\text{suffix}(S) \cap Au = \{au, bu\}$. This means that $S(u,u)$ contains u-u arrows of both a-form and b-form. We consider two possibilities:

+ Extending X at u by the letter a on the left: $X' = X - \{u\} \cup \{au\}$. Choose in $S(u,u)$ a u-u arrow w of b-form. Consider an arrow w_n with n large enough. Because w is X-atomic, w avoids $X - \{u\}$. It follows from the minimality of X that X is an infix set. Hence, by Lemma 1.1, the u-u arrows w_n avoid $X - \{u\}$. Because w is X-atomic and w is of b-form, w_n avoids au. Thus w_n avoids X', therefore X' is an unavoidable set.

+ Extending X at u by the letter b on the left $X' = X - \{u\} \cup \{bu\}$. In a similar way we can prove that X' is not an unavoidable.

Thus, X can not be extended on the left to obtain a new n-minimal unavoidable set.

Now we prove that set $X' = X - \{u\} \cup \{au\} \cup \{bu\}$ is a $(n+1)$-minimal unavoidable set. Obviously, every word long enough which avoids both au and bu also avoids u (note that the alphabet A consists of only two letters a and b). Therefore X' is also an unavoidable set. Now we prove that X' is n+1-minimal.

Since $S_X(u,u) \neq \emptyset$ and $\text{suffix}(S) \cap Au = \{au, bu\}$, we have $S_{X'}(au, au) \neq \emptyset$ and $S_{X'}(bu, bu) \neq \emptyset$. Because X is a minimal unavoidable set, it follows by Proposition 1.2 that, $S_X(v,v) \neq \emptyset$, for all $v \in X$, $v \neq u$. It is easily seen that all the words avoiding u also avoid au and bu. This implies that all the words avoiding X also avoid X'. Hence $\emptyset \neq S_X(v,v) \subseteq S_{X'}(v,v)$. Thus, for all $x \in X'$, $S_{X'}(x,x) \neq \emptyset$. From Proposition 1.2. it follows that X' is n+1-minimal.

(b) Let $X' = X - \{u\} \cup \{au\}$. We shall prove that X' is n-minimal unavoidable set.

Indeed, by Consequence 1.1, $\exists N_0$: $\forall w \in A^+$, $|w| \geq N_0$, if w avoids $X - \{u\}$ then w meets u. Consider an arbitrary word v avoiding $X - \{u\}$ with $|v| \geq 2N_0$, we have $v = v_1 v_2$, where $|v_1|, |v_2| \geq N_0$. Since v, v_1, v_2 meet u and avoid $X - \{u\}$, there is a factor w of v which belongs to $S_X(u,u)$. Because au is a suffix of w, v meets au. Thus, X' is unavoidable. Since X is minimal, there are infinitely many words v with $|v| \geq N_0$ which avoid $X - \{u\}$ and do not avoid u. Now, if $|v| \geq 2N_0$ then v avoids $X - \{u\}$ and v meets au. Therfore X' is minimal.

Since $S_X(u,u)$ consists of only one word w, whose suffix is au, $S_{X'}(au, au)$ has the same form, namely $|S_{X'}(u,u)| = 1$ and $\text{suffix}(S_{X'}(u,u)) \cap Au = \{pu\}$ for $p = a$ or $p = b$. Hence $X'' = X' - \{au\} \cup \{pau\}$ is also n-minimal. Continuing this argument we can confirm that X can be infinitely extended on the left at u with the letter a at the first step.

The proof of (c), (d), (e), (f) are similar. ∎

We consider now some basic transformations on any finite set $X \subset A^*$.

Let us note that if $S_X(u, u) = \emptyset$ then u is redundant in X and we can delete it.

Now we consider the following transformations on an unavoidable set X.

+ Left extension by a, denoted La: transforming X into $X' = X - \{u\} \cup \{au\}$, for some u in X with $l_u = La$ or $l_u = Lia$. The transformation Lb is defined similarly.

+ Right extension by a, denoted Ra: transforming X into $X' = X - \{u\} \cup \{ua\}$, for some u in X with $r_u = Ra$ or $r_u = Ria$. The transformation Lb is defined in a similarly way.

+ Left extension by a, b, denoted Lab: transforming X into $X' = X - \{u\} \cup \{au, bu\}$, for some u in X with $l_u = Lab$.

+ Right extension by a, b, denoted Rab: transforming X into $X' = X - \{u\} \cup \{ua, ub\}$, for some u in X with $r_u = Rab$.

+ Left cutting, denote LC: transforming X into X', where X' is obtained from $X - \{u\} \cup \{u'\}$ with $u' = A^{-1}u$, by deleting all words in it which contain u' as a proper factor.

+ Right cutting, denote RC: transforming X into X', where X' is obtained from $X - \{u\} \cup \{u'\}$ with $u' = uA^{-1}$, by deleting all words in it which contain u' as a proper factor.

+ Deletion, denote by D: transforming X into X', where X' is obtained from X by deleting consecutively all redundant words in X.

For brevity, the transformations La, Lb, Lab, (Ra, Rb, Rab) are called commonly *transformation L* (*transformation R*, respectively).

As a direct consequence of Theorem 2.1 we obtain

Consequence 2.1. *Applying transformation L and R on minimal unavoidable sets leads to minimal unavoidable sets again.*

The proofs of the following lemmas are easy and therefore omitted.

Lemma 2.1. *Applying the transformations LC, RC and D on unavoidable sets lead to unavoidable sets again.*

Lemma 2.2. *The initial unavoidable set $A = \{a, b\}$ can be obtained from any finite unavoidable set by finitely many applications of the transformations LC, RC and D.*

Lemma 2.3. *Let X and Y be minimal unavoidable sets such that $Y = X - \{u\} \cup \{u'\}$, where u is a proper factor of u'. Then Y can be obtained from X by a finite number of applications of the transformations L and R.*

The following lemma is somewhat more complex and need some verification.

Lemma 2.4. *Let X be a minimal unavoidable set. If Y can be obtained from X by applying first some transformation RC or LC, and then some transformation*

D, then X can be obtained from Y by a finite number of applications of the transformations L, R, LC, RC and D.

Proof. In the case $|X| = |Y|$ the assertion is true by Lemma 2.3. So we treat only the case $|X| > |Y|$. By duality, it suffices to check only the case applying RC on some word x in X. Without loss of generality we may assume $x = ua$. When the first transformation is RC, by applying RC we get $u \in Y$. By using lemmas above, the following facts can be verified step by step:

+ The right label of u in Y is Rab

+ Each element deleted from X to get Y must be appeared in some ub-ub arrow avoiding Y.

+ Each u-u arrow avoiding $Y - \{u\}$ has to meet some deleted element in $Z = X - Y$.

+ For any x in Z, each long enough x-x arrow avoiding $Y \cup \{ua\}$ has to meet ub.

+ By applying the extension Rab on Y at u in Y, we get X', $X' = Y - \{u\} \cup \{ua, ub\}$, and then by other transformations R and L we can obtain a new minimal unavoidable set Y' so that every deleted element x in Z is a factor of some ub-ub arrow in Y', $|Y'| \geq |X|$.

+ Y' can be also obtained from X by applying some transformations R and L.

+ Applying transformations RC, LC, D on Y' first at ub-ub arrows and then at factors of them we can get X. ■

Now the following main result of this section can be easily proved basing on the above mentioned lemmas.

Theorem 2.2. *Every minimal unavoidable set over the alphabet $A = \{a, b\}$ can be obtained from the initial minimal unavoidable set A by taking finitely many transformations R, L, LC, RC and D.*

It is not difficult to see that the result remains valid for any finite alphabet.

Consequence 2.2. *Every minimal unavoidable set over any finite alphabet A can be obtained from the initial minimal unavoidable set A by applying finitely many transformations R, L, LC, RC and D.*

3. Generating of Counter-examples for Ehrenfeucht's and Haussler's Conjectures

Now we present a computer computer program generating finite minimal unavoidable set starting from the alphabet A. This allows to find out easily counter-examples for Ehrenfeucht's and Haussler's conjectures in one or two hours. The correctness of the procedure can be verified directly basing on Theorem 2.2.

Procedure 3.1. (interactive mode)

1. Input any minimal unavoidable set, say X_0;

2. $X = X_0$;
3. Calculating $Asg_X(u)$ for all u in X;
4. exitOK=0;
5. While exitOK=0 do
6. Begin
   a) SelectOne(Transformation);
   b) Case Transformation of
      L_k: taking one left extension step of type L on x_k in X;
      R_k: taking one right extension step of type R on x_k in X;
      LC_k: cutting one letter on the left of x_k in X by an action of type LC_k;
      RC_k: cutting one letter on the right of x_k in X by an action of type RC_k;
      D_k: deleting x_k in X; { whenever x_k is redundant };
      Write: Save list X to a file;
      Copy: Copy list X to a buffer to re-use later if needed;
      Exit: exitOK=1;
   EndCase;
7. EndWhile;
8. EndProc;

We exhibit below some counter-examples for Ehrenfeucht's and Haussler's conjectures, which have been obtained with the aid of Precedure 3.1.

Example 3.1. Counter-examples for Haussler's conjecture:

$X_n = \{aa, bbb, (ab)^n a, (ab)^n b, bbabb, bba(ba)bb, ..., bba(ba)^{n\text{-}2}bb\}$, $n = 2, 3, ...$

Generating X_2 is presented in Table 3.1.

Remark that the maximal wordlength of X_n is $2n + 1$ which is much larger than $n = |X|$. In [4] it is showed that there are infinitely many reduced unavoidable sets X whose maximal wordlength is of the size $O(c^n)$, where $c > 1$ is a constant.

Table 3.1. Generating counter-example for Haussler Conjecture from $A = \{a, b\}$

Step	Left label	Right label	Unavoidable sets	Next operation
1	Lia Lib	Ria Rib	a b	R1
2	Lia Lab	Ria Rab	aa b	 R2
3	Lia Lia Lib	Ria Rib Rib	aa ba bb	 R3
4	Lia Lab Lib	Ria Rb Rib	aa ba bbb	 R2

Step	Left label	Right label	Unavoidable sets	Next operation
5	Lab Lab Lib	Rab Rab Rib	aa bab bbb	L2
6	Lab Lib Lia Lib	Rab Ria Rib Rib	aa abab bbab bbb	R2
7	Lab Lib La Lib	Rab Rib Rab Rib	aa ababa bbab bbb	R2
8	Lab Lib La Lib	Rab Ria Rab Rib	aa ababab bbab bbb	R3
9	Lab Lib Lia Lia Lib	Rab Ria Rib Ria Rib	aa ababab bbaba bbabb bbb	R3
10	Lab Lib Lia Lia Lib	Rab Ria Rib Ria Rib	aa ababab bbabab bbabb bbb	R3
11	Lab Lb Lia Lia Lib	Rab Rab Ria Ria Rib	aa ababab bbababb bbabb bbb	R2
12	Lab Lib Lib Lia Lia Lib	Rab Rib Ria Ria Ria Rib	aa abababa abababb bbababb bbabb bbb	

Example 3.2. Counter-examples for Ehrenfeucht's conjecture:

a) $X = \{a^3, b^4, ab^3ab, ab^2ab, abab, b^2a^2b^2, ba^2ba^2b\}$
(see Table 3.2). This counter-examples has been obtained first by Rozas [6].

b) $X = \{a^4, b^4, ba^3b, ba^2ba^2b, baba, b^2a^2b^2, bab^2a, bab^3a\}$,
c) $X = \{a^3, b^4, a^2ba^2b, baba, b^2a^2b^2, bab^2a, bab^3a\}$...

Table 3.2. Generating counter-example for Ehrenfeut conjecture from $\{a, b\}$

Step	Left label	Right label	Unavoidable sets	Next operation
1	Lia Lib	Ria Rib	a b	R1
2	Lia Lab	Ria Rab	aa b	 L2
3	Lia Lab Lib	Ria Ra Rib	aaa ab bb	 R3
4	Lia Lab Lib	Ria Rab Rib	aaa ab bbb	 R3
5	Lia Lab Lib	Ria Rab Rib	aaa ab bbbb	 R2
6	Lia Lab Lab Lib	Ria Rab Rab Rib	aaa aba abb bbbb	 R2
7	Lia Lia Lib Lab Lib	Ria Rib Ria Rab Rib	aaa abaa abab abb bbbb	 R4
8	Lia Lia Lib Lab Lab Lib	Ria Rib Ria Rab Ra Rib	aaa abaa abab abba abbb bbbb	 R5
9	Lia Lia Lib Lab Lab Lab	Ria Rib Ria Rab Rab Rab	aaa abaa abab abba abbba bbbb	 R4

Step	Left label	Right label	Unavoidable sets	Next operation
10	Lia	Ria	aaa	
	Lia	Rib	abaa	
	Lib	Ria	abab	
	Lia	Rib	abbaa	R4
	Lib	Rib	abbab	
	Lab	Rab	abbba	
	Lab	Rab	bbbb	
11	Lab	Rab	aaa	
	Lia	Rib	abaa	R2
	Lib	Ria	abab	
	Lia	Rib	abbaab	
	Lib	Rib	abbab	
	Lab	Rab	abbba	
	Lab	Rab	bbbb	
12	Lab	Rab	aaa	
	Lia	Ria	abaab	L2
	Lib	Ria	abab	
	Lia	Rib	abbaab	
	Lib	Rib	abbab	
	Lab	Rab	abbba	
	Lab	Rab	bbbb	
13	Lab	Rab	aaa	
	Lib	Ria	aabaab	
	Lab	Ra	abab	
	Lia	Rib	abbaab	
	Lab	Rab	abbab	
	Lab	Rab	abbba	R6
	Lab	Rab	bbbb	
14	Lab	Rab	aaa	
	Lia	Rib	baabaab	
	Lab	Ra	abab	
	Lia	Rib	abbaab	LC4
	Lab	Rab	abbab	
	Lab	Rab	abbba	
	Lab	Rab	bbbb	
15	Lab	Rab	aaa	
	Lia	Ria	baabaab	
	Lab	Ra	abab	
	Lia	Rib	bbaab	
	Lab	Rab	abbab	
	Lab	Rb	abbba	R6
	Lab	Rab	bbbb	

Step	Left label	Right label	Unavoidable sets	Next operation
16	Lia	Ria	aaa	
	Lib	Rib	aabaa	R2
	Lab	Ra	abab	
	Lab	Rb	bbaa	
	Lab	Rab	abbab	
	Lab	Rab	abbbab	
	Lab	Rab	bbbb	
17	Lab	Rab	aaa	
	Lib	Ria	aabaab	L2
	Lab	Ra	abab	
	Lab	Rb	bbaa	
	Lab	Rab	abbab	
	Lab	Rab	abbbab	
	Lab	Rab	bbbb	
18	Lab	Rab	aaa	
	Lia	Ria	baabaab	
	Lab	Ra	abab	
	Lab	Rb	bbaa	R4
	Lab	Rab	abbab	
	Lab	Rab	abbbab	
	Lab	Rab	bbbb	
19	Lab	Rab	aaa	
	Lia	Ria	baabaab	
	Lab	Ra	abab	
	Lab	Rb	bbaab	R4
	Lab	Rab	abbab	
	Lab	Rab	abbbab	
	Lab	Rab	bbbb	
20	Lab	Rab	aaa	
	Lab	Rab	baabaab	
	Lab	Rab	abab	
	Lab	Rab	bbaabb	
	Lab	Rab	abbab	
	Lab	Rab	abbbab	
	Lab	Rab	bbbb	

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