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Amenable Locally Compact Foundation Semigroups

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Abstract. Let S be a locally compact Hausdorff topological semigroup, and $M(S)$ be the Banach algebra of all bounded regular Borel measures on S. Let $M_a(S)$ be the space of all measures $\mu \in M(S)$ such that both mapping $x \mapsto \delta_x * |\mu|$ and $x \mapsto |\mu| * \delta_x$ from S into $M(S)$ are weakly continuous.

In this paper, we present a few results in the theory of amenable foundation semigroups. A number of theorems are established about left invariant mean of a foundation semigroup. In particular, we establish theorems which show that $M_a(S)^*$ has a left invariant mean. Some results were previously known for groups.

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1. Introduction

Let S be a locally compact Hausdorff topological semigroup and $M(S)$ the Banach algebra of all bounded regular Borel measures on S with total variation norm and convolution $\mu * \nu$, $\mu, \nu \in M(S)$ as multiplication where

$$\int f d\mu * \nu = \int \int f(xy) d\mu(x) d\nu(y) = \int \int f(xy) d\nu(y) d\mu(x)$$

for $f \in C_\circ(S)$ the space of all continuous functions on S which vanish at infinity. (see for example [5, 11] or [13]). Let $M_\circ(S)$ be the set of all probability measures in $M(S)$. Let $M_a(S)$ ([1, 5, 12]) denote the space of all measures $\mu \in M(S)$ such that both mappings $x \mapsto \delta_x * |\mu|$ and $x \mapsto |\mu| * \delta_x$ from S into $M(S)$ are weakly continuous. A semigroup S is called a *foundation semigroup* if $\bigcup\{\text{supp}\mu;\ \mu \in M_a(S)\}$ is dense in S. In this paper, we may assume that S is a foundation locally compact Hausdorff topological semigroup with identity e. Note that $M_a(S)$ is a closed two-sided L-ideal of $M(S)$ [5]. We also note that for $\mu \in M_a(S)$ both mappings $x \mapsto \delta_x * |\mu|$ and $x \mapsto |\mu| * \delta_x$ from S into $M(S)$ are norm continuous [5]. It is known that $M_a(S)$ admits a bounded approximate identity [11].

We know that $M_a(S)$ is a Banach algebra with total variation norm and convolution, so we can define the first Arens product on $M_a(S)^{**}$, i.e., for $F, G \in M_a(S)^{**}$ and $f \in M_a(S)^*$

$$\langle FG, f\rangle = \langle F, Gf\rangle,\ \langle Gf, \mu\rangle = \langle G, f\mu\rangle,\ \langle f\mu, \nu\rangle = \langle f, \mu * \nu\rangle$$

, where $\mu, \nu \in M_a(S)$. For $\mu \in M_a(S)$, $\nu \in M(S)$ and $f \in M_a(S)^*$, we define $\langle f\nu, \mu\rangle = \langle f, \nu * \mu\rangle$ and $\langle \nu, f\mu\rangle = \langle f, \mu * \nu\rangle$. In [6] the author defined $B = M_a(S)^* M_a(S)$ which is a Banach subspace of $M_a(S)^*$. Clearly $M(S) \subseteq B^*$.

We denote by $LUC(S)$ the space of all $f \in C_b(S)$ (the space of bounded continuous complex-valued functions on S) for which the mapping $x \mapsto L_x f$ (where $L_x f(y) = f(xy)(y \in S)$) from S into $C_b(S)$ is norm continuous. The author [6] recently proved that the mapping $T : LUC(S) \to B$ given by $\langle T(f), \mu\rangle = \int f(x) d\mu(x)$ is an isometric isomorphism of $LUC(S)$ onto B.

Denote by 1 the element in $M_a(S)^*$ such that $\langle 1, \mu\rangle = \mu(S)$ ($\mu \in M_a(S)$). A linear functional $M \in M_a(S)^{**}$ is called a *mean* if $\langle M, f\rangle \geq 0$ whenever $f \geq 0$ and $\langle M, 1\rangle = 1$. Obviously, every probability measure μ in $M_\circ(S) \bigcap M_a(S)$ is a mean. A mean M on $M_a(S)^*$ is called *topologically left invariant mean* if $\langle M, f\mu\rangle = \langle M, f\rangle$ for any $\mu \in M_\circ(S)$ and $f \in M_a(S)^*$. A mean M on $M_a(S)^*$ is a *left invariant mean* if $\langle M, f\delta_x\rangle = \langle M, f\rangle$ for any $x \in S$ and $f \in M_a(S)^*$. Obviously, a topologically left invariant mean on $M_a(S)^*$ is also a left invariant mean on $M_a(S)^*$ (for more on invariant mean on locally compact semigroup, the reader is referred to ([2, 4, 13, 14])).

Finally, we denote by $P(S)$ the convex set formed by the probability measures in $M_a(S)$, that is, all $\mu \in M_a(S)$ for which $\langle 1, \mu\rangle = 1$ and $\mu \geq 0$.

We shall follow Ghaffari [8] and Wong [18, 19] for definitions and terminologies not explained here. We know that topologically left invariant mean on $M(S)^*$ have been studied by Riazi and Wong in [16] and by Wong in [18, 19]. They also went further and for several subspaces X of $M(S)^*$, have obtained a number of interesting and nice results. Also, Junghenn [10] studied topological left amenability of semidirect product.

In this paper, among other things, we obtain a necessary and sufficient condition for $M_a(S)^*$ to have a topologically left invariant mean.

2. Main Results

Our starting point of this section is the following lemma whose proof is straightforward.

Lemma 2.1. *A linear functional M on $M_a(S)^*$ is a mean on $M_a(S)^*$ if and only if any pair of the following conditions hold:*
(1) *M is nonnegative, that is, $\langle M, f\rangle \geq 0$ whenever $f \geq 0$.*
(2) *$\langle M, 1\rangle = 1$.*
(3) *$\|M\| = 1$.*

Lemma 2.2. *A linear functional M on $M_a(S)^*$ is a mean on $M_a(S)^*$ if and only if*

$$\inf\{\langle f, \mu\rangle;\ \mu \in P(S)\} \leq \langle M, f\rangle \leq \sup\{\langle f, \mu\rangle;\ \mu \in P(S)\},$$

for every $f \in M_a(S)^*$ with $f \geq 0$.

Proof. The statement follows directly from Lemma 2.1. ∎

For a locally compact abelian group G, $M_a(G) = L^1(G)$, $M_a(G)^* = L^\infty(G)$ and $f\delta_x = L_x f$ for any $f \in L^\infty(G)$ and $x \in G$. Also, if $\varphi \in L^1(G)$, $f\varphi = \tilde{\varphi} * f$, where $\tilde{\varphi}(x) = \varphi(x^{-1})$. Granirer in [9] has shown that for a nondiscrete abelian locally compact group G, there is a left invariant mean on $L^\infty(G)$ which is not a topologically left invariant mean on $L^\infty(G)$.

In the following theorem, we will show that every left invariant mean on B is a topologically left invariant mean on B.

Theorem 2.1. *Let M be a mean on B. Then M is a topologically left invariant mean on B if and only if M is a left invariant mean on B.*

Proof. It is clear that every topologically left invariant mean on B is a left invariant mean on B.

To prove the converse, let $\mu \in M_\circ(S)$, $f \in B$ and $\epsilon > 0$. By [6, Lemma 2.3] there exists a combination of members of $M_\circ(S)$, that is, $t_1\delta_{x_1} + \cdots + t_n\delta_{x_n}$ in which $x_i \in S$, $t_i \geq 0$ and $\sum_{i=1}^n t_i = 1$, such that

$$\|f\mu - \sum_{i=1}^n t_i f\delta_{x_i}\| < \epsilon.$$

So $|\langle M, f\mu\rangle - \langle M, f\rangle| < \epsilon$. It follows that $\langle M, f\mu\rangle = \langle M, f\rangle$, i.e., M is a topologically left invariant mean on B. ∎

Let M be a left invariant mean on $M_a(S)^*$. There exists a net (μ_α) in $P(S)$ such that for every $x \in S$, $\delta_x * \mu_\alpha - \mu_\alpha \to 0$ in the weak topology. For every finite subset $\{x_1, ..., x_n\}$ of S, it is easy to find a net (ν_α) in $P(S)$ such that $\|\delta_{x_i} * \nu_\alpha - \nu_\alpha\| \to 0$ for $1 \leq i \leq n$. An argument similar to the proof of Theorem 2.2 in [7] shows that, there is a net (μ_α) in $P(S)$ such that, $\|\delta_x * \mu_\alpha - \mu_\alpha\| \to 0$ for every $x \in S$.

Let there exist a net (μ_α) in $P(S)$ such that $\|\delta_x * \mu_\alpha - \mu_\alpha\| \to 0$ whenever $x \in S$. The net (μ_α) admits a subnet (μ_β) converging to a mean N on $M_a(S)^*$ in the weak* topology. For all $f \in M_a(S)^*$ and $x \in S$,

$$\begin{aligned}\langle N, f\delta_x\rangle &= \lim_\beta\langle \mu_\beta, f\delta_x\rangle = \lim_\beta\langle f, \delta_x * \mu_\beta\rangle \\ &= \lim_\beta\langle f, \mu_\beta\rangle = \langle N, f\rangle,\end{aligned}$$

that is, N is a left invariant mean on $M_a(S)^*$.

If M is a topologically left invariant mean on $M_a(S)^*$, as above we can find a net (μ_α) in $P(S)$ such that $\|\mu*\mu_\alpha - \mu_\alpha\| \to 0$ for all $\mu \in M_\circ(S)$. An argument similar to the proof of Theorem 2.2 in [7] shows that, there is a net (μ_α) in $P(S)$ such that for every compact subset K of S, $\|\mu * \mu_\alpha - \mu_\alpha\| \to 0$ uniformly over all μ in $M_\circ(S)$ which are supported in K.

Theorem 2.2. *The following statements are equivalent:*
(1) *$M_a(S)^*$ has a left invariant mean.*
(2) *(Riter's condition) for every compact subset K of S and every $\epsilon > 0$, there exists $\mu \in P(S)$ such that $\|\delta_x * \mu - \mu\| < \epsilon$ whenever $x \in K$.*
(3) *(Riter's condition) for every finite subset F of S and every $\epsilon > 0$, there exists $\mu \in P(S)$ such that $\|\delta_x * \mu - \mu\| < \epsilon$ whenever $x \in F$.*

Note that this is Proposition 6.12 in [15], which was proved for groups. However, our proof is completely different.

Proof. Let $M_a(S)^*$ have a left invariant mean. Then B has a left invariant mean. By Theorem 2.1, B has a topologically left invariant mean. So, there exists a net (μ_α) in $P(S)$ such that $\lim_\alpha \|\mu * \mu_\alpha - \mu_\alpha\| = 0$ whenever $\mu \in P(S)$. Choose $\nu \in P(S)$ and let $\nu_\alpha = \nu * \mu_\alpha$, $\alpha \in I$. It is easy to see that $\lim_\alpha \|\delta_x * \nu_\alpha - \nu_\alpha\| = 0$ for all $x \in S$ $(*)$.

Let K be a compact subset of S and let $\epsilon > 0$. For any $x \in S$, there exists a nighbourhood U_x of x such that $\|\delta_x * \nu - \delta_y * \nu\| < \epsilon$ whenever $y \in U_x$. We may determine a subset $\{x_1, ..., x_n\}$ in S such that $K \subseteq \bigcup_{i=1}^n U_{x_i}$ and $\|\delta_{x_i} * \nu - \delta_y * \nu\| < \epsilon$ whenever $y \in U_{x_i}$ $(i = 1, ..., n)$. By $(*)$ there exists $\alpha_\circ \in I$ such that for any $i \in \{1, ..., n\}$, $\|\delta_{x_i} * \nu_{\alpha_\circ} - \nu_{\alpha_\circ}\| < \epsilon$. For any $x \in K$, there exists $i \in \{1, ..., n\}$ such that $x \in U_{x_i}$. Then we have

$$\begin{aligned}\|\delta_x * \nu_{\alpha_\circ} - \nu_{\alpha_\circ}\| &\le \|\delta_x * \nu_{\alpha_\circ} - \delta_{x_i} * \nu_{\alpha_\circ}\| + \|\delta_{x_i} * \nu_{\alpha_\circ} - \nu_{\alpha_\circ}\| \\ &< \|\delta_x * \nu * \mu_{\alpha_\circ} - \delta_{x_i} * \nu * \mu_{\alpha_\circ}\| + \epsilon \\ &< \|\delta_x * \nu - \delta_{x_i} * \nu\| + \epsilon < 2\epsilon.\end{aligned}$$

Thus (1) implies (2).

(2) implies (3) is easy.

Now, assume that (3) holds. We will show that $M_a(S)^*$ has a left invariant mean. To every finite subset F in S and each $\epsilon > 0$, we associate the nonempty subset

$$\Omega_{F,\epsilon} = \{\mu \in P(S);\ \|\delta_x * \mu - \mu\| < \epsilon \quad \text{for all } x \in F\}.$$

We know that the weak* closure $\overline{\Omega_{F,\epsilon}}$ of $\Omega_{F,\epsilon}$ is compact (see Theorem 3.15 in [17]). Since the family

$$\{\Omega_{F,\epsilon};\ \epsilon > 0,\ F \text{ is a finite subset in } S\}$$

has the finite intersection property, therefore there exists $M \in M_a(S)^{**}$ such that

$$M \in \bigcap_{F,\epsilon} \overline{\Omega_{F,\epsilon}}.$$

Choose $f \in M_a(S)^*$, $x \in S$ and $\epsilon > 0$. The set of all $N \in M_a(S)^{**}$ such that $|\langle M, f\delta_x\rangle - \langle N, f\delta_x\rangle| < \epsilon$ and $|\langle M, f\rangle - \langle N, f\rangle| < \epsilon$ is a weak* neighborhood of M. Therefore $\Omega_{\{x\},\epsilon}$ contains such an μ. We have $|\langle M, f\rangle - \langle \mu, f\rangle| < \epsilon$ and $|\langle M, f\delta_x\rangle - \langle \mu, f\delta_x\rangle| < \epsilon$. So that

$$\begin{aligned}|\langle M, f\delta_x\rangle - \langle M, f\rangle| &\le |\langle M, f\delta_x\rangle - \langle \mu, f\delta_x\rangle| + |\langle \mu, f\delta_x\rangle - \langle \mu, f\rangle| \\ &\quad + |\langle M, f\rangle - \langle \mu, f\rangle| \le \epsilon + |\langle \delta_x * \mu - \mu, f\rangle| + \epsilon \\ &< 2\epsilon + \epsilon\|f\|.\end{aligned}$$

Since ϵ was arbitrary, we see $\langle M, f\delta_x\rangle = \langle M, f\rangle$. This shows that M is a left invariant mean on $M_a(S)^*$. ∎

In the following theorem, we establish a characterization of amenability terms of limits of averaging operators.

Theorem 2.3. *If $\mu \in M_a(S)$, we define*

$$d_l(\mu) = \inf\{\|\mu * \eta\|,\ \eta \in M_\circ(S)\}.$$

$M_a(S)^*$ has a topologically left invariant mean if and only if $d_l(\mu) = |\mu(S)|$ for all $\mu \in M_a(S)$.

Proof. Let $M_a(S)^*$ have a topologically left invariant mean. Let $\mu \in M_a(S)$ and $\epsilon > 0$. For every $\eta \in M_\circ(S)$, we have

$$\|\mu * \eta\| \ge |\mu * \eta(S)| = |\mu(S)|.$$

It follows that $d_l(\mu) \ge |\mu(S)|$. On the other hand, there exists a compact subset K in S such that $|\mu|(S \setminus K) < \epsilon$. By Theorem 2.2, there exists a measure ν in $P(S)$ such that $\|\delta_x * \nu - \nu\| < \epsilon$ whenever $x \in K$. For every $f \in M_a(S)^*$, by Lemma 2.1 in [6], we can write

$$\begin{aligned}|\langle f, \mu * \nu\rangle - \mu(S)\langle f, \nu\rangle| &= \Big|\int \langle f, \delta_x * \nu\rangle\ d\mu(x) - \mu(S)\langle f, \nu\rangle\Big| \\ &= \Big|\int \langle f, \delta_x * \nu\rangle - \langle f, \nu\rangle\ d\mu(x)\Big| \\ &\le \Big|\int_K \langle f, \delta_x * \nu\rangle - \langle f, \nu\rangle\ d\mu(x)\Big| \\ &\quad + \Big|\int_{S\setminus K} \langle f, \delta_x * \nu\rangle - \langle f, \nu\rangle\ d\mu(x)\Big| \\ &\le \|f\| \int_K \|\delta_x * \nu - \nu\| d|\mu|(x) + 2\|f\||\mu|(S \setminus K) \\ &\leqslant \epsilon\|f\|\|\mu\| + 2\|f\||\mu|(S \setminus K).\end{aligned}$$

It follows that

$$\|\mu * \nu - \mu(S)\nu\| \leqslant \epsilon\|\mu\| + 2|\mu|(S \setminus K),$$

and so

$$\|\mu * \nu\| \leqslant \epsilon(\|\mu\| + 2) + |\mu(S)|.$$

As $\epsilon > 0$ may be chosen arbitrary,

$$\inf\{\|\mu * \eta\|;\ \eta \in M_\circ(S)\} = |\mu(S)|.$$

Conversely, suppose that $d_l(\mu) = |\mu(S)|$ for all $\mu \in M_a(S)$. Let $\epsilon > 0$, $\mu_1, ..., \mu_n \in M_\circ(S)$. Since $\mu_1 - \delta_e(S) = 0$, there exists a measure $\nu_1 \in P(S)$ such that $\|\mu_1 * \nu_1 - \nu_1\| < \epsilon$. Since $\mu_2 * \nu_1 - \nu_1(S) = 0$, there exists a measure $\nu_2 \in P(S)$ such that $\|\mu_2 * \nu_1 * \nu_2 - \nu_1 * \nu_2\| < \epsilon$. Proceeding in this way, we produce $\eta \in P(S)$ such that

$$\|\mu_i * \eta - \eta\| < \epsilon \qquad (1 \le i \le n).$$

An argument similar to the proof of Theorem 2.2 shows that $M_a(S)^*$ has a topologically left invariant mean. ■

Let S be a locally compact semigroup. A left Banach S-module A is a Banach space A which is a left S-module such that:

(1) $\|x.a\| \le \|a\|$ for all $a \in A$ and $x \in S$.

(2) for all $x, y \in S$ and $a \in A$, $x.(y.a) = (xy).a$.

(3) for all $a \in A$, the map $x \mapsto x.a$ is continuous from S into A.

We define similarly a right dual S-module structure on A^* by putting $\langle f.x, a\rangle = \langle f, x.a\rangle$. Define

$$\langle x.F, f\rangle = \langle F, f.x\rangle,$$

for all $x \in S$, $f \in A^*$ and $F \in A^{**}$. If $\mu \in M(S)$, $f \in A^*$ and $a \in A$, we define

$$\langle f.\mu, a\rangle = \int \langle f, x.a\rangle d\mu(x).$$

We also define $\langle \mu.F, f\rangle = \langle F, f.\mu\rangle$, for all $\mu \in M(S)$, $f \in A^*$ and $F \in A^{**}$.

By the *weak* * *operator topology* on $\mathcal{B}(A^{**})$, we shall mean the weak* topology of $\mathcal{B}(A^{**})$ when it is identified with the dual space $(A^{**} \bigotimes A^*)^*$. We denote by $\mathcal{P}(A^{**})$ the closure of the set $\{T_\mu;\ \mu \in P(S)\}$ in the weak* operator topology, where $T_\mu \in \mathcal{B}(A^{**})$ is defined by $T_\mu(F) = \mu.F$ for all $F \in A^{**}$.

Theorem 2.4. *The following two statements are equivalent:*

(1) *$M_a(S)^*$ has a topologically left invariant mean.*

(2) *For each left Banach S-module A, there exists $T \in \mathcal{P}(A^{**})$ such that $T_\mu T = T$ for all $\mu \in P(S)$.*

Proof. Let $M_a(S)^*$ have a topologically left invariant mean. There exists a net (μ_α) in $P(S)$ such that $\|\mu * \mu_\alpha - \mu_\alpha\| \to 0$ for each $\mu \in P(S)$. Hence we may find $T \in \mathcal{B}(A^{**})$ with $\|T\| \le 1$ and a subnet (μ_β) of (μ_α) such that $T_{\mu_\beta} \to T$ in the weak* operator topology. For every $\mu \in P(S)$ and $F \in A^{**}$, we have

$$\begin{aligned}\|T_\mu T_{\mu_\beta}(F) - T_{\mu_\beta}(F)\| &= \|T_{\mu*\mu_\beta}(F) - T_{\mu_\beta}(F)\| \\ &\le \|\mu * \mu_\beta - \mu_\beta\|\|F\| \to 0.\end{aligned}$$

Consequently $T_\mu T = T$.

To prove the converse, let $A = M_a(S)$. If $\mu \in A$ and $x \in S$, let $x.\mu = \delta_x * \mu$. It is easy to see that A is a left Banach S-module. By assumption, there exists a net (μ_α) in $P(S)$ such that $T_{\mu_\alpha} \to T$ in the weak* operator topology of $\mathcal{B}(A^{**})$. We may assume by passing to a subnet if necessary that $\mu_\alpha \to M$ in the weak* topology of A^{**}. Let (e_β) be a bounded approximate identity of $M_a(S)$ bounded by 1 [11], and let E be a weak*-cluster point of (e_β). For every $\mu \in P(S)$ and $f \in M_a(S)^*$, we have

$$\begin{aligned}\langle M, f\mu\rangle = \langle M, E(f\mu)\rangle &= \langle ME, f\mu\rangle = \langle T(E), f\mu\rangle \\ &= \langle \mu T(E), f\rangle = \langle T_\mu T(E), f\rangle \\ &= \langle T(E), f\rangle = \langle M, f\rangle.\end{aligned}$$

Consequently M is a topologically left invariant mean. ■

Let $\mathcal{V}$ be a locally convex Hausdorff topological vector space and let $\mathcal{Z}$ be a compact convex subset of $\mathcal{V}$. An action of $M_a(S)$ on $\mathcal{V}$ is a bilinear mapping $T : M_a(S) \times \mathcal{V} \to \mathcal{V}$ denoted by $(\mu, v) \mapsto T_\mu(v)$ such that $T_{\mu*\nu} = T_\mu o T_\nu$ for any $\mu, \nu \in M_a(S)$. We say that $\mathcal{Z}$ is $P(S)$-invariant under the action $M_a(S) \times \mathcal{V} \to \mathcal{V}$, if $T_\mu(\mathcal{Z}) \subseteq \mathcal{Z}$ for any $\mu \in P(S)$.

Theorem 2.5. *The following two statements are equivalent:*

(1) *$M_a(S)^*$ has a topologically left invariant mean.*

(2) *For any separately continuous action $T : M_a(S) \times \mathcal{V} \to \mathcal{V}$ of $M_a(S)$ on $\mathcal{V}$ and any compact convex $P(S)$-invariant subset $\mathcal{Z}$ of $\mathcal{V}$, there is some $z \in \mathcal{Z}$ such that $T_\mu(z) = z$ for all $\mu \in P(S)$.*

Proof. Let $M_a(S)^*$ have a topologically left invariant mean. If M is any topologically left invariant mean on $M_a(S)^*$, the weak* density of $P(S)$ in the set of all means on $M_a(S)^*$ insures that we can find weak* convergent net $(\mu_\alpha) \subseteq P(S)$ such that $\mu_\alpha \to M$. Consider the net $T_{\mu_\alpha}(z)$ where $z \in \mathcal{Z}$ is arbitrary but fixed. By compactness of $\mathcal{Z}$, we can assume $T_{\mu_\alpha}(z) \to z_\circ$ in $\mathcal{Z}$, passing to a subnet if necessary. If $x^* \in \mathcal{V}^*$, we consider the mapping $f : M_a(S) \to \mathbb{C}$ given by $\langle f, \mu\rangle = \langle x^*, T_\mu(z)\rangle$. It is easy to see that $f \in M_a(S)^*$. For every $\mu \in P(S)$, we have

$$\begin{aligned}\langle x^*, z_\circ\rangle &= \lim_\alpha \langle x^*, T_{\mu_\alpha}(z)\rangle = \lim_\alpha \langle f, \mu_\alpha\rangle = \lim_\alpha \langle \mu_\alpha, f\rangle \\ &= \langle M, f\rangle = \langle M, f\mu\rangle = \lim_\alpha \langle \mu_\alpha, f\mu\rangle = \lim_\alpha \langle f, \mu * \mu_\alpha\rangle \\ &= \lim_\alpha \langle x^*, T_{\mu*\mu_\alpha}(z)\rangle = \lim_\alpha \langle x^*, T_\mu o T_{\mu_\alpha}(z)\rangle = \lim_\alpha \langle x^* o T_\mu, T_{\mu_\alpha}(z)\rangle \\ &= \langle x^* o T_\mu, z_\circ\rangle = \langle x^*, T_\mu(z_\circ)\rangle.\end{aligned}$$

So $T_\mu(z_\circ) = z_\circ$ for every $\mu \in P(S)$, i.e., $z_\circ$ is a fixed point under the action of $P(S)$.

To prove the converse, let $\mathcal{V}= M_a(S)^{**}$ with weak* topology. We define an action $T : M_a(S)\times M_a(S)^{**} \to M_a(S)^{**}$ by putting $T_\mu(F) = \mu F$ for $\mu \in M_a(S)$ and $F \in M_a(S)^{**}$. Then clearly T is a separately continuous action of $M_a(S)$ on $M_a(S)^{**}$.

Let $\mathcal{Z}$ be the convex set of all means on $M_a(S)^*$. We know that the set $\mathcal{Z}$ is convex and weak* compact in $M_a(S)^{**}$. Clearly $\mathcal{Z}$ is $P(S)$-invariant under T. By assumption, there exists $M \in \mathcal{Z}$, which is fixed under the action of $P(S)$, that is $\mu M = M$ for every $\mu \in P(S)$. It follows that M is a topologically left invariant mean on $M_a(S)^*$. This completes our proof. ■

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