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Some Remarks on Set-Valued Minty Variational Inequalities

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Abstract. The paper generalizes to variational inequalities with a set-valued formulation some results for scalar and vector Minty variational inequalities of differential type. It states that the existence of a solution of the (set-valued) variational inequality is equivalent to an increasing-along-rays property of the set-valued function and implies that the solution is also a point of efficiency (minimizer) for the underlying set-valued optimization problem. A special approach is proposed in order to treat in a uniform way the cases of several efficient points. Applications to a-minimizers (absolute or ideal efficient points) and w-minimizers (weakly efficient points) are given. A comparison among the commonly accepted notions of optimality in set-valued optimization and these which appear to be related with the set-valued variational inequality leads to two concepts of minimizers, called here point minimizers and set minimizers. Further the role of generalized (quasi)convexity is highlighted in the process of defining a class of functions, such that each solution of the set-valued optimization problem solves also the set-valued variational inequality. For a-minimizers and w-minimizers it appears to be useful $*$-quasiconvexity and C-quasiconvexity for set-valued functions.

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1. Introduction

Variational inequalities (for short, VI) provide suitable mathematical models for a range of practical problems, see e.g.[3] or [25]. Vector VI were introduced first in [16] and studied intensively. For a survey and some recent results we refer to [2, 15, 17, 26]. Stampacchia VI and Minty VI (see e.g. [36, 31]) are the most investigated types of VI. In both formulations the differential type plays a crucial role in the study of equilibrium models and optimization. In this framework, Minty VI characterize more qualified equilibria than Stampacchia VI. This means that, when a solution of a Minty VI exists, then the associated primitive function has some regularity properties. In [7] for scalar Minty VI of differentiable type we observe that the primitive function increases along rays (IAR property). We try to generalize this result to vector VI firstly in [9] and then in [7]. In [13] the problem has been studied to define a general scheme, which allows to copy with various type of efficient solution defining for each proper VI of Minty type.

The present paper is an attempt to apply these results also to set-valued optimization problems.

We prove, within the framework of set-valued optimization, that solutions of Minty VI, optimal solution and some monotonicity along rays property are related to each other. This result is developed in a general setting, which allows to recover ideal minimizer and weak minimizer as a special case. Other type of optimal solutions to a set-valued optimization problem can also be readily available within the same scheme. Moreover we introduce the notions of a set a-minimizer and set w-minimizer and compare them to well known notions of a-minimizer and w-minimizer for set-valued optimization. Wishing to distinguish a class of functions, for which each solution of the set-valued optimization problem solves also the set-valued variational inequality, we define generalized quasiconvex set-valued function. In the case of a-minimizers and w-minimizers the classes of $*$-quasiconvex and C-quasiconvex set-valued functions are involved.

In Sec. 2 we pose the problem and define a set-valued VI raising the scheme from [10]. In Sec. 3 we develop for set-valued problems the more flexible scheme from [13]. In Secs. 4 and 5 we give applications of the main result to a-minimizers and w-minimizers. Sec. 6 discusses generalized quasiconvexity of set-valued functions associated to the set-valued VI.

As a whole, like in [32] we base our investigation on methods of nonsmooth analysis.

2. Notation and Setting

In the sequel X denotes a real linear space and K is a convex set in X. Further Y is a real topological vector space and $C \subset Y$ is a closed convex cone.

In [7] we consider the scalar case $Y = \boldsymbol{R}$ and investigate the scalar (generalized) Minty VI of differential type

$$f'(x, x^0 - x) \leqslant 0, \quad x \in K, \tag{1}$$

were $f'(x, x^0 - x)$ is the Dini directional derivative of the function $f : K \to \boldsymbol{R}$ at x in direction $x^0 - x$. For $x \in K$ and $u \in X$ feasible we define the Dini derivative

$$f'(x, u) = \liminf_{t \to 0^+} \frac{1}{t} \ (f(x + tu) - f(x)) \tag{2}$$

as an element of the extended real line $\overline{\boldsymbol{R}} = \boldsymbol{R} \cup \{-\infty\} \cup \{+\infty\}$. Here u feasible means that the set $\{t > 0 \mid x + tu \in K\}$ has zero as a cluster accumulating point.

Theorem 2.1.[7] *Let K be a set in a real linear space and let the function $f : X \to \mathbb{R}$ be radially l.s.c. on the rays starting at $x^0 \in \ker K$. Then x^0 is a solution of the Minty VI* (1) *if and only if f increases along rays starting at x^0. In consequence, each such solution x^0 is a global minimizer of f.*

Recall that $f : K \to \boldsymbol{R}$ is said radially l.s.c. on the rays starting at x^0 (as usual l.s.c. stands for *lower semi-continuous*) if, for all $u \in X$, the function $t \to f(x^0 + tu)$ is l.s.c. on the set $\{t \geq 0 \mid x^0 + tu \in K\}$. We write then $f \in$ RLSC(K, x^0). In a similar way we can introduce other "radial notions". We write also $f \in$ IAR(K, x^0) if f increases along rays starting at x^0, the latter means that for all $u \in X$ the function $t \to f(x + tu)$ is increasing on the set $\{t \geq 0 \mid x^0 + tu \in K\}$. We call this property IAR. The kernel $\ker K$ of K is defined as the set of all $x^0 \in K$, for which $x \in K$ implies that $[x^0, x] \subset K$, where $[x^0, x] = \{(1-t)x^0 + tx \mid 0 \leqslant t \leqslant 1\}$ is the segment determined by x^0 and x. Obviously, for a convex set $\ker K = K$. Sets with nonempty kernel are star-shaped and play an important role in abstract convexity [34]. Theorem 2.1 deals with sets K which are not necessarily convex, hence it occurs the possibility $\ker K \neq K$. For simplicity we confine in this paper the considerations to a convex set K, so the case $x^0 \notin \ker K$ does not occur (see [7]).

In [10] we generalize some results of [7] to a vector VI of the form

$$f'(x, x^0 - x) \cap (-C) \neq \emptyset, \quad x \in K, \tag{3}$$

where the Dini derivative $f'(x, u)$ is defined as

$$f'(x, u) = \operatorname*{Limsup}_{t \to 0^+} \frac{1}{t} (f(x + tu) - f(x)) \tag{4}$$

and the Limsup is taken in the sense of Painlevé-Kuratowski [1].

To generalize this result to vector optimization means (see [13]) *to keep as given the well established notions of minimizer (ideal, efficient, weak-efficient,...) and to develope a VI problem and an IAR concept which allows to recover Theorem* 2.1 *in conjunction with any concept of minimizer fixed in advance.*

The underlying global minimizers are ideal efficient points, which often are not the appropriate points of efficiency for practical reason (many vector optimization problems do not possess such solutions). In order to be able to copy with other points of efficiency, in [13] we proposed a scheme based on scalarization. The vector VI is replaced with a system of scalar VI.

In this paper we focus on the more general set-valued optimization problem

$$\min_C F(x), \quad x \in K\,, \tag{5}$$

where $F : K \rightsquigarrow Y$. The squiggled arrow $\rightsquigarrow$ denotes a set-valued function (for short, svf) with nonempty values. Like in [1] the solutions to (5) (minimizers) are defined as pairs (x^0, y^0), $y^0 \in F(x^0)$. In this paper we deal with global minimizers and next we recall some definitions.

The pair (x^0, y^0), $y^0 \in F(x^0)$, is said to be a w-minimizer (weakly efficient point) if $F(K) \cap \big(y^0 - \text{int}\, C\big) = \emptyset$. The pair (x^0, y^0), $y^0 \in F(x^0)$, is said to be an e-minimizer (efficient point) if $F(K) \cap \big(y^0 - (C \setminus \{0\})\big) = \emptyset$. Obviously, when $C \neq Y$ each e-minimizer is a w-minimizer. The pair (x^0, y^0), $y^0 \in F(x^0)$, is said to be an a-minimizer (absolute or ideal efficient point) if $F(K) \subset y^0 + C$.

For a given set $M \subset Y$ we define the w-frontier (weakly efficient frontier) $w\text{-Min}_C M = \{y \in M \mid M \cap (y - \text{int} C) = \emptyset\}$. The e-frontier (efficient frontier) is defined by $e\text{-Min}_C M = \{y \in M \mid M \cap (y - C \setminus \{0\}) = \emptyset\}$. The a-frontier (absolute or ideal frontier) is defined by $a\text{-Min}_C M = \{y \in M \mid M \subset y + C\}$.

Let us underline that the a-frontier with respect to a pointed cone C, if not empty, is a singleton. Indeed, if y^1 belongs to the a-frontier $a\text{-Min}_C M$, we have $y^2 - y^1 \in C$ for any $y^2 \in M$. If also y^2 is in the a-frontier $a\text{-Min}_C M$, we have $y^1 - y^2 \in C$. With regard to C pointed, the two inclusions give $y^1 = y^2$.

It is straightforward, that if (x^0, y^0) is a minimizer of one of the mentioned types, then y^0 belongs to the respective efficient frontier of $F(x^0)$.

When F reduces to a single-valued function $f : K \to Y$, then we deal with the vector optimization problem

$$\min_C f(x)\,, \quad x \in K. \tag{6}$$

To say that the couple $(x^0, f(x^0))$ is a w-minimizer, e-minimizer or a-minimizer, amounts to say that x^0 is respectively a w-minimizer, e-minimizer or a-minimizer (see [29]).

Dini derivatives for set-valued functions have been studied in [12, 24]. We recall the Dini derivative of svf $F : K \rightsquigarrow Y$ at (x, y), $y \in F(x)$, in the feasible direction $u \in X$ is

$$F'(x, y; u) = \operatorname*{Limsup}_{t \to 0^+} \frac{1}{t}(F(x + tu) - y). \tag{7}$$

This turns out to have similar applications to (5) as the Dini derivative for the vector problem (6) (see e.g. [12, 18]). This motivates the question, whether a kind of VI defined through the Dini derivative $F'(x, y; u)$ reveals similar relation between solutions, increasing-along-rays property, and global minimizers, as the one expressed in Theorem 2.1 and its extensions to vector problems.

Following the scheme developed in [10] as a starting point we could propose the VI

$$F'(x, y; x^0 - x) \cap (-C) \neq \emptyset, \quad x \in K,\ \ y \in F(x). \tag{8}$$

We call a solution of (8) a point $x^0 \in K$, such that for all $x \in K$ and all $y \in F(x)$ the property in (8) holds. The vector VI (3) is indeed a particular case of (8).

Remark 2.1. As for the terminology, let us underline that both VI (3) and (8) involve set-valuedness (in fact (3) applies the set-valued Dini derivative of the vector function f). We refer to (3) as vector VI as related to the vector optimization problem (6), while (8) is a set-valued VI as related to the set-valued problem (5). Both (3) and (8) design as a solution only points x^0 in the domain space. This does not affect the relations with vector optimization, where the point x^0 can be eventually recognized as a minimizer. Instead, for set-valued problem (5) the point x^0 could be at most only one component of a minimizer, since, as commonly accepted, the minimizers are defined as pairs (x^0, y^0), $y^0 \in F(x^0)$. This may lead to the attempt to redefine the notion of a minimizer, as we discuss further.

The positive polar cone of C is denoted by $C' = \{\xi \in Y^* \mid \langle \xi, y\rangle \geq 0, y \in C\}$. Here Y^* is the topological dual space of Y. Recall that, for Y locally convex space and C closed convex cone, it holds $(C')' = C$. Here the second positive polar cone is defined by $C'' = \{y \in Y \mid \langle \xi, y\rangle \geq 0, \xi \in C'\}$.

Theorem 2.2. *Let X be a real linear space, $K \subset X$ be a convex set, Y be a locally convex space, and $C \subset Y$ be a closed convex cone. Let $F : K \rightsquigarrow Y$ be a svf with convex and weakly compact values. Assume that for each $\xi \in C'$ the function $\varphi_\xi : K \to \boldsymbol{R}$, $\varphi_\xi(x) = \min\langle \xi, F(x)\rangle$ is l.s.c. and suppose that $x^0 \in K$ is a solution of the set-valued VI* (8)*. Then F possesses the following IAR property: for all $u \in X$, and all $0 \leqslant t_0 < t_1$, such that $x^0 + t_i u \in K$ for $i = 0, 1$, it holds $F(x^0+t_1u) \subset F(x^0+t_0u)+C$. In consequence, $F(x) \subset F(x^0)+C$ for all $x \in K$, and, when $F(x^0) = \{y^0\}$ is a singleton, the pair (x^0, y^0) is an a-minimizer for the set-valued problem* (5).

The proof of this theorem is in Sec. 4. Still, let us underline that in the case when F is a single-valued function we have as a special case Theorem 3, Sec. 3 in [10].

Theorem 2.2 states that if x^0 is a solution of (8), then in the case of a singleton $F(x^0) = \{y^0\}$ the pair (x^0, y^0) is an a-minimizer of the set-valued problem (5). Generally, when $F(x^0)$ is not a singleton, the following example shows that it may not exist a point $y^0 \in F(x^0)$, such that the pair (x^0, y^0) is an a-minimizer of (5).

Example 2.1. Let $X = \boldsymbol{R}$, $K = \boldsymbol{R}_+ := [0, +\infty)$, $Y = \boldsymbol{R}^2$, and

$$C = \{(y_1, y_2) \in \boldsymbol{R}^2 \mid 0 \leqslant y_1 < +\infty,\ -y_1 \leqslant y_2 \leqslant y_1\}.$$

Define the set-valued function $F : K \rightsquigarrow Y$ by $F(x) = \{x\} \times [-x-1, x+1]$. Then $x^0 = 0$ is a solution of the set-valued VI (8), since for $x \geq 0$, $y = (y_1, y_2)$ with $y_1 = x$ and $-x - 1 \leqslant y_2 \leqslant x + 1$ the set-valued derivative $F'(x, y; x^0 - x) =$

$F'(x, y; -x)$ is given by

$$F'(x, y; -x) = \begin{cases} \{-x\} \times [x, +\infty), & y_2 = -x - 1, \\ \{-x\} \times (-\infty, +\infty), & -x - 1 < y_2 < x + 1, \\ \{-x\} \times (-\infty, -x], & y_2 = x + 1. \end{cases}$$

At the same time $a\text{-Min}_C F(x^0) = \emptyset$, hence there is no $y^0 \in F(x^0)$, such that (x^0, y^0) is an a-minimizer of F.

However, when x^0 is a solution of (8) the IAR property yields that $F(x) \subset F(x^0) + C$ for all $x \in K$. To observe this we must put $u = x - x^0$, $t_0 = 0$, $t_1 = 1$. The above inclusion in the case when $F = f$ is single-valued, shows exactly that x^0 is an a-minimizer for the vector problem (6). Therefore, in the set-valued case as in the vector case, we still may claim some optimality of x^0. Namely the whole set $F\left(x^0\right)$ is in some sense optimal with respect to any other set of images $F(x)$. We refer to this property by x^0 is a set a-minimizer of F, defining the point $x^0 \in K$ to be set a-minimizer of F if $F(x) \subset F(x^0) + C$ for all $x \in K$. The set $F(x^0)$ can be called set a-minimal value of F at x^0. Introducing the notion of a set a-minimizer, we may refer now to the previously defined a-minimizers (x^0, y^0), $y^0 \in F(x^0)$, as point a-minimizers. Then y^0 can be called a point a-minimal value of F at x^0.

Remark 2.2. A concept of solution to set-valued optimization problem which take into account the sets of images can be found also in [28, 33].

Theorem 2.1 says also, that when the scalar function f is IAR at x^0, then x^0 is a solution of the considered VI. Similar reversal in Theorem 2.2 is not true, even for a single-valued function, that is for the vector case $F = f$. We observe this on the following example.

Example 2.2. Let $X = \boldsymbol{R}$, $K = [0, 1]$, $Y = \boldsymbol{R}$, $C = \boldsymbol{R}_+$. Let $f : K \to Y$, be any increasing singular function, for instance the well known in the real functions theory Cantor scale. Then f is continuous and increasing-along-rays starting at $x^0 = 0$. At the same time x^0 is not a solution of the VI (3).

To see this, note that VI (3) is now the scalar VI

$$f'(x, x^0 - x) \cap (-\boldsymbol{R}_+) \neq \emptyset\,, \quad x \in K\,, \tag{9}$$

where the derivative $f'(x, x^0 - x)$ is defined as a set in $\boldsymbol{R}$ through (4). At the points from the support S of f, which are not end points of an interval being a component of connectedness for the set $K \setminus S$, we have $f'(x, x^0 - x) = \emptyset$. Therefore x^0 is not a solution of VI (3).

Example 2.2 does not contradict Theorem 2.1. In fact, because of the use of infinite element, the derivative (2) is different for applications than (4). In consequence, VI (1) is not equivalent to (9).

To guarantee the reversal of Theorem 2.2 in the vector case $F = f$, in [10], we introduce infinite elements in the image space Y in a way well motivated by the VI, and modify the VI (3). Actually, when $Y = \boldsymbol{R}$, like in Example 2.2, the modified VI coincides with the scalar VI (1).

Here, with regard to eventual reversal of Theorem 2.2, we could try to follow the same approach for the set-valued VI (8). However, we prefer instead to generalize from vector VI to set-valued VI the more flexible scheme from [13], and this is the main task of the paper. We do this in the next section.

3. The Approach Through Scalarization

The vector problem (6) with a function $f : K \to Y$ can be an underlying optimization problem to different VI problems, one possible example was (3). In [13] we follow a more general approach. Let Ξ be a set of functions $\xi : Y \to \boldsymbol{R}$. For $x^0 \in \ker K$ (to pose the problem we need not assume that K is convex) put $\Phi(\Xi, x^0)$ to be the set of all functions $\phi : K \to \boldsymbol{R}$ such that $\phi(x) = \xi(f(x) - f(x^0))$ for some $\xi \in \Xi$ (we may write also ϕ_ξ instead of ϕ to underline that ϕ is defined through ξ). Instead of a single VI we consider the system of scalar VI

$$\phi'(x, x^0 - x) \leqslant 0\,, \quad x \in K\,, \quad \text{for all} \quad \phi \in \Phi(\Xi, x^0)\,. \tag{10}$$

A solution of (10) is any point x^0, which solves all the scalar VI of the system.

Now we say that f is increasing-along-rays with respect to Ξ (Ξ-IAR) at x^0 along the rays starting at $x^0 \in K$, and write $f \in \Xi\text{-IAR}\,(K, x^0)$, if $\phi \in \text{IAR}\,(K, x^0)$ for all $\phi \in \Phi(\Xi, x^0)$. We say that $x^0 \in K$ is a Ξ-minimizer of f on K if x^0 is a minimizer on K of each of the scalar functions $\phi \in \Phi(\Xi, x^0)$. We say that the function f is radially Ξ-l.s.c. at the rays starting at x^0, and write $f \in \Xi\text{-RLSC}\,(K, x^0)$, if all the functions $\phi \in \Phi(\Xi, x^0)$ satisfy $\phi \in \text{RLSC}\,(K, x^0)$. Note that the set Ξ plays the role of scalarizing the problem (i.e. it reduces a vector valued problem to a family of scalar valued problems).

Since system (10) consists of independent VI, we can apply Theorem 2.1 to each of them, getting in such a way the following result.

Theorem 3.1. [13] *Let K be a convex set in a real linear space X and Ξ be a set of functions $\xi : Y \to \boldsymbol{R}$ on a topological vector space Y. Let a function $f : K \to Y$ satisfy $f \in \Xi$-RLSC(K, x^0) at the point $x^0 \in K$. Then x^0 is a solution of the system of VI* (10) *if and only if $f \in \Xi$-IAR(K, x^0). In consequence, any solution $x^0 \in K$ of* (10) *is a Ξ-minimizer of f.*

Despite when dealing with VI in the vector case an ordering cone should be given in advance, see e.g. [14, 16], C does not appear explicitly neither in the system of VI (10) nor in the statement of the theorem. Therefore, the result of Theorem 3.1 depends on the set Ξ, but not on C directly. Actually, since the VI is related to a vector optimization problem, the cone C is given in advance because of the nature of the problem itself. The adequate system of VI claims then for a reasonable choice of Ξ depending in some way on C. In such a case the result in Theorem 3.1 depends implicitly on C through Ξ.

So, the cone C need not be given in advance, still any set Ξ as described above defines a Ξ-minimizer as a notion of a minimizer related to the underlying vector problem (6). Choosing different sets Ξ we get a variety of minimizers, which can be associated to the vector problem (6).

When $\Xi = \{\xi^0\}$ is a singleton, then Theorem 3.1 easily reduces to Theorem 2.1, where f should be substituted by $\phi : K \to \boldsymbol{R}$, $\phi(x) = \xi^0(f(x) - f(x^0))$, and the VI (1) by a single scalar VI of the form (10). Obviously, now f radially Ξ-l.s.c. means that ϕ is radially l.s.c., $f \in \Xi\text{-IAR}\,(K, x^0)$ means that $\phi \in \text{IAR}\,(K, x^0)$, x^0 a Ξ-minimizer of f means that x^0 is a minimizer of ϕ.

The importance of Theorem 3.1 is based on possible applications with different sets Ξ. At least two such cases can be stressed. The first case is when $\Xi = C'$, where $C \subset Y$ is the given in advance closed convex cone. Then the result is closely related to VI (3), the Ξ-minimizers turn to be a-minimizers, and the Ξ-IAR property is the one called IAR$^+$ in [10]. The second case is when Y is a normed space, C is a closed convex cone in Y. The dual space Y^* is also a normed space endowed with the norm $\|\xi\| = \sup_{y \in Y \setminus \{0\}} \langle \xi, y \rangle / \|y\|$ for $\xi \in Y^*$. Let $\Xi = \{\xi^0\}$ consists of the single function $\xi^0 : Y \to \boldsymbol{R}$ given by

$$\xi^0(y) = \sup\{\langle \xi, y \rangle \mid \xi \in C',\ \|\xi\| = 1\}\,. \tag{11}$$

In fact $\xi^0(y) = D(y, -C)$ is the so called oriented [20, 21] distance from the point y to the cone $-C$. The oriented distance $D(y, A)$ from a point $y \in Y$ to a set $A \subset Y$ is defined by $D(y, A) = d(y, A) - d(y, Y \setminus A)$. Here $d(y, A) = \inf\{\|y - a\| \mid a \in A\}$. It is shown in [19] that for a convex set A it holds

$$D(y, A) = \sup_{\|\xi\|=1} \big(\langle \xi, y \rangle - \sup_{a \in A} \langle \xi, a \rangle\big),$$

which when C is a convex cone gives $D(y, -C) = \xi^0(y)$. With the choice $\Xi = \{\xi^0\}$ the Ξ-minimizers turn to be w-minimizers of (6) and $f \in \Xi\text{-IAR}\,(K, x^0)$ means that the oriented distance $D(f(x) - f(x^0), -C)$ is increasing along the rays starting at x^0.

Our main task is now to generalize Theorem 3.1 and its applications to a suitable VI problem having the set-valued problem (5) as an underlying set-valued optimization problem.

To accomplish this task as in the vector case we suppose that a set Ξ of functions $\xi : Y \to \boldsymbol{R}$ is given. We deal now with the svf $F : K \rightsquigarrow Y$. For $x^0 \in K$ put $\Phi(\Xi, x^0)$ to be the set of all functions $\phi : K \to \overline{\boldsymbol{R}}$, such that

$$\phi(x) = \sup_{y^0 \in F(x^0)} \inf_{y \in F(x)} \xi(y - y^0)\,. \tag{12}$$

As in the vector case, we say that F is increasing-along-rays with respect to Ξ, (for short Ξ-IAR) at x^0 along the rays starting at x^0, and we write $F \in \Xi\text{-IAR}\,(K, x^0)$, if $\phi \in \text{IAR}\,(K, x^0)$ for all $\phi \in \Phi(\Xi, x^0)$. We say, that the svf F is radially Ξ-l.s.c. at the rays starting at x^0, and we write $F \in \Xi\text{-RLSC}\,(K, x^0)$, if all the functions $\phi \in \Phi(\Xi, x^0)$ satisfy $\phi \in \text{RLSC}\,(K, x^0)$. We say, that $x^0 \in K$ is a Ξ-minimizer of F on K, if x^0 is a minimizer on K of each of the scalar functions $\phi \in \Phi(\Xi, x^0)$.

Obviously, when F is single-valued, the functions ϕ in (12) are the same as those previously defined for the vector problem (6) with $f = F$. The properties of a function to be Ξ-IAR or Ξ-l.s.c. do not change their meaning. Neither does the notion of a Ξ-minimizer.

The Ξ-minimizer of the svf $F : K \rightsquigarrow Y$ is a point $x^0 \in K$ in the original space X. By similarity with the notions of set a-minimizers and point a-minimizers, we may refer to x^0 as set Ξ-minimizer of F with $F(x^0)$ being the corresponding set Ξ-minimal value. Now a point Ξ-minimizer of F can be defined as a pair (x^0, y^0), $y^0 \in F(x^0)$, with $x^0 \in K$, such that x^0 is a set Ξ-minimizer of F, and $y^0 \in F(x^0)$ is such that

$$\inf_{y \in F(x^0)} \xi(y - y^0) = \sup_{\bar{y} \in F(x^0)} \inf_{y \in F(x^0)} \xi(y - \bar{y}) \quad \text{for all} \quad \xi \in \Xi\,. \tag{13}$$

In this case y^0 can be called a point Ξ-minimal value of F at x^0.

Obviously, when $F(x^0) = \{y^0\}$ is a singleton, equality (13) is satisfied. Therefore, in this case x^0 is a set Ξ-minimizer if and only if (x^0, y^0) is a point Ξ-minimizer. In the sequel, when we deal with Ξ-minimizers, we write explicitly *set* Ξ*-minimizers* or *point* Ξ*-minimizers*, putting sometimes the words *set* or *point* in parentheses, when they can be missed by default.

Dealing with the set-valued problem (5), again as in the case of a vector problem (6) the system (10) is taken to be the scalarized VI problem. Only now it corresponds to the underlying set-valued problem (5) and the functions ϕ are defined by (12). By applying Theorem 2.1 to each scalar VI in (10), we get easily the following result.

Theorem 3.2. *Let K be a convex set in a real linear space X and Ξ be a set of functions $\xi : Y \to \boldsymbol{R}$ on a topological vector space Y. Let $x^0 \in K$ and suppose that all the functions $\phi \in \Phi(\Xi, x^0)$, being defined by* (12)*, are finite. Let a svf $F : K \rightsquigarrow Y$ satisfy $F \in \Xi\text{-RLSC}\,(K, x^0)$. Then x^0 is a solution of the system of VI* (10) *if and only if $F \in \Xi\text{-IAR}\,(K, x^0)$. In consequence, any solution $x^0 \in K$ of* (10) *is a (set) Ξ-minimizer of F. Moreover, if $F(x^0) = \{y^0\}$ is a singleton, then (x^0, y^0) is a point Ξ-minimizer of F.*

Obviously, Theorem 3.1 is now a corollary of Theorem 3.2. Applications of Theorem 3.2 can be based on special choices of the set Ξ. In the next sections we show applications to a-minimizers and w-minimizers.

4. Application to a-Minimizers

As usual let X be a linear space and $K \subset X$ be a convex set in X. We assume that the topological vector space Y is locally convex and denote by Y^* its dual space. Let C be a closed convex cone in Y with positive polar cone $C' = \{\xi \in Y^* \mid \langle \xi, y \rangle \geq 0,\ y \in C\}$. Due to the Separation Theorem for locally convex spaces, see Theorem 9.1 in [35], we have $C = \{y \in Y \mid \langle \xi, y \rangle \geq 0,\ \xi \in C'\}$. Let a svf $F : K \rightsquigarrow Y$ be given, with values $F(x)$ being convex and weakly compact. Consider the system of VI (10) with $\Xi = C'$. Now $\Phi(\Xi, x^0)$ is the set of functions

$\phi : K \to \boldsymbol{R}$ defined for all $x \in K$ by

$$\phi(x) = \max_{y^0 \in F(x^0)} \min_{y \in F(x)} \langle \xi,\, y - y^0 \rangle = \min_{y \in F(x)} \langle \xi,\, y \rangle - \min_{y^0 \in F(x^0)} \langle \xi,\, y^0 \rangle \tag{14}$$

for some $\xi \in C'$. Due to the weak compactness of the values of F the minimum and the maximum in the above formula are attained, and the values of ϕ are finite.

The property $F \in \Xi\text{-IAR}\,(K, x^0)$ means that for arbitrary $u \in X$ and $0 \leqslant t_1 < t_2$ in the set $\{t \geq 0 \mid x^0 + tu \in K\}$, it holds $F(x^0 + t_2 u) \subset F(x^0 + t_1 u) + C$. We call this property IAR$^+$ and write $F \in \text{IAR}^+(K, x^0)$ following [10], where similar convention is done for vector functions.

To show this, we put for brevity $x^1 = x^0 + t_1 u$, $x^2 = x^0 + t_2 u$. Suppose that $F \in \Xi\text{-IAR}\,(K, x^0)$, but $F(x^2) \not\subset F(x^1) + C$. Then there exists $y^2 \in F(x^2)$, such that $y^2 \notin F(x^1) + C$. The set $F(x^1) + C$ is convex as the sum of two convex sets, and weakly closed (hence closed) as the sum of a weakly compact and a weakly closed set. The separation theorem implies the existence of $\xi^0 \in Y^*$, such that $\langle \xi^0, y^2 \rangle < \langle \xi^0, y^1 + c \rangle$ for all $y^1 \in F(x^1)$ and $c \in C$. Since C is a cone, we get from here $\xi^0 \in C'$, and $\langle \xi^0, y^2 \rangle < \langle \xi^0, y^1 \rangle$ for all $y^1 \in F(x^1)$. Since $F(x^1)$ is weakly compact, we get from here that there exists $\epsilon > 0$, such that $\langle \xi^0, y^2 - y^0 \rangle + \varepsilon < \langle \xi^0, y^1 - y^0 \rangle$ for all $y^1 \in F(x^1)$ and $y^0 \in F(x^0)$. Therefore for all $y^0 \in F(x^0)$ it holds (further dealing with infima and suprema, we may confine in fact to minima and maxima)

$$\inf_{y \in F(x^2)} \langle \xi^0, y - y^0 \rangle + \varepsilon \leqslant \langle \xi^0, y^2 - y^0 \rangle + \varepsilon \leqslant \inf_{y^1 \in F(x^1)} \langle \xi^0, y^1 - y^0 \rangle .$$

Taking a supremum in $y^0 \in F(x^0)$ in the above inequality, we get $\phi(t_2) + \varepsilon \leqslant \phi(t_1)$, where $\phi \in \Phi(\Xi, x^0)$ is the function corresponding to ξ^0. The obtained inequality contradicts the assumption $F \in \Xi\text{-IAR}(K, x^0)$.

Conversely, let in the above notation we have $F(x^2) \subset F(x^1) + C$. Fix $\xi \in C'$. Let $y^2 \in F(x^2)$. The above inclusion shows that there exists $y^1 \in F(x^1)$, such that $\langle \xi, y^2 - y^1 \rangle \geq 0$, whence for arbitrary $y^0 \in F(x^0)$ it holds

$$\inf_{y \in F(x^1)} \langle \xi, y - y^0 \rangle \leqslant \langle \xi, y^1 - y^0 \rangle \leqslant \langle \xi, y^2 - y^0 \rangle .$$

With account that $y^2 \in F(x^2)$ is arbitrary, we get that, for all $y^0 \in F(x^0)$, it holds

$$\inf_{y \in F(x^1)} \langle \xi, y - y^0 \rangle \leqslant \inf_{y \in F(x^2)} \langle \xi, y - y^0 \rangle .$$

Taking the supremum in $y^0 \in F(x^0)$, we obtain $\phi(t_1) \leqslant \phi(t_2)$, where $\phi \in \Phi(\Xi, x^0)$ is the function corresponding to ξ. Since $\xi \in C'$ is arbitrary, we get $F \in \Xi\text{-IAR}\,(K, x^0)$.

The point $x^0 \in K$ is a (set) Ξ-minimizer of F if and only if $F(x) \subset F(x^0) + C$ for all $x \in K$. We call the point x^0 satisfying this inclusion a set a-minimizer, which is justified by the following. The pair (x^0, y^0), $y^0 \in F(x^0)$, is a point Ξ-minimizer of F if and only if (x^0, y^0) is a (point) a-minimizer of F.

Indeed, put $x^1 = x^0$ and $x^2 = x$. Now the proof of the set-case property comes by repeating word in word the preceding reasoning. The case of a point Ξ-minimizer is investigated similarly.

If for some $\xi \in C'$ the function $\varphi_\xi : K \to \boldsymbol{R}$, $\varphi_\xi(x) = \min\langle \xi,\, F(x)\rangle$ is l.s.c., then also the function $\phi \in \Phi(\Xi, x^0)$ corresponding to ξ is l.s.c.

Indeed, the representation

$$\phi(x) = \max_{y^0 \in F(x^0)} \min_{y \in F(x)} \langle \xi, y - y^0\rangle = \min_{y \in F(x)} \langle \xi, y\rangle - \min_{y^0 \in F(x^0)} \langle \xi, y^0\rangle$$

shows that ϕ differs from φ_ξ by a constant.

We collect these results in the following corollary of Theorem 3.2.

Corollary 4.1. *Let X be a real linear space, $K \subset X$ be a convex set, Y be a locally convex space, and $C \subset Y$ be a closed convex cone. Let $F : K \rightsquigarrow Y$ be a svf with convex and weakly compact values. Assume that for each $\xi \in C'$ the function $\varphi_\xi : K \to \boldsymbol{R}$, $\varphi_\xi(x) = \min\langle \xi,\, F(x)\rangle$ is l.s.c. Consider the system of VI* (10) *with $\Xi = C'$. Then $x^0 \in K$ is a solution of* (10) *if and only if $F \in \mathrm{IAR}^+(K, x^0)$. In consequence, any solution $x^0 \in K$ of* (10) *is a set a-minimizer of F. In the case when $F(x^0) = \{y^0\}$ is a singleton the point (x^0, y^0) is a point Ξ-minimizer of F and hence a (point) a-minimizer. Moreover, if x^0 is a solution of* (10)*, then (x^0, y^0), $y^0 \in F(x^0)$, is a (point) a-minimizer of F if and only if $y^0 \in a\text{-}\mathrm{Min}_C F(x^0)$.*

To prove Theorem 2.2, the following proposition is crucial.

Proposition 4.1. *Suppose that the hypotheses of Theorem* 2.2 *are satisfied. In particular let $F : K \rightsquigarrow Y$ be a svf with convex and weakly compact values, which is used to construct the set-valued VI* (8)*. Suppose that x^0 is a solution of* (8)*. Then x^0 is also a solution of the system of VI* (10) *determined by the set $\Xi = C'$ as it is described in Corollary* 4.1*.*

Proof. Fix $x \in K$. Let $y \in F(x)$ be arbitrary. Since x^0 is a solution of set-valued VI (10), there exists

$$z \in F'(x, y; x^0 - x) \cap (-C) = \operatorname*{Limsup}_{t \to 0^+} \frac{1}{t}(F(x + t(x^0 - x)) - y) \cap (-C).$$

Let $z = \lim_n (1/t_n)(y^n - y)$ with $t_n \to 0^+$ and $y^n \in F(x + t_n(x^0 - x))$. From $z \in -C$ we get, that for arbitrary $\xi \in C'$ it holds

$$0 \geq \langle \xi, z\rangle = \lim_n \Big\langle \xi, \frac{1}{t_n}(y^n - y)\Big\rangle \geq \liminf_{t \to 0^+} \min_{\bar{y} \in F(x + t(x^0 - x))} \Big\langle \xi, \frac{1}{t_n}(\bar{y} - y)\Big\rangle.$$

Since this inequality is true for arbitrary $y \in F(x)$, we get

$$0 \geq \liminf_{t \to 0^+} \max_{y \in F(x)} \min_{\bar{y} \in F(x+t(x^0-x))} \Big\langle \xi, \frac{1}{t}(\bar{y} - y) \Big\rangle = \phi'(x, x^0 - x),$$

where $\phi \in \Phi(\Xi, x^0)$ is the function corresponding to ξ. We have used in fact, that for all x^1, $x^2 \in K$ it holds

$$\begin{aligned} &\phi(x^2) - \phi(x^1) = \\ &= \Big(\min_{y^2 \in F(x^2)} \langle \xi, y^2 \rangle - \min_{y^0 \in F(x^0)} \langle \xi, y^0 \rangle \Big) - \Big(\min_{y^1 \in F(x^1)} \langle \xi, y^1 \rangle - \min_{y^0 \in F(x^0)} \langle \xi, y^0 \rangle \Big) \\ &= \min_{y^2 \in F(x^2)} \langle \xi, y^2 \rangle - \min_{y^1 \in F(x^1)} \langle \xi, y^1 \rangle = \max_{y^1 \in F(x^1)} \min_{y^2 \in F(x^2)} \langle \xi, y^2 - y^1 \rangle. \end{aligned}$$

Thus, since $\xi \in C'$ in the above reasoning was arbitrary, we have obtained, that $\phi'(x, x^0 - x) \leqslant 0$ for all $\phi \in \Phi(\Xi, x^0)$. Therefore x^0 is a solution of the system of VI (10). ■

Now we see, that if the hypotheses of Theorem 2.2 are satisfied, also the hypotheses of Corollary 4.1 are satisfied. Therefore Theorem 2.2 follows from Corollary 4.1.

In Example 2.1 the point $x^0 = 0$ is a solution of the set-valued VI (10) and hence of (10). Therefore, according to Corollary 4.1 it is a set a-minimizer. However there do not exist point a-minimizers (x^0, y^0), $y^0 \in F(x^0)$, since the efficient a-frontier of $F(x^0)$ is empty.

In Example 2.2 we have $C' = \boldsymbol{R}_+$. Therefore the system (10) becomes

$$\xi \cdot f'(x, x^0 - x) \leqslant 0\,, \quad \xi \geq 0,$$

which is obviously equivalent to the single VI $f'(x, x^0 - x) \leqslant 0$. Here the directional derivative $f'(x, x^0 - x)$ is taken in the sense of (2). The function f is increasing, hence $f \in \mathrm{IAR}\,(K, x^0)$ with $x^0 = 0$, and therefore according to the reversal of Corollary 4.1 the point x^0 is a solution of (10). This follows also straightforward from properties of increasing functions. In particular at points x in the support S of f, we have $f'(x, x^0 - x) = -\infty < 0$. These are not end points of an interval being a component of connectedness for the set $K \setminus S$. As we have seen at these points the set-valued VI (10), which actually now is (9), is not satisfied. To prove Theorem 2.2, we have seen that each solution of (8) is a solution of (10). The above reasoning shows that the converse is not true. Consequently, while Corollary 4.1 admits a reversal, that is the IAR property implies existence of a solution, Theorem 2.2 does not.

As for Theorem 3.1, one may assume Ξ to be defined prior than the cone C. So let an arbitrary set Ξ in Y^* be given. We show how this may affects Corollary 4.1.

Now $\Phi(\Xi, x^0)$ is the set of functions defined by (14) for some $\xi \in \Xi$. Define the cone $C_\Xi = \{y \in Y \mid \langle \xi, y \rangle \geq 0 \text{ for all } \xi \in \Xi\}$. Its positive polar cone is $C'_\Xi = \mathrm{clconvcone}\Xi$. We note that, despite Ξ might be a proper subset of C'_Ξ, the set of the solutions of the system of VI (10) coincides with the set of the solutions of the system of VI obtained from (10) by replacing Ξ with C'_{Ξ}. However, the new system allows to recover the case already described in Corollary 4.1 with the cone C_{Ξ} replacing C. Therefore, we get the following proposition, which in fact generalizes Corollary 4.1.

Proposition 4.2. *Let X be a real linear space, $K \subset X$ be a convex set, Y be a locally convex space, and $\Xi \subset Y^*$ be an arbitrary set. Let $F : K \rightsquigarrow Y$ be a svf with convex and weakly compact values. Assume that for each $\xi \in \Xi$ the function $\varphi_\xi : K \to \boldsymbol{R}$, $\varphi_\xi(x) = \min\langle \xi, F(x)\rangle$ is l.s.c. Then the system of VI (10) with $\phi \in \Phi(\Xi, x^0)$ is equivalent to the similar system of VI, in which Ξ is replaced with C'_{Ξ}. Therefore the conclusions of Corollary* 4.1 *hold with cone C replaced with the cone C_{Ξ}.*

This proves that, given an arbitrary set Ξ, we shall always find a suitable ordering cone C_{Ξ} by which we define optimality in problem (5). Let now assume that a closed and convex cone C in Y is given in advance.

With respect to Proposition 4.2, if we choose $\Xi \subset C'$ such that $C' = \text{cone}\Xi$, say e.g. Ξ is a base of C', then $C_{\Xi} = C$, and we have the conclusions of Corollary 4.1.

Often in optimization with constraints it happens to deal with the set $\Xi = \{\xi \in C' \mid \langle \xi, y^0\rangle = 0\}$, where $y^0 \in C$. Then C_{Ξ} is the contingent cone (see e.g. [1]) of C at y^0, at least when Y is a normed space.

Another particular case is when $\Xi = \{\xi^0\}$, $\xi^0 \in C'$, is a singleton. Then (10) reduces to a single VI $\phi'(x, x^0 - x) \leqslant 0$ with $\phi(x) = \min\langle \xi^0, F(x)\rangle - \min\langle \xi^0, F(x^0)\rangle$ to which Theorem 2.1 can be directly applied. Now x^0 is a Ξ-minimizer of F if $\min\langle \xi^0, F(x^0)\rangle \leqslant \min\langle \xi^0, F(x)\rangle$, $x \in K$. In vector optimization, that is when $F = f$ is single-valued, the points x^0 satisfying this condition are called linearized through $\xi^0 \in C'$ efficient points. The same could be said with respect to the set-valued problem.

5. Application to w-Minimizers

As usual here X is a real linear space and K is a convex set in X. Let Y be a normed space and C be a closed convex cone in Y. Suppose that a svf $F : K \rightsquigarrow Y$ is given. We choose now $\Xi = \{\xi^0\}$ to be a singleton with function $\xi^0 : Y \to \boldsymbol{R}$ being the oriented distance $\xi^0(y) = D(y, -C)$, $y \in Y$, from the point y to the cone $-C$ given by (11). Now the system of VI (10) is a single VI with function $\phi : K \to \boldsymbol{R}$ given by

$$\begin{aligned}\phi(x) &= \sup_{y^0 \in F(x^0)} \inf_{y \in F(x)} D(y - y^0, -C) \\ &= \sup_{y^0 \in F(x^0)} \inf_{y \in F(x)} \sup\{\langle \xi, y - y^0\rangle \mid \xi \in C',\ \|\xi\| = 1\}.\end{aligned} \tag{15}$$

The oriented distance $D(M, A)$ from a set $M \subset Y$ to the set $A \subset Y$ can be defined by $D(M, A) = \inf\{D(y, A) \mid y \in M\}$. With this definition the function

ϕ in (15) is represented as

$$\phi(x) = \sup_{y^0 \in F(x^0)} D(F(x) - y^0, -C), \quad x \in K. \tag{16}$$

The following proposition gives a characterization of the w-minimizers of the set-valued problem (5) in terms of the oriented distance.

Proposition 5.1. [12] *The pair (x^0, y^0), $y^0 \in F(x^0)$, is a w-minimizer of set-valued problem (5) with $C \neq Y$ if and only if $\varphi(x^0) = 0$ and x^0 is a minimizer for the scalar function*

$$\varphi : K \to \boldsymbol{R}, \quad \varphi(x) = D(F(x) - y^0, -C). \tag{17}$$

Let $F \in \Xi\text{-IAR}(K, x^0)$. Then for arbitrary $u \in X$ and $0 \leqslant t_1 < t_2$ in the set $\{t \geq 0 \mid x^0 + tu \in K\}$, there exists $y^1 \in F(x^0 + t_1 u)$, such that $F(x^0 + t_2 u) \cap (y^1 - \operatorname{int} C) = \emptyset$. In particular, when $F(x^0) = \{y^0\}$ is a singleton, then the point (x^0, y^0) is a w-minimizer of F. Also, when $F(x^0) = \{y^0\}$ is a singleton, the property $F \in \Xi\text{-IAR}\,(K, x^0)$ means in fact that the oriented distance $D(F(x) - y^0, -C)$, $x \in K$, is increasing along the rays starting at x^0.

To prove this we put $x^1 = x^0 + t_1 u$, $x^2 = x^0 + t_2 u$. Now $F \in \Xi\text{-IAR}\,(K, x^0)$ means that $\phi(x^1) \leqslant \phi(x^2)$, where ϕ is the function (16). Assume that the claimed property is not true. Then there exist some $0 \leqslant t_1 < t_2$, such that for all $y^1 \in F(x^1)$ it holds $F(x^2) \cap (y^1 - \text{int}C) \neq \emptyset$. Fix $y^1 \in F(x^1)$. Then there exists $y^2 \in F(x^2)$ such that $y^2 \in y^1 - \text{int}C$. Consequently there exists $\varepsilon > 0$ such that $D(y^2 - y^1, -C) \leqslant -\varepsilon$. Therefore for all $\xi \in C'$, $\|\xi\| = 1$, and all $y^0 \in F(x^0)$ we have $\langle \xi, y^2 - y^0 \rangle \leqslant \langle \xi, y^1 - y^0 \rangle - \varepsilon$, whence

$$D(F(x^2) - y^0, -C) \leqslant D(F(x^1) - y^0, -C) - \varepsilon.$$

Taking the supremum in $y^0 \in F(x^0)$ we get $\phi(x^2) \leqslant \phi(x^1) - \varepsilon$, which contradicts the inequality $\phi(x^2) \geq \phi(x^1)$. Since the claim holds also with $t_1 = 0$, while $x^2 = x$ can be an arbitrary point of K, we get then that for each $x \in K$ there exists $y^0 \in F(x^0)$ such that $F(x) \cap (y^0 - \text{int}C)$. When $F(x^0) = \{y^0\}$ is a singleton, then the same y^0 serves for all $x \in K$, hence $F(K) \cap (y^0 - \text{int}C) = \emptyset$, that is (x^0, y^0) is a w-minimizer.

If the point $x^0 \in K$ is a (set) Ξ-minimizer of F, then for each $x \in K$ there exists $y^0 \in F(x^0)$ such that $F(x) \cap (y^0 - \text{int}C) = \emptyset$. In particular, when $F(x^0) = \{y^0\}$ is a singleton, then the point (x^0, y^0) is a w-minimizer of F.

Actually, to prove this claim we repeat the reasonings above.

The following definition seems now natural. We say that $x^0 \in K$ is a set w-minimizer of the set-valued problem (5) with a svf $F : K \rightsquigarrow Y$ if for each $x \in K$ there exists $y^0 \in F(x^0)$ such that $F(x) \cap (y^0 - \text{int}C) = \emptyset$. When $F(x^0) = \{y^0\}$ is a singleton, this condition turns into $F(K) \cap (y^0 - \text{int}C) = \emptyset$ and is equivalent to (x^0, y^0) is a w-minimizer. In the case when F is single-valued, then x^0 is a set w-minimizer of the set-valued problem (5), as defined here, if and only if x^0 is a w-minimizer for the vector problem (6). Introducing the notion of a set w-minimizer of a set-valued problem, we call point w-minimizers the pair

(x^0, y^0) of minimizers. Let us underline again, that while the set w-minimizer of the set-valued problem (5) is a point $x^0 \in K$, the point w-minimizer is a pair (x^0, y^0), $y^0 \in F(x^0)$.

Straightforward from the definition of the set w-minimizer we see that if (x^0, y^0) is a (point) w-minimizer of the set-valued problem (5), then x^0 is a set w-minimizer. When $F(x^0) = \{y^0\}$ is a singleton, then $w\text{-Min}_C F(x^0) = \{y^0\}$ and (x^0, y^0) is a (point) w-minimizer of (5). An interesting question is whether a similar property holds without the assumption that $F(x^0)$ is a singleton, that is whether when x^0 is a set w-minimizer of (5) and $y^0 \in w\text{-Min}_C F(x^0)$ it holds that (x^0, y^0) is a (point) w-minimizer of (5). The following example gives a negative answer to this question. Moreover, we see an example where the efficient w-frontier of $F(x^0)$ is nonempty and still the set of the (point) w-minimizers is empty.

Example 5.1. Let $X = K = \boldsymbol{R}$, $Y = \boldsymbol{R}^2$ with the Euclidean norm, and $C = \boldsymbol{R}^2_+$. Define the set-valued function $F : X \rightsquigarrow Y$ by

$$F(x) = \begin{cases} \{(1-t, 1+t) \mid -1 \leqslant t \leqslant 1\}, & x = 0, \\ \{(x, -x)\}, & x \neq 0. \end{cases}$$

Then for $x^0 = 0$ it holds $F \in \Xi\text{-IAR}\,(K, x^0)$ and x^0 is a set w-minimizer of the set-valued problem (5). Simultaneously for each $y^0 \in F(x^0)$ it holds $y^0 \in w\text{-Min}_C F(x^0)$, but there does not exist $y^0 \in F(x^0)$ such that (x^0, y^0) to be a (point) w-minimizer of (5).

An easy calculation gives that $\phi(x) = |x|$ for $x \in \boldsymbol{R}$, whence obviously ϕ is increasing along the rays starting at x^0 and x^0 is a set w-minimizer. While obviously each point $y^0 = (1-t, 1+t) \in \boldsymbol{R}^2$, $-1 \leqslant t \leqslant 1$, belongs to the efficient w-frontier of $F(x^0)$, we have $F(x) \subset y^0 - \text{int}C$ for all x in the set $(-1-t, 1-t) \setminus \{0\} \subset \boldsymbol{R}$, whence (x^0, y^0) is not a (point) w-minimizer of F.

As a complement we have the following result.

Proposition 5.2. *Let $x^0 \in K$ be a set w-minimizer of the set-valued problem (5) with $C \neq Y$ and a svf $F : K \rightsquigarrow Y$, and let $y^0 \in w\text{-}Min_C F(x^0)$. If*

$$\phi(x) = \sup_{\bar{y}^0 \in F(x^0)} D(F(x) - \bar{y}^0, -C) = D(F(x) - y^0, -C) \quad \textit{for all } x \in K \setminus \{x^0\},$$

then (x^0, y^0) is a (point) w-minimizer of (5).

Proof. Consider the function φ in (17). We calculate the value $\varphi(x^0)$ and $\phi(x^0)$. We have

$$\begin{aligned} \varphi(x^0) &= D(F(x^0) - y^0, -C) \leqslant \phi(x^0) \\ &= \sup_{y \in F(x^0)} D(F(x^0) - y, -C) \leqslant \sup_{y \in F(x^0)} D(y - y, -C) = 0. \end{aligned}$$

In general $\phi(x^0)$ need not be zero. For instance, if $F(x^0) = Y$, and then $w\text{-Min}_C F(x^0) = \emptyset$, we have $\phi(x^0) = -\infty$. The nonemptiness of $w\text{-Min}_C F(x^0)$

changes the situation. From $y^0 \in w\text{-Min}_C F(x^0)$ we have $y - y^0 \notin -\text{int}C$ for all $y \in F(x^0)$, whence

$$\varphi(x^0) = D(F(x^0) - y^0, -C) = \inf_{y \in F(x^0)} D(y - y^0, -C) \geq 0\,.$$

Now the inequalities $\varphi(x^0) \leqslant \phi(x^0) \leqslant 0 \leqslant \varphi(x^0)$ give $\varphi(x^0) = \phi(x^0) = 0$. Since x^0 is a set w-minimizer and according to the made assumptions for all $x \in K \setminus \{x^0\}$ we have

$$\varphi(x) = \phi(x) \geq \phi(x^0) = \varphi(x^0) = 0.$$

Therefore the function $\varphi(x)$ satisfies the hypotheses of Proposition 5.1. In consequence (x^0, y^0) is a (point) w-minimizer of the set-valued problem (5). ■

Remark 5.1. By analogy with the notion of a set w-minimizer one can define the notion of a set e-minimizer. In spite that in the sequel the set e-minimizers are not used, we present for completeness the definition. We say that $x^0 \in K$ is a set e-minimizer of the set-valued problem (5) with a svf $F : K \rightsquigarrow Y$ if for each $x \in K$ there exists $y^0 \in F(x^0)$ such that $F(x) \cap (y^0 - (C \setminus \{0\})) = \emptyset$. When $F(x^0) = \{y^0\}$ is a singleton, this condition turns into $F(K) \cap (y^0 - (C \setminus \{0\})) = \emptyset$ and is equivalent to (x^0, y^0) is an e-minimizer. In the case when F is single-valued, and then we write $F = f$, then x^0 is a set e-minimizer of the set-valued problem (5) if and only if x^0 is an e-minimizer for the vector problem (6). Introducing the notion of a set e-minimizer of a set-valued problem, we will call point e-minimizers the defined earlier e-minimizers.

Now we discuss the l.s.c. properties of F.

Let F have weakly compact values. Suppose that the functions $\varphi_\xi : K \to \boldsymbol{R}$, $\varphi_\xi(x) = \min\langle \xi, F(x)\rangle$, are l.s.c. uniformly on the set $\{\xi \in C' \mid \|\xi\| = 1\}$. Then the function ϕ in (16) is l.s.c., and moreover $\phi \in \text{RLSC}\,(K, x^0)$. Confining to F with weakly compact values, this condition admits some relaxation.

To show the above property fix $\bar{x} \in K$, and take $\varepsilon > 0$. Then there exists a neighborhood U of $\bar{x}$, such that for all $x \in U \cap K$, and all $\xi \in C'$ with $\|\xi\| = 1$, it holds

$$\min\langle \xi, F(x)\rangle > \min\langle \xi, F(\bar{x})\rangle - \varepsilon.$$

This inequality can be written also as

$$\forall\, y \in F(x) : \exists\, \bar{y} \in F(\bar{y}) : \langle \xi, y\rangle > \langle \xi, \bar{y}\rangle - \varepsilon,$$

hence, for all $y^0 \in F(x^0)$, it holds

$$\forall\, y \in F(x) : \exists\, \bar{y} \in F(\bar{y}) : \langle \xi, y - y^0\rangle > \langle \xi, \bar{y} - y^0\rangle - \varepsilon.$$

Because of the uniformity, the above inequality is true for all $\xi \in C'$, $\|\xi\| = 1$, whence for all $y \in F(x)$, there exists $\bar{y} \in F(\bar{y})$, such that

$$\begin{aligned} D(y - y^0, -C) &= \sup_{\xi \in C',\, \|\xi\|=1} \langle \xi, y - y^0\rangle > \sup_{\xi \in C',\, \|\xi\|=1} \langle \xi, \bar{y} - y^0\rangle - \varepsilon \\ &\geq \min_{\bar{y} \in F(\bar{y})} D(\bar{y} - y^0, -C) = D(F(\bar{x}) - y^0, -C) - \varepsilon. \end{aligned}$$

Since the above inequality is true for all $y \in F(x)$, we get

$$D(F(x) - y^0, -C) = \min_{y \in F(x)} D(y - y^0, -C) \geq D(F(\bar{x}) - y^0, -C) - \varepsilon.$$

This inequality is true for all $y^0 \in F(x^0)$, whence finally we get the claimed lower semi-continuity

$$\sup_{y^0 \in F(x^0)} D(F(x) - y^0, -C) \geq \sup_{y^0 \in F(x^0)} D(F(\bar{x}) - y^0, -C) - \varepsilon.$$

We collect these claims into the following corollary of Theorem 3.2.

Corollary 5.1. *Let K be a convex set in a real linear space X, Y be a normed space with $C \subset Y$ a closed convex cone, and a svf $F : K \rightsquigarrow Y$ have weakly compact values. If $\Xi = \{\xi^0\}$ is a singleton given by $\xi^0 : Y \to \boldsymbol{R}$, $\xi^0(y) = D(y, -C)$, and $x^0 \in K$, then the system of VI* (10) *consists of a single VI with a function ϕ given by* (16)*, which in the case when $F(x^0) = \{y^0\}$ is a singleton, is simply the oriented distance $\phi(x) = D(F(x) - y^0, -C)$. Suppose that all the functions $\varphi_\xi : K \to \boldsymbol{R}$, $\varphi_\xi(x) = \min\langle \xi, F(x)\rangle$, are l.s.c. uniformly on the set $\{\xi \in C' \mid \|\xi\| = 1\}$. Then x^0 is a solution of the VI* (10) *if and only if $\phi \in IAR(K, x^0)$. In consequence, any solution $x^0 \in K$ of* (10) *is a set w-minimizer of F. In the case when $F(x^0) = \{y^0\}$ is a singleton, then the point (x^0, y^0) is a (point) w-minimizer of F. Moreover, if $x^0 \in K$ is a solution of* (10) *and y^0 satisfies the hypotheses of Proposition 5.2, then the point (x^0, y^0) is a (point) w-minimizer of F.*

In Example 5.1 for $x^0 = 0$ we have $F \in \Xi$-IAR, therefore x^0 is a solution of (10) and x^0 is a set w-minimizer.

Let us underline, that the property (x^0, y^0) is a w-minimizer is invariant when equivalent norms in Y are considered. On the contrary, the property $\phi \in$ IAR(K, x^0) is norm-dependent, which in [11] is observed for vector functions.

6. Generalized Quasiconvexity

In Theorem 3.2 as a result of the equivalence of the properties of $x^0 \in \ker K$ to be a solution of the system of VI (10) with $\phi \in \Phi(\Xi, x^0)$ defined by (12) and $F \in \Xi$-IAR(K, x^0) we see that x^0 is a (set) Ξ-minimizer of F. However the next example shows that not any Ξ-minimizer of a svf F is a solution of the system of VI (10).

Example 6.1. Let $X = \boldsymbol{R}$, $Y = \boldsymbol{R}^2$, $C = \boldsymbol{R}^2_+$. Let $S = \{(t, (1-t)), t \in [0,1]\}$ and consider the scalar function $f : \boldsymbol{R}^2 \to \boldsymbol{R}$ given by

$$f(x_1, x_2) = \begin{cases} x_1^2 x_2^2, & x_1 \geq 0 \text{ or } x_2 \geq 0, \\ 0, & x_1 < 0 \text{ and } x_2 < 0. \end{cases}$$

Let $F : X \rightsquigarrow Y$ be a svf defined by $F(x) = f(x)S$ and let $\Xi = C'$. The point $x^0 = 0$ is a set Ξ-minimizer for F and also a solution of the system of VI (10).

At the same time, points x with $x_1 < 0$ and $x_2 < 0$ are also set Ξ-minimizers for F, but are not solutions of system (10).

Our task in this section is to identify a class of svf F such that for a point $x^0 \in K$ to be a solution of the system of VI (10) and to be a set Ξ-minimizer of F are equivalent. The generalized quasiconvexity for svf is the key to solve this problem.

We say that $f : K \to \boldsymbol{R}$ is radially quasiconvex along the rays starting at $x^0 \in K$ if the restriction of f to any such ray is quasiconvex. If this property is satisfied we write $f \in \text{RQC}(K, x^0)$.

The following assertion is a straightforward consequence of the definitions of quasiconvexity.

Theorem 6.1. *A function $f : K \to \boldsymbol{R}$ defined on a set K in a real linear space is quasiconvex if and only if $f \in RQC(K, x^0)$ for all points $x^0 \in K$.*

The following theorem (see e.g. [13]) establishes the equivalence of the properties that x^0 is a solution of the scalar VI (1) and x^0 is a minimizer of f.

Theorem 6.2. [13] *Let K be a set in a real linear space and let a function $f : X \to \mathbb{R}$ have the property $f \in \text{RQC}(K, x^0)$ at $x^0 \in K$. If x^0 is a minimizer of f, then x^0 is a solution of the scalar VI* (1)*. In particular, if f is quasiconvex, then any minimizer of f is a solution of VI* (1)*.*

In [13] we have extended Theorem 6.1 to vector generalized quasiconvex functions. In this section we deal with a similar task in the case of a svf F.

To generalize Theorem 6.2 from the scalar VI (1) to the system of VI (10) with $\phi \in \Phi(\Xi, x^0)$ defined by (12) we introduce Ξ-quasiconvexity.

Let Ξ be a set of functions $\xi : Y \to \boldsymbol{R}$. For $x^0 \in K$ define the functions $\phi \in \Phi(\Xi, x^0)$ as in (12). For any such x^0 we say that the svf F is radially Ξ-quasiconvex along the rays starting at x^0, and write $F \in \Xi\text{-RQC}(K, x^0)$, if $\phi \in \text{RQC}(K, x^0)$ for all $\phi \in \Phi(\Xi, x^0)$. We say that F is Ξ-quasiconvex if K is convex and $f \in \Xi\text{-RQC}(K, x^0)$ for all $x^0 \in K$.

The following theorem generalizes Theorem 6.2. The proof follows straightforward from Theorem 6.2 and is omitted.

Theorem 6.3. *Let K be a set in a real linear space and Ξ be a set of functions $\xi : Y \to \boldsymbol{R}$ on a topological vector space Y. Let a svf $F : K \rightsquigarrow Y$ have the property $F \in \Xi\text{-RQC}(K, x^0)$ at the point $x^0 \in K$. If x^0 is a (set) Ξ-minimizer of F, then x^0 is a solution of the system of VI* (10)*. In particular, if F is Ξ-quasiconvex, then any (set) Ξ-minimizer of F is a solution of* (10)*.*

Theorem 6.3 opens the problem, given Ξ, to characterize Ξ-quasiconvex functions and to compare Ξ-quasiconvexity with the usual notion of convexity. We consider this problem in two major cases.

For simplicity, we may assume, from now on that the svf has weakly compact values.

The case $\Xi = C'$.

When $\Xi = C'$ it holds $F \in \Xi\text{-RQC}\,(K, x^0)$ at $x^0 \in K$ if all the functions ϕ defined in (14) are radially quasiconvex along the rays starting at x^0. The svf F is Ξ-quasiconvex if the functions (14) are quasiconvex for each $x^0 \in K$.

In [27] a svf $F : K \subseteq X \rightsquigarrow Y$ is said to be $*$-quasiconvex when for each $\xi \in C'$ the function

$$\tilde{\phi}(x) = \min_{y \in F(x)} \langle \xi, y \rangle \tag{18}$$

is quasiconvex on K (for deeper insight we refer to [37]). A radial variant of $*$-quasiconvexity is introduced straightforward. Recalling (14), we get immediately that when $\Xi = C'$ the svf $F : K \rightsquigarrow Y$ is Ξ-quasiconvex if and only if it is $*$-quasiconvex.

Recalling Corollary 4.1, it becomes clear that the following corollary of Theorem 6.3 holds true.

Corollary 6.1. *Let K be a set in a real linear space and C be a closed convex cone in a real topological vector space Y. Let a svf $F : K \rightsquigarrow Y$ be radially $*$-quasiconvex along the rays starting at $x^0 \in K$. If x^0 is a set a-minimizer of F, then x^0 is a solution of the system of VI* (10)*. In particular, if F is $*$-quasiconvex, then any set a-minimizer of F is a solution of* (10)*.*

Recall that a svf $F : K \rightsquigarrow Y$ is said to be C-quasiconvex on the convex set $K \subset X$ if for each $y \in Y$, the set $\{x \in K \,|\, y \in F(x) + C\}$ is convex. Similarly, we call F radially C-quasiconvex along the rays starting at $x^0 \in K$, if the restriction of F on each such ray is C-quasiconvex. It is well known (see e.g. [27]), that the class of (radially) C-quasiconvex functions is broader than that of (radially) $*$-quasiconvex functions.

The following proposition (see Proposition 3.1 and Theorem 3.1 in [5]), shows that diminishing eventually the set Ξ, we still can get equivalence of Ξ-quasiconvexity and C-quasiconvexity. In its formulation we apply the following notions.

We say that the pair (Y, C) is directed, if for arbitrary $y^1, y^2 \in Y$, there exists $y \in Y$, such that $y - y^1 \in C$ and $y - y^2 \in C$. If Y is a Banach space, and the closed convex cone C has a nonempty interior, then the pair (Y, C) is directed. There are, however, important examples (see e.g. [4]) of directed pairs in which $\operatorname{int} C = \emptyset$. Given a set $P \subset Y$, a point $x \in P$ is said to be an extreme point of P, when there does not exist any couple of different points $x^1, x^2 \in P$, such that x is expressed as a convex combination with positive coefficients of x^1 and x^2. Recall also that a vector $\xi \in C'$ is said to be an extreme direction of C' when $\xi \in C' \backslash \{0\}$ and for all $\xi^1, \xi^2 \in C'$ such that $\xi = \xi^1 + \xi^2$, there exist positive reals λ_1, λ_2 for which $\xi^1 = \lambda_1 \xi$, $\xi^2 = \lambda_2 \xi$. We denote by $\operatorname{ext} P$ the set of extreme points of P and by $\operatorname{extd} C'$ the set of extreme directions for C'.

Proposition 6.1. [5] *Let Y be a Banach space, and C be a closed convex cone in Y, such that the pair (Y, C) is directed.*

i) *If F is C-quasiconvex then F is Ξ-quasiconvex with $\Xi = extd\, C'$.*
ii) *Suppose that C' is the weak-$*$ closed convex hull of $extd\, C'$ and assume the svf F is such that a-$Min_C F(x)$ is nonempty for every $x \in K$. If F is Ξ-quasiconvex with $\Xi = extd\, C'$ then it is C-quasiconvex.*

Obviously a "radial version" of Proposition 6.1 can be obtained straightforward.

We need the following lemma.

Lemma 6.1. *Let K be a set in a real linear space and Y be a Banach space. Let C be a closed convex cone in Y, such that (Y, C) is directed and C' has a weak-$*$ compact convex base Γ (these assumptions hold in particular when C is a closed convex cone with nonempty interior) and let functions ϕ be defined by* (12). *Let a svf $F : K \rightsquigarrow Y$ be such that $F \in \Xi\text{-RLSC}\,(K, x^0)$ for $\Xi = C'$ and for every $x \in K$ the set a-$\mathrm{Min}_C F(x)$ is nonempty. Then the system of VI* (10) *with $\Xi = C'$ is equivalent to the system*

$$\phi'(x,\, x^0 - x) \leqslant 0\,, \quad x \in K\,, \quad \textit{for all} \quad \phi \in \Phi(\Gamma \cap\; extd\, C', x^0) \tag{19}$$

where $\Phi(\Gamma \cap\; extd\, C', x^0)$ is defined by (12).

Proof. Since $\Gamma \cap\; \mathrm{extd}\, C' \subset C'$, we see that if x^0 is a solution of (10), then x^0 is a solution of (19). To prove the reverse inclusion, observe that according to the Krein-Milman Theorem, $C' = \mathrm{cl\,cone\ \ co}\,(\Gamma \cap \mathrm{extd}\, C')$. Assume x^0 is a solution of (19). Hence for $\xi \in \Gamma \cap \mathrm{extd}\, C' \subset C'$ functions

$$\phi(x) = \max_{y^0 \in F(x^0)} \min_{y \in F(x)} \langle \xi,\, y - y^0 \rangle = \min_{y \in F(x)} \langle \xi,\, y \rangle - \min_{y^0 \in F(x^0)} \langle \xi,\, y^0 \rangle \tag{20}$$

are increasing along rays starting at x^0 and this is equivalent to $\tilde{\phi}(x)$ increasing along rays at x^0. This means that for $x \in K$ and $0 < t_1 < t_2$ it holds

$$\begin{aligned} \tilde{\phi}(x^0 + t_1(x - x^0)) &= \min_{y \in F(x^0 + t_1(x - x^0))} \langle \xi, y \rangle \\ \leqslant \tilde{\phi}(x^0 + t_2(x - x^0)) &= \min_{y \in F(x^0 + t_2(x - x^0))} \langle \xi, y \rangle . \end{aligned}$$

Let $\tilde{\xi}_n$ be a sequence in $\mathrm{cone\,co}\,(\Gamma \cap\; \mathrm{extd}\, C')$. Hence for every positive integer n there exists a positive integer l_n, positive numbers λ_n, $\alpha_{n,i}$, $i = 1, \cdots, l_n$ with $\sum_{i=1}^{l_n} \alpha_{n,1} = 1$ and vectors $\xi_{n,i} \in (\Gamma \cap \mathrm{extd}\, C')$ such that $\tilde{\xi}_n = \lambda_n \sum_{i=1}^{l_n} \alpha_{n,i} \xi_{n,i}$. From the previous inequalities we have

$$\begin{aligned} \lambda_n \sum_{i=1}^{l_n} \alpha_{n,i} \min_{y \in F(x^0 + t_1(x - x^0))} \langle \xi_{n,i}, y \rangle &\leqslant \lambda_n \sum_{i=1}^{l_n} \alpha_{n,i} \min_{y \in F(x^0 + t_2(x - x^0))} \langle \xi_{n,i}, y \rangle \\ &\leqslant \min_{y \in F(x^0 + t_2(x - x^0))} \langle \lambda_n \sum_{i=1}^{l_n} \alpha_{n,i} \xi_{n,i}, y \rangle . \end{aligned}$$

Since the set a-$\mathrm{Min}_C F(x^0 + t_1(x - x^0))$ is nonempty, we can find a vector $z \in F(x^0 + t_1(x - x^0))$ such that $F(x^0 + t_1(x - x^0)) \subseteq z + C$. Hence for every $\xi \in C'$ we get

$$\min_{y \in F(x^0 + t_1(x - x^0))} \langle \xi, y \rangle = \langle \xi, z \rangle$$

and it follows that

$$\min_{y \in F(x^0 + t_1(x - x^0))} \langle \tilde{\xi}_n, y \rangle \leqslant \min_{y \in F(x^0 + t_2(x - x^0))} \langle \tilde{\xi}_n, y \rangle \, .$$

For $\tilde{\xi}_n \to \tilde{\xi} \in C'$, we get

$$\min_{y \in F(x^0 + t_1(x - x^0))} \langle \tilde{\xi}, y \rangle \leqslant \min_{y \in F(x^0 + t_2(x - x^0))} \langle \tilde{\xi}, y \rangle,$$

which proves that for $\Xi = C'$, functions $\phi \in \Phi(\Xi, x^0)$ are increasing along rays starting at x^0. Since $F \in \Xi$-$\mathrm{RLSC}\,(K, x^0)$ for $\Xi = C'$, the proof is completed applying Theorem 2.1. ∎

The nonemptiness assumption of a-$\mathrm{Min}_C F(x)$ is essential in order Lemma 6.1 holds true, as shown by Example 3.1 in [5].

Now, as an application of Theorem 6.3 and Proposition 6.1, we get the following result.

Corollary 6.2. *Let K be a set in a real linear space and Y be a Banach space. Let C be a closed convex cone in Y, such that (Y, C) is directed and C' has a weak-$*$ compact convex base Γ (these assumptions hold in particular when C is a closed convex cone with nonempty interior). Let a svf $F : K \to Y$ be radially C-quasiconvex along the rays starting at $x^0 \in K$ and assume that $F \in \Xi$-$\mathrm{RLSC}\,(K, x^0)$ and for every $x \in K$ the set a-$\mathrm{Min}_C F(x)$ is nonempty. If x^0 is a set a-minimizer of F, then x^0 is a solution of the system of VI* (10). *In particular, if F is C-quasiconvex, then each set a-minimizer of F is a solution of* (10).

Proof. From Lemma 6.1 we know the system of VI (19) is equivalent to system (6.1) with $\Xi = C'$. Suppose that F is radially C-quasiconvex along the rays starting at $x^0 \in \ker K$. This assumption according to Proposition 6.1 is equivalent to the condition $F \in \Xi - \mathrm{RQC}\,(K, x^0)$, with $\Xi = \Gamma \cap \mathrm{extd}\, C'$ (replacing $\mathrm{extd}\, C'$ with $\Gamma \cap \mathrm{extd}\, C'$ does give any change). Therefore, the condition that x^0 is a set a-minimizer of F is equivalent to the condition that x^0 is a Ξ-minimizer (with $\Xi = \Gamma \cap \mathrm{extd}\, C'$). The Ξ-quasiconvexity of F and Theorem 6.3 give that x^0 is a solution of the system of VI (19). Finally, the equivalence of (19) and (10) gives that x^0 is a solution of the system of VI (10), with $\Xi = C'$. ∎

The case $\Xi = \{\xi^0\}$ with $\xi^0(y) = D(y, -C)$

Now the following corollary of Theorem 6.3 has place.

Corollary 6.3. *Let K be a set in a real linear space, Y be a normed space, C be a closed convex cone in the normed space Y, and $\Xi = \{\xi^0\}$ with ξ^0 given by* (11)*.*

Let a svf $F : K \rightsquigarrow Y$ be radially $*$-quasiconvex along the rays starting at $x^0 \in K$. If x^0 is a set w-minimizer of F, then x^0 is a solution of the system of VI (10)*. In particular, if F is $*$-quasiconvex, then any set w-minimizer of F is a solution of* (10)*.*

Proof. The proof is an immediate consequence of Theorem 6.3, since it is enough to observe that if F is $*$-quasiconvex, then $F \in \Xi - \mathrm{RQC}\,(K, x^0)$ with $\Xi = \{\xi^0\}$. Indeed, recalling (15) and using the Minimax Theorem, we have

$$\begin{aligned}\phi(x) &= \sup_{y^0 \in F(x^0)} \inf_{y \in F(x)} \sup_{\xi \in C',\, \|\xi\|=1} \langle \xi, y - y^0 \rangle \\ &= \sup_{y^0 \in F(x^0)} \inf_{y \in F(x)} \sup_{\xi \in C',\, \|\xi\|\leqslant 1} \langle \xi, y - y^0 \rangle \\ &= \sup_{y^0 \in F(x^0)} \sup_{\xi \in C',\, \|\xi\|\leqslant 1} \inf_{y \in F(x)} \langle \xi, y - y^0 \rangle\end{aligned}$$

so that ϕ is the supremum of radially quasiconvex functions and hence is radially quasiconvex. ■

Since the class of (radially) C-quasiconvex functions is broader than that of (radially) $*$-quasiconvex functions, it arises naturally the question whether a result similar to Corollary 6.1 holds under radial C-quasiconvexity assumptions. We are going to show that in this case a result analogous to Corollary 6.1 holds when Y is a Banach space and its dual Y^* is endowed with a suitable norm, equivalent to the original one. From now on we assume that C is a closed convex cone in the normed space Y with both $\operatorname{int} C \neq 0$ and $\operatorname{int} C' \neq 0$. Fix $c \in \operatorname{int} C$. The set $G = \{\xi \in C' | \langle \xi, c\rangle = 1\}$ is a weak-$*$ compact convex base for C' [22]. Let $\tilde{B} = \operatorname{conv}\{G \cup (-G)\}$. Since $\tilde{B}$ is a balanced, convex, absorbing and bounded set, with $0 \in \operatorname{int} \tilde{B}$ (here we apply $\operatorname{int} C' \neq \emptyset$), the Minkowski functional $\gamma_{\tilde{B}}(y) = \{\lambda \in \boldsymbol{R} |\, \lambda > 0\,, y \in \lambda \tilde{B}\}$ is a norm on Y^*, see e.g. [38, 22]. We denote this norm by $\|\cdot\|_1$. Since $\operatorname{int} \tilde{B} \neq \emptyset$ and $\tilde{B}$ is bounded, it is easily seen that the norm $\|\cdot\|_1$ is equivalent to the original norm $\|\cdot\|$ in Y^*.

Theorem 6.4. *Let K be a convex set in a linear space and Y be a normed space. Let a svf $F : K \to Y$ be radially C-quasiconvex along the rays starting at $x^0 \in K$ and assume Y^* is endowed with the norm $\|\cdot\|_1$. If x^0 is a set w-minimizer of F, then the function $\phi(x)$ defined by (15) is radially quasiconvex along the rays starting at x^0* (i.e. $F \in \Xi - \mathrm{RQC}\,(K, x^0)$ for $\Xi = \{\xi^0\}$ *and ξ^0 defined by* (11)). *Hence x^0 is a solution of VI (10), with ϕ given by (15). In particular, if F is C-quasiconvex, then any set w-minimizer of F is a solution of VI (10).*

Proof. To prove the theorem it is enough to show that if F is radially C-quasiconvex and Y^* is endowed by the norm $\|\cdot\|_1$, then the function $\phi(x)$ defined by (15) is radially C-quasiconvex along the rays starting at x^0. Indeed, we recall that

$$\phi(x) = \sup_{y^0 \in F(x^0)} \inf_{y \in F(x)} \sup\{\langle \xi, y - y^0 \rangle \,|\, \xi \in C', \|\xi\|_1 = 1\}, \tag{21}$$

and we observe that $\{\xi \in C' | \, \|\xi\|_1 = 1\} = G$. Hence the supremum over $\{\xi \in C', \|\xi\|_1 = 1\}$ is attained, since G is weak-$*$ compact. Observe further that since x^0 is a set w-minimizer of F, then $\phi(x^0) = 0$.

For $\varepsilon > 0$, we have

$$\begin{aligned}
&\{x \in K | \phi(x) \leqslant \varepsilon\} \\
&= \{x \in K | \sup_{y^0 \in F(x^0)} \inf_{y \in F(x)} \max\{\langle \xi, y - y^0 \rangle \,|\, \xi \in C', \|\xi\|_1 = 1\} \leqslant \varepsilon\} \\
&= \{x \in K | \sup_{y^0 \in F(x^0)} \inf_{y \in F(x)} \max\{\langle \xi, y - y^0 \rangle \,|\, \xi \in C', \|\xi\|_1 = 1\} \\
&\leqslant \varepsilon \max\{\langle \xi, k \rangle \,|\, \xi \in C', \|\xi\|_1 = 1\}\} \\
&= \{x \in K | \sup_{y^0 \in F(x^0)} \inf_{y \in F(x)} \max\{\langle \xi, y - y^0 - \varepsilon k \rangle \,|\, \xi \in C', \|\xi\|_1 = 1\} \leqslant 0\} \\
&= \{x \in K | \, \forall y^0 \in F(x^0), y^0 \in F(x) - \varepsilon k + C\}.
\end{aligned}$$

Since F is C-quasiconvex on K, this last set is convex for every $\varepsilon > 0$ and so the level set of $\phi(x)$, $\{x \in K | \, \phi(x) \leqslant \varepsilon\}$ is convex too. It follows that $\phi(x)$ is quasiconvex with x^0 as minimizer over K and hence is in the class IAR (K, x^0), which completes the proof. ■

The following example shows that the previous theorem is not true without the assumption that Y^* is endowed with the norm $\| \cdot \|_1$.

Example 6.2. Let $X = \boldsymbol{R}$, $K = [-1, 1]$, $Y = \boldsymbol{R}^2$, $C = \boldsymbol{R}^2_+$. Let $F : K \rightsquigarrow Y$ be the svf defined by

$$F(x) = \begin{cases} (1,\, 0) & x = -1 \\ \operatorname{conv}\{(1,\, x+1),\, (1,\, 1)\}\,, & -1 \leqslant x \leqslant 0\,, \\ \operatorname{conv}\{(1-x,\, 1),\, (1,\, 1)\}\,, & 0 \leqslant x \leqslant 1\,, \\ (0,\, 1), & x = 1\,. \end{cases}$$

Then F is C-quasiconvex but not $*$-quasiconvex. The point $x^0 = -1$ is a set w-minimizer of the set-valued problem (5). If Y is endowed with the norm $\| \cdot \|$ in which the unit ball is the parallelogram $\operatorname{conv}\{(0, 1), (-2, 2), (0, -1), (2, -2)\}$, then x^0 is not a solution of the respective VI (10). At the same time according to Theorem 6.4 the point x^0 is a solution of the VI (10) obtained on the base of the norm $\| \cdot \|_1$.

The svf F is C-convex, since for each $y = (y_1, y_2) \in Y$ the set

$$\{x \in K \mid y \in F(x) + C\} = \begin{cases} [-1,\, 1]\,, & y_1 \geq 1,\, y_2 \geq 1\,, \\ [-1,\, y_2 - 1]\,, & y_1 \geq 1,\, 0 \leqslant y_2 < 1\,, \\ [-y_1 + 1,\, 1]\,, & 0 \leqslant y_1 < 1,\, y_2 \geq 1\,, \\ \emptyset\,, & \begin{cases} y_1 < 0 \text{ or } y_2 < 0 \\ \text{or } y_1 < 1,\, y_2 < 1\,, \end{cases} \end{cases}$$

is an interval and hence convex.

The svf F is not $*$-quasiconvex, since for $\xi = (1, 1) \in C' = \boldsymbol{R}^2_+$ the function

$$\min_{y \in F(x)} \langle \xi, y \rangle = \begin{cases} 2 + x, & -1 \leqslant x \leqslant 0, \\ 2 - x, & 0 \leqslant x \leqslant 1, \end{cases}$$

is not quasiconvex as a function of $x \in K$.

The dual norm to $\|\cdot\|$ in Y^* is determined by its unit ball being the parallelogram conv$\{(3/2, 1), (1/2, 1), (-3/2, -1), (-1/2, -1)\}$. With this observation, denoting by $y^0 = (1, 0)$ the unique value of $F(x^0)$, it is easy to calculate, that the oriented distance with respect to the the norm $\|\cdot\|$ is

$$\phi(x) = D(f(x) - f(x^0), -C) = \begin{cases} 1 + x, & -1 \leqslant x \leqslant 0, \\ 1 - x/2, & 0 \leqslant x \leqslant 1. \end{cases}$$

The function ϕ is not increasing along rays starting at x^0, hence x^0 is not a solution of the respective VI (10), which is in fact

$$\phi'(x, x^0 - x) \equiv \begin{cases} x^0 - x \leqslant 0, & -1 \leqslant x \leqslant 0, \\ -\frac{1}{2}(x^0 - x) \leqslant 0, & 0 < x \leqslant 1. \end{cases}$$

The norm $\|\cdot\|_1$ in Y^* with the choice $c = (1, 1)$ (see the proof of Corollary 6.3) is the ℓ^1 norm $\|\xi\| = |\xi_1| + |\xi_2|$ being dual to the norm ℓ^∞ of the original space $\|y\| = \max(|y_1|, |y_2|)$. With respect to this norm, the oriented distance, written with a subscript to distinguish it from the case when the norm $\|\cdot\|$ is applied, is $\phi_1(x) \equiv D_1(F(x) - y^0, -C) = 1$, $-1 \leqslant x \leqslant 1$. Obviously ϕ_1 is increasing along the rays starting at x^0, hence x^0 is a solution of the respective VI (10). The latter is in fact the trivial one $\phi_1'(x, x^0 - x) \equiv 0 \leqslant 0$, $-1 \leqslant x \leqslant 1$.

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