Vietnam Journal of Mathematics **35**:1 (2007) 21–32

Vietnam Journal
of
MATHEMATICS

On a Probability Metric Based on Trotter Operator

Tran Loc Hung

Hue College of Science, Hue University, 77 Nguyen Hue, Hue, Vietnam

Received December 19, 2005
Revised June 25, 2006

Abstract. The main purpose of this paper is to present a probability metric based on well-known Trotter's operator. Some estimations related to the rates of convergence via Trotter metric are established.

2000 Mathematics Subject Classification: 60G50, 60E10 25, 32U 05.
Keywords: Probability metric, Trotter operator, rates of convergence, weak law of large numbers, quicksort algorithm.

1. Introduction

During the last several decades the probability metric approach has risen to become one of the most important tools available for dealing with certain types of large scale problems.

In the solution of a number of problems of probability theory the method of distance function has attracted much attention and it has successfully been used lately as Abramov [1], Butzer and Kirschfink [4], Dudley [6] and [7], Kirschfink [12], Rachev [20] and Zolotarev [26 - 31].

The essence of this method is based on the knowledge of the properties of metrics in spaces of random variables as well as on the principle according to which in every problem of the approximating type a metric as a comparison measure must be selected in accordance with the requirements to its properties.

In recent years several results of applied mathematics and informatics have been established by using the probability metric approach. Results of this nature may be found in Gibbs and Edward [9], Hutchinson and Ludger [11] and Ralph and Ludger [16 - 18], Hwang and Neininger [l0], Mahmound and Neininger [13].

The main purpose of the present note is to introduce a probability metric which is based on well-known Trotter's operator. Some approximations of the rates of convergence via Trotter metric are indicated.

This paper is organized as follows. Sec. 2 deals with some well-known probability metrics. Sec. 3 reviews definition and properties of Trotter's operator. The definition of the Trotter metric basing on Trotter operator and some its connections with different probability metrics are described in Sec. 4. Sec. 5 shows some estimations related to the rates of convergence via Trotter metric. It is worth pointing out that all proofs of theorems of this section utilize Trotter's idea from [25] and the method used in this section is the same as in [2 - 4, 12, 14, 15, 21]. The received results in Sec. 5 are extensions of that given in [23, 24]. It should be noted that the results for dependent random variables have been obtained by Butzer and Kirschfink in [3], Butzer, Kirschfink and Schulz in [4], Kirschfink in [12]. However, this idea is due to Trotter, who has presented an elementary proof of a central limit theorem (see [25] for more details). After presenting Trotter's method, some analogous results concerning the proofs of limit theorems and the rates of convergence in limit theorems for independent random variables were demonstrated by Renyi [21], Feller [8], Molchanov [14], Butzer, Hahn, Westphal, Kirschfink and Schulz [2 - 4], Muchanov [15], Rychlich and Szynal [22] and Hung [23, 24]. The concluding remarks will be taken up in the last section.

2. Probability Metrics

Before stating the main results of this paper we review the definitions and properties of some well-known probability metrics. We will denote by Ψ the set of random variables defined on a probability space $(\Omega, \mathcal{A}, P)$.

Definition 2.1. *The mapping $d : \Psi \times \Psi \to [0, \infty)$ is called a probability metric, denoted by $d(X, Y)$, if*

i. $P(X = Y) = 1$ *implies* $d(X, Y) = 0$,

ii. $d(X, Y) = d(Y, X)$ *for random variables X and Y,*

iii. $d(X, Y) \leq d(X, Z) + d(Z, Y)$ *for random variables X, Y and Z in Ψ.*

Definition 2.2. *A metric d is called simple if its values are determined by a pair of marginal distributions P_X and P_Y. In all other cases d is called composed.*

It should be noted that, for a simple metric the following forms are equivalent

$$d(X, Y) = d(P_X, P_Y) = d(F_X, F_Y).$$

Definition 2.3. *A metric d is called ideal of order $s \geq 0$ on a subspace $\Psi^* \subset \Psi$, if for $X, Y, Z \in \Psi^*$ with X and Y independent of Z, and $c \neq 0$, the following two properties hold*

i. *regularity:* $d(X + Z, Y + Z) \leq d(X, Y)$,

ii. *homogeneity:* $d(cX, cY) \leq |\, c\,|^s d(X, Y)$.

An interesting consequence of the regularity and homogeneity properties is the semi additivity of the metric d: Let $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ be two collections of independent random variables, then one has for X, Y with real numbers $c_j, 1 \leq j \leq n$

$$d(\sum_{j=1}^{n} X_j, \sum_{j=1}^{n} Y_j) \leq \sum_{j=1}^{n} |c_j|^s d(X_j, Y_j).$$

We now turn to some examples for illustration of well-known probability metrics

1. Kolmogorov metric (Uniform metric). Let us consider the state space $\Omega = \mathbb{R} = (-\infty, +\infty)$, then the Kolmogorov metric is defined by

$$d_K(F, G) := \sup_{t \in \mathbb{R}} \Big| F(t) - G(t) \Big|. \tag{2.1}$$

The Kolmogorov metric assumes values in $[0, 1]$, and is invariant under all increasing one-to-one transformations of the line.

2. Levy metric. Let the state space $\Omega = \mathbb{R} = (-\infty, +\infty)$, then the Levy metric is defined by

$$d_L(F, G) = \inf_{\delta > 0} \Big\{ G(x - \delta) - \delta \leq F(x) \leq G(x + \delta) + \delta, \forall x \in \mathbb{R} \Big\}. \tag{2.2}$$

The Levy metric assumes values in $[0, 1]$. While not easy to compute, the Levy metric does metrize weak convergence of measures on $\mathbb{R}$. This metric is a simple metric.

3. Prokhorov (or Levy-Prokhorov) metric. Let μ and ν be two Borel measures on the metric space (S, d), then the Prokhorov metric d_P is given by

$$d_P(\mu, \nu) := \inf_{\epsilon > 0} \Big\{ \mu(A) \leq \nu(A^\epsilon) + \epsilon, \text{ for all Borel sets } A \in (S, d) \Big\}, \tag{2.3}$$

where $A^\epsilon := \{y \in S; \exists x \in A : d(x, y) < \epsilon\}$.

The Prokhorov metric d_P assumes values in $[0, 1]$. It is possible to show that this metric is symmetric in μ, ν. This metric was defined by Prokhorov as the analogue of the Levy metric for more general spaces. This metric is theoretically important because it metrizes weak convergence on any separable metric space. Moreover, $d_P(\mu, \nu)$ is precisely the minimum distance "in probability" between random variables distributed according to μ, ν.

4. Zolotarev metric. The Zolotarev metric for distributions F_X and F_Y is defined by

$$d_Z(X, Y) := \sup \Big\{ \Big| E[f(X) - f(Y)] \Big|; f \in D_1(s; r + 1; C(\mathbb{R})) \Big\}, \tag{2.4}$$

here $C(\mathbb{R})$ is the set of all real-valued, bounded, uniformly continuous functions defined on the reals $\mathbb{R} = (-\infty, +\infty)$, endowed with the norm

$$\| f \| = \sup_{t\in\mathbb{R}} |f(t)|.$$

Furthermore, for $r \in \mathbb{N}$ we set $C^o(\mathbb{R}) = C(\mathbb{R})$,

$$C^r(\mathbb{R}) := \{f \in C(\mathbb{R}) : f^{(j)} \in C(\mathbb{R}), 1 \leq j \leq r, r \in \mathbb{N}\}.$$

and

$$D_1(s; r+1; C(\mathbb{R})) := \Big\{ f \in C^r(\mathbb{R}); \big| f^{(r)}(x) - f^{(r)}(y) \big| \leq \big| x - y \Big|^s \Big\}.$$

It should be noted that $C^r(\mathbb{R}) \subset D_1(s; r+1; C(\mathbb{R})) \subset C(\mathbb{R})$,

The Zolotarev metric $d_Z(X, Y)$ is an ideal metric of order 3, i. e. we have for Z independent of (X, Y) and $c \neq 0$,

$$d_Z(X + Z, Y + Z) \leq d_Z(X, Y)$$

and

$$d_Z(cX, cY) = |c|^3 d_Z(X, Y).$$

It is easy to see that, for X_j and Y_j being pairwise independent,

$$d_Z\left(\sum_{j=1}^{n} X_j, \sum_{j=1}^{n} Y_j\right) \leq \sum_{j=1}^{n} d_Z(X_j, Y_j).$$

It is well known that convergence in d_Z implies weak convergence and it plays a great role in some approximation problems. For general reference and properties of d_Z we refer to Zolotarev in [26 - 31] or to Gibbs and Edward in [9], Hutchinson and Ludger in [11] and Ralph and Ludger in [16 - 18].

In addition, we also illustrate some relationships among probability metrics in (2.1), (2.2) and (2.3) as follows (cf. [9]).

1. For probability measures μ, ν on $\mathbb{R}$ with distribution functions F, G,

$$d_L(F, G) \leq d_K(F, G).$$

2. If $G(x)$ is absolutely continuous (with respect to Lebesgue measures), then

$$d_K(F, G) \leq \Big(1 + \sup_x |G^{'}(x)|\Big).d_L(F, G).$$

3. For probability measures on $\mathbb{R}$,

$$d_L(F, G) \leq d_P(F, G).$$

3. The Trotter Operator

In order to present an elementary proof that a sequence $\{X_n,\, n \geq 1\}$ of random variables satisfies the central limit theorem, a linear operator was mainly introduced by Trotter [25]. The operator of Trotter to be dealt with in the present section can be called the characteristic operator (or Trotter's operator). We recall some definitions and properties of the Trotter operator from [2, 12, 21, 25].

Definition 3.1. *By the Trotter operator of a random variable X we mean the mapping $T_X : C(\mathbb{R}) \to C(\mathbb{R})$ such that*

$$T_X f(t) := E[f(X+t)], \quad t \in \mathbb{R},\ f \in C(\mathbb{R}). \tag{3.1}$$

The norm of $f \in C(\mathbb{R})$ needs to be recalled as

$$\| f \| = \sup_{t\in\mathbb{R}} |f(t)|.$$

We need in the sequel the following properties of the Trotter operator (see [2, 12, 21, 25] for more details).

At first, the operator T_X is a positive linear operator satisfying the inequality

$$\| T_X f \| \leq \| f \|,$$

for each $f \in C(\mathbb{R})$.

The equation $T_X f = T_Y f$ for every $f \in C(\mathbb{R})$, provided that X and Y are identically distributed random variables.

The condition

$$\lim_{n\to+\infty} \| T_{X_n} f - T_X f \| = 0 \quad \text{for} \quad f \in C(\mathbb{R}),$$

implies that

$$\lim_{n\to+\infty} F_{X_n}(x) = F_X(x),$$

for all $x \in C(F)-$ the set of all continuous point of F.

Let X and Y be independent random variables, then

$$T_{X+Y}(f) = T_X(T_Y f) = T_Y(T_X f),$$

for each $f \in C(\mathbb{R})$.

Moreover, if $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ are independent random variables (in each group) and $X_1, X_2, \ldots, X_n$ are independent of $Y_1, Y_2, \ldots, Y_n$, then for each $f \in C(\mathbb{R})$, we have

$$\| T_{\sum_{i=1}^n X_i} f - T_{\sum_{i=1}^n Y_i} f \| \leq \sum_{i=1}^n \| T_{X_i} f - T_{Y_i} f \| .$$

and

$$\| T_X^n - T_Y^n \| \leq n \| T_X f - T_Y f \| .$$

For the proofs of these properties we refer the reader to Trotter [25] and Butzer, Hahn, Westphal [2], Molchanov [14] or Renyi [21] for more details.

The modulus of continuity we denote by

$$\omega(f; \delta) := \sup_{|h|<\delta} \| f(.+h) - f(.) \|, \quad f \in C(\mathbb{R}),\ \delta > 0.$$

Of course, we have

$$\lim_{\delta \to 0} \omega(f;\delta) = 0$$

and for each $\lambda > 0$,

$$\omega(f;\lambda\delta) \leq (1+\lambda)\omega(f;\delta).$$

The detailed discussions of the properties of the modulus of continuity can be found in [2-4].

4. The Trotter Metric

In this section the definition and properties of a probability metric basing on Trotter operator are considered. Some relationships with well-known probability metrics are established, too.

Definition 4.1. *The Trotter metric $d_T(X,Y;f)$ of two random variables X and Y related to a function f is defined by*

$$d_T(X,Y;f) = \sup_{t \in \mathbb{R}} \Big\{ \big| Ef\big(X+t\big) - Ef\big(Y+t\big) \big| ; f \in C^r(\mathbb{R}) \Big\}.$$

The most important properties of the Trotter metric are summarized in the following. The proofs are easy to get from the properties of the Trotter operator (see [2, 12, 14, 25] for more details).

1. $d_T(X,Y;f)$ is a probability metric.
   It is easy to see that, if $P(X=Y)=1$ then

$$\sup_t \Big\{ \big| Ef\big(X+t\big) - Ef\big(Y+t\big) \big| ; f \in C^r(\mathbb{R}) \Big\} = 0,$$

in Definition 2.1 we have i) holds. The condition ii) is trivial, and the condition iii) follows from triangle-inequality.

2. $d_T(X,Y;f)$ is not a ideal metric because neither regularity nor homogeneity holds.
3. If $d_T(X,Y;f) = 0$ for $f \in C^r(\mathbb{R})$, then $F_X = F_Y$.
4. Let $\{X_n, n \geq 1\}$ be a sequence of random variables and X be a random variable. Then, for all $x \in C(F)$,

$$\lim_{n \to +\infty} F_{X_n}(x) = F_X(x)$$

if

$$\lim_{n \to +\infty} d_T(X_n, X; f) = 0, \quad \text{for} \quad f \in C^r(\mathbb{R}).$$

5. Let $X_1, X_2, \dots, X_n$ and $Y_1, Y_2, \dots, Y_n$ be two collections of independent random variables, then

$$d_T\left(\sum_{j=1}^n X_j, \sum_{j=1}^n Y_j; f \right) \leq \sum_{j=1}^n d_T\big(X_j, Y_j; f\big).$$

6. In the case when $X_1, X_2, \dots, X_n$ and $Y_1, Y_2, \dots, Y_n$ are two collections of independent identically distributed random variables, then

$$d_T\Big(\sum_{j=1}^{n} X_j, \sum_{j=1}^{n} Y_j; f\Big) \le n d_T\big(X_1, Y_1; f\big).$$

7. If N is a positive integer-valued random variable independent of

$$X_1, X_2, \dots, X_n \quad \text{and} \quad Y_1, Y_2, \dots, Y_n,$$

then

$$d_T\Big(\sum_{j=1}^{N} X_j, \sum_{j=1}^{N} Y_j; f\Big) \le \sum_{n=1}^{\infty} P(N=n) \sum_{j=1}^{n} d_T\big(X_j, Y_j; f\big).$$

A special interest in approximation problems is the connection between the Trotter metric and other well known metric such as the d_Z metric in (2.4), and Prokhorov-metric d_P in (2.3), who metrizes weak convergence. We have the following (see for more details in [1, 4, 9, 11]).

8.

$$c_s \sup\{d_T(X, Y; f)^{1/(1+s)}; f \in D_1(s; r+1; C(\mathbb{R})\} \ge d_P(\mid X \mid, \mid Y \mid),$$

where c_s is a constant .

9. (Recall Theorem 8, [4])

$$d_T(X, Y; f) \le E[|X - Y|^s],\ 0 < s \le 1,$$

where

$$f \in D_s = \begin{cases} f \in C(\mathbb{R}) \cap Lip(\alpha) \\ f^r \in C(\mathbb{R}) \cap Lip(\alpha), s = r + \alpha, r \ge 1, \alpha \in (0,1], s > 1. \end{cases}$$

10. (Recall from Lemma 2, [26])

$$d_Z(X, Y) \le \frac{\Gamma(1+\alpha)}{\Gamma(1+s)} \Big\{E|X|^s + E|Y|^s\Big\} \quad \text{with } s > 0,$$

where $s = r + \alpha, r \ge 1, \alpha \in (0, 1]$.

11. (cf.[11]) Let $s = r + \alpha, r \in \mathbb{N} \cup \{0\}, \alpha \in (0, 1]$, then there exists a constant c_s, such that for X and Y,

$$d_P^{1+s}(|X|, |Y|) \le c_s d_Z(X, Y).$$

12. (cf. [11]) In comparison with the Zolotarev metric d_Z, there holds

$$\sup\big\{d_T(X, Y; f); f \in D_1(s; r+1; C(\mathbb{R}))\big\} = d_Z(X, Y).$$

5. Applications

The above relationships will help to solve some approximation problems in theory of limit theorems via Trotter metric.

First at all, we recall a well-known theorem due to Petrov (see [25, Theorem 28, page 349]), which related to the rate of convergence in weak law of large numbers.

Theorem Petrov. [25] *Let $\{X_n, n \geq 1\}$ be a sequence of identically independent distributed (i.i.d.) random variables with zero means and finite r-th absolute moments $E(| X_j |^r) < +\infty$ for $r \geq 1$ and for $j = 1, 2, \dots n$. Then,*

$$P(|S_n| > \epsilon) = o(n^{-(r-1)}), \quad \text{as } n \to +\infty,$$

where $S_n = n^{-1} \sum_{j=1}^{n} X_j$.

We are now interested in the rate of convergence of the Trotter metric to zero,

$$d_T(S_n; X^0; f) \to 0 \quad \text{as } n \to +\infty.$$

Theorem 5.1. *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with zero expectation and finite r-th absolute moments $E(| X_j |^r) < +\infty$ for $r \geq 1$ and for $j = 1, 2, \dots n$. Then, for every $f \in C^r(\mathbb{R})$, we have the following estimation*

$$d_T(S_n; X^0; f) = o(n^{-(r-1)}), \quad \text{as } n \to +\infty. \tag{5.1}$$

Proof. By the same method used in [23], since $f \in C^r(\mathbb{R})$, we have the Taylor expansion

$$f(n^{-1}X_j + t) = \sum_{k=0}^{r} \frac{f^{(k)}(t)}{k!} n^{-k} X_j^k + (r!)^{-1} \big[f^{(r)}(t + \theta_1 n^{-1} X_j) - f^{(r)}(t)\big](n^{-1}X_j)^r,$$

where $0 < \theta_1 < 1$.

Taking the expectation of both sides of the last equation, we have

$$\begin{aligned} E\big[f(n^{-1}X_j + t)\big] &= \sum_{k=0}^{r} \frac{f^{(k)}(t)}{k!} n^{-k} E(X_j)^k \\ &+ (r!)^{-1} \int_{\mathbb{R}} \big[f^{(r)}(t + \theta_1 n^{-1} x) - f^{(r)}(t)\big](n^{-1}x)^r dF_{X_j}(x), \end{aligned}$$

where $0 < \theta_1 < 1$.

Then

$$\begin{aligned} \big|E\big[f(n^{-1}X_j + t)\big] - f(t)\big| &\leq \sum_{k=1}^{r} \big[(k! n^k)^{-1} \| f^{(k)} \| E|X_j|^k\big] \\ &+ \big[(r! n^r)^{-1}\big] \int_{\mathbb{R}} \big|f^{(r)}(t + \theta_1 n^{-1} x) - f^{(r)}(t)\big| . |x|^r dF_{X_j}(x), \end{aligned} \tag{5.2}$$

where $\| f^{(k)} \| = \sup_{t \in \mathbb{R}} |f^{(k)}(t)|, \ 1 \leq k \leq r$.

Since $f \in C^r(\mathbb{R})$, it follows that $\| f^{(k)} \| \leq M_1 = \text{const}$, and because $E|X_j|^k < +\infty$ for $k = 1, 2, \dots, r$, we get

$$\sum_{k=1}^{r}\left[(k!n^k)^{-1}\| \; f^{(k)} \; \|E|X_j|^k\right] = o(1), \quad \text{as} \;\; n \to +\infty. \tag{5.3}$$

Subsequently, by estimating the integral of right hand side of (5.2), we get

$$\begin{aligned}
&[(r!n^r)^{-1}]\int_{\mathbb{R}}|f^{(r)}(t+\theta_1 n^{-1}x)-f^{(r)}(t)|.|x|^r dF_{X_j}(x)\\
&=[(r!n^r)^{-1}]\int_{|x|\le n\delta(\epsilon)}|f^{(r)}(t+\theta_1 n^{-1}x)-f^{(r)}(t)|.|x|^r dF_{X_j}(x)\\
&\quad+[(r!n^r)^{-1}]\int_{|x|>n\delta(\epsilon)}|f^{(r)}(t+\theta_1 n^{-1}x)-f^{(r)}(t)|.|x|^r dF_{X_j}(x) = I_1 + I_2.
\end{aligned}$$

Because $f \in C^r(\mathbb{R})$, so for every $\epsilon > 0$, there is $\delta(\epsilon) > 0$, such that, for $|n^{-1}x| \le \delta(\epsilon)$, we find

$$|f^{(r)}(t+\theta_1 n^{-1}x) - f^{(r)}(t)| < \epsilon.$$

It follows that

$$I_1 \le \epsilon\int_{\mathbb{R}}|x|^r dF_{X_j}(x) = \epsilon E|X|^r. \tag{5.4}$$

Since $E|X|^r < +\infty$, so we get, for every $\epsilon > 0$, and for n sufficiently large, we obtain

$$I_2 \le 2\epsilon\| \; f^{(k)} \; \|. \tag{5.5}$$

Combining (5.4) and (5.5) and since ϵ is arbitrary positive number, so we have

$$\sup_t|Ef(n^{-1}X_j+t) - f(t)| = o(n^{-r}) \quad \text{as} \quad n \to +\infty. \tag{5.6}$$

Then we get, for $f \in C^r(\mathbb{R})$, using the properties of d_T,

$$d_T(S_n; X^0; f) \le n d_T(n^{-1}X_j; n^{-1}X_j^0; f).$$

We get the complete proof $d_T(S_n; X^0; f) = o(n^{-(r-1)})$ as $n \to +\infty$. ■

Let now $\{N_n; n \ge 1\}$ be a sequence of random variables which assume only positive integer values and which are supposed to obey the relation

$$N_n \to +\infty \quad \text{(in probability)} \quad \text{as} \;\; n \to +\infty$$

and

$$P(N_n = n) = p_n \ge 0; \quad \sum_{n=1}^{+\infty} p_n = 1.$$

Suppose that the $N_n, n \ge 1$ are independent of random variables $X_1, X_2, \ldots$. Then we can deduce from Theorem 5.1 the following result.

Theorem 5.2. *Let $\{X_n; n \ge 1\}$ be a sequence of i.i.d. random variables with zero expectation and let for $r \ge 1, j = 1, 2, \ldots, E|X_j|^r < +\infty$. Let further*

$\{N_n; n \geq 1\}$ be a sequence of positive integer-valued random variables satisfying the above conditions. Then, for every $f \in C^r(\mathbb{R})$, the relation

$$d_T(S_{N_n}; X^0; f) = o(E(N_n)^{-(r-1)}) \quad as \;\; n \to +\infty \tag{5.7}$$

is valid.

Proof. The proof rests upon the inequality of property 7, Sec. 4 and (5.1) using the same method as the proof of Theorem 5.1. ■

Theorem 5.3. *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with mean zero and $0 < Var(X_j) = \sigma^2 \leq M_2 < +\infty$, for every $j = 1, 2, \dots n$. Then, for every $f \in C(\mathbb{R})$, we have the following estimation*

$$d_T(S_n; X^0; f) \leq (2 + M_2)\omega(f; n^{-\frac{1}{2}}). \tag{5.8}$$

Proof. We first observe that $E(S_n) = 0$, and

$$Var(S_n) = E(S_n^2) = \frac{\sigma^2}{n}.$$

Let us denote $\lambda = \left[\frac{|S_n|}{\delta}\right] + 1, \quad \forall \delta > 0$. For $f \in C(\mathbb{R})$, using the properties of the modulus of continuity of the function f, we have

$$|f(S_n + t) - f(t)| \leq \omega(f; \lambda\delta) \leq (1 + \lambda)\omega(f; \delta).$$

Clearly,

$$\begin{aligned} d_T(S_n; X^0; f) &\leq \omega(f; \delta)E(1 + \lambda) \leq \omega(f; \delta)(1 + E(\lambda^2)) \\ &\leq \omega(f; \delta)(2 + \frac{E(S_n^2)}{\delta^2}) \leq \omega(f; \delta)(2 + \frac{\sigma^2}{n\delta^2}). \end{aligned}$$

The complete proof follows by taking $\delta = n^{-\frac{1}{2}}$ and $\sigma^2 \leq M_2$. ■

Remark 5.1. By taking $r = 1$ from (5.1) we get the weak law of large in Khinchin form (see [8, 19, 21]).

Remark 5.2. By taking $r = 1$ from (5.7) we get the random weak law of large.

Remark 5.3. Because of (5.8), using the fact that $\omega(f; n^{-\frac{1}{2}}) \to 0$ as $n \to +\infty$, the weak law of large in Chebyshev form (see [8, 15, 17]) will be received.

6. Concluding Remarks

We conclude this paper with the following comments, and the interested reader is referred to [16] for more details. Let X_n be a sequence of the numbers of key comparisons needed by the Quick sort algorithm to sort an array of n randomly permuted items satisfies $X_o = 0$ and the recursion

$$X_n \stackrel{\mathrm{d}}{=} X_{I_n} + X'_{n-1-I_n} + n - 1, \quad n \geq 1,$$

where $\stackrel{d}{=}$ denotes equality in distribution, $(X_n), (X_n^{'}), I_n$ are independent, I_n are uniformly distributed on $\{0, 1, \ldots n-1\}$, and $X_k \sim X_k^{'}, k \geq 0$, where $\sim$ also denotes equality of distributions.

The distribution of the number of key comparisons X_n of the Quick sort algorithm needed to sort an array of n randomly permuted items is known to converge after normalization in distribution as $n \to \infty$. Recently, some estimates for the rate of the convergence were upper estimates $0(n^{-\frac{1}{2}})$ in the minimal l_p metric, $p \geq 1$, and $0(n^{-\frac{1}{2+\epsilon}})$ for the Kolmogorov metric for all $\epsilon > 0$ as well as the lower estimates $0(\frac{\ln(n)}{n})$ for the l_p metric, $p \geq 2$, and $0(\frac{1}{n})$ for the Kolmogorov metric.

It is to be noticed that some indication was given that $0\left(\frac{\ln n}{n}\right)$ might be the right order of the rate of convergence for many metrics. And this conjecture for the Zolotarev metric d_Z was confirmed in [16]. An interesting question can be raised for further study whether the same order of the rate of convergence in Trotter metric d_T can be found? We shall take this up in the next paper.

Acknowledgments. The author would like to take this opportunity to thank Professor Sh. K. Formanov from V. I. Romanovski Institute of Mathematics (Tashkent, Uzbekistan) and Professor Troush N.N. from Belarus State University (Minsk, Belarus) for their excellent advice and remarks leading to the writing of this paper.

References

1. V.A. Abramov, General limit theorem in M-scheme, *Theory Prob. Appl.* **32** (1987) 549–552 (Russian).
2. P.L. Butzer, L. Hahn, and U. Westphal, On the rate of approximation in the central limit theorem, *J. Approx. Theory* **13** (1975) 327–340.
3. P.L. Butzer, H. Kirschfink, and D. Schulz, An extension of the Lindeberg-Trotter operator-theoretic approach to limit theorems for dependent random variables, *Acta Sci. Math.* **51** (1987) 423–433.
4. P.L. Butzer and H. Kirschfink, General limit theorems with o-rates and Markov processes under pseudo-moment conditions, *Zeitschrift fur Analysis und ihre Anwendungen* **7** (1988) 280–307.
5. R. Cioczek and Szynal, On the convergence rate in terms of the Trotter operator in the central limit theorem without moment conditions, *Bull. Polish Acad. Sci. Math.* **35** (1987) 617–627.
6. R.M. Dudley, Distances of probability measures and random variables, *Ann. Math. Statist.* **39** (1968) 1563–1572.
7. R.M. Dudley, *Probabilities and Metrics: convergence of laws on metric spaces, with a view to statistical testing* (Lect. Notes Series: N45), Aarhus Uni, Aarhus, 1976.
8. W. Feller, *Introduction to Theory Probability and Its Applications*, 1967, Vol. 2.
9. Alison L. Gibbs and Su Francis Edward, *On Choosing and Bounding Probability Metrics*, Manuscript version 2002, pp. 1–21.

10. Hsien-Kuel Hwang and Ralph Neininger, *Phase Change of Limit Law in the Quick sort Recurrence under Varying Toll Functions*, Manuscript version April 12, 2002, pp. 1–34.
11. John E. Hutchinson and Ruschendorf Ludger, *Random Fractals and Probability Metrics*, Manuscript version 2002, 1–21.
12. H. Kirschfink, The generalized Trotter operator and weak convergence of dependent random variables in different probability metrics, *Results in Mathematics* **15** (1989) 294–323.
13. Hosam M. Mahmoud and Ralph Neininger, *Distribution of Distances in Random Binary Search Trees*, Manuscript version April 24, 2002, pp. 1–20.
14. A. B. Molchanov, *Central Limit Theorem for Large Deviations*, Moscow, 1975 (Russian).
15. M. V. Muchanov, Limit theorem for Abel sums, *Izvestia Acad. Sci. Turcmenistan* **5987** (1987) 19–23 (Russian).
16. Neininger Ralph and Ruschendorf Ludger, *Rates of convergence for Quick sort*, Manuscript version February 5, 2002, 1–10.
17. Neininger Ralph and Ruschendorf Ludger, *On the Contraction Method with Degenerate Limit Equation*, Manuscript version October 17, 2002, pp. 1–19.
18. Neininger Ralph and Ruschendorf Ludger, *A general Limit Theorem for Recursive Algorithms and Combinatorial Structures*, Manuscript version December 17, 2002, pp. 1–37.
19. V. V. Petrov, *Sums of Random Variables*, Moscow, 1972 (Russian).
20. S. T. Rachev, *Probability Metrics and the Stability of Stochastic Models*, 1991.
21. A. Renyi, *Probability Theory*, Amsterdam, 1970.
22. Z. Rychlik and D. Szynal, On the rate of approximation in the random-sum central limit theorem, *Theory Prob. Appl.* **24** (1979) 614–620.
23. Tran Loc Hung, Applications of the operator methods in the law of large numbers, *Vietnam J. Math.* **11** (1983) 20–24.
24. Tran Loc Hung, The Trotter's operator method in the law of large numbers with random index, *Vietnam J. Math.* **2** (1988) 4–9.
25. H. F. Trotter, An elementary proof of the central limit theorem, *Arch. Math. Basel* **10** (1959) 226–234.
26. V. M. Zolotarev, Approximation of the distribution of sum of independent variables with values in infinite-dimensional spaces, *Theory Prob. Appl.* **21** (1976) 741–758 (Russian).
27. V. M. Zolotarev, Metric distances in spaces of random variables, *Math. Uspekhi* **101** (1976) 417–454 (Russian).
28. V. M. Zolotarev, Ideal metrics in the problems of probability theory and mathematical statistics, *Austral. J. Statist.* **21** (1979) 193–208.
29. V. M. Zolotarev, Probability metrics, *Theory Prob. Appl.* **28** (1983) 278–302.
30. V. M. Zolotarev and V. V. Kalashnikov *Stability Problems for Stochastic Models*, Proceedings of the 6th International Seminar held in Moscow, USSR, April 1982, Springer-Verlag, 1983.
31. V. M. Zolotarev, *Modern Theory of Summation of Random Variables*, Utrecht, the Netherlands, 1997.