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Irreducible Quadratic Perturbation of Spatial Oscillator

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Abstract. In this paper, we consider the irreducible quadratic perturbation for the third dimensional linear oscillator. Using the Poincare method, we investigate conditions guaranteeing existence (lack) of periodic solutions. Also, we study the role of the iterative derivatives of the displacement function on constitution of periodic solutions, the type of the stability and global bifurcation of the system.

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1. Introduction

Third order differential equations are recently the subject of much research, specially, because of their role in modeling of natural phenomena, spatial oscillatory systems are of great importance. These kinds of equations arise in biology [9, 12] and physical behaviour of a fluid [2, 3, 10, 17]. Although, there are a few papers for the persistence of the periodic solutions [4-8, 19], but for almost all of them, the existence of a family of periodic solutions for a primary system is assumed. Therefore, the major problem is still finding periodic solutions for the primary system. Because of the topological characteristics of the three-dimensional space, the investigation of periodic solutions for the nonlinear third-order differential equations is a difficult problem. The three dimensional linear oscillator appears in some phenomena such as turbulent fluid dynamics [1, 16]. The concept of this oscillatory system is adopted from linear oscillation in a plane, which is modeled by the equation

$$\ddot{x} + \omega^2 x = 0.$$

This introduces an object moving on an ellipse in $xy-$plane. Differentiating the above equation, we obtain

$$\dddot{x} + \omega^2 \dot{x} = 0, \tag{1}$$

which is known as the three dimensional linear oscillator and introduces an object moving on an ellipse in $xyz-$space. In paper [14], the authors considered the equation

$$\dddot{x} + \omega^2 \dot{x} = \mu(\frac{\partial f(x,\dot{x})}{\partial x}\dot{x} + \frac{\partial f(x,\dot{x})}{\partial \dot{x}}\ddot{x}). \tag{2}$$

They showed that the system can be reduced to a second order differential equation, furthermore, if $\frac{\partial^2 f}{\partial x \partial \dot{x}}(0,0) \neq 0$, then (2) has infinitely many periodic solutions making a cylinder along the x-axis. Also, they considered the case $f(x,\dot{x}) = ax^2 + bx\dot{x} + c\dot{x}^2$ and imposed conditions on a, b and c, such that the system has infinitely many homoclinic orbits and periodic solutions. They left the case that the equation (2) cannot be reduced to a planar system. The irreducible quadratic perturbation of (1), i.e.

$$\dddot{x} + \omega^2 \dot{x} = Ax^2 + B\dot{x}^2 + C\ddot{x}^2 + ax\dot{x} + bx\ddot{x} + c\dot{x}\ddot{x},$$

can be written as below

$$\dddot{x} + \omega^2 \dot{x} = \underbrace{Ax^2 + (B-b)\dot{x}^2 + C\ddot{x}^2}_{irreducable\ terms} + \underbrace{\frac{\partial f(x,\dot{x})}{\partial x}\dot{x} + \frac{\partial f(x,\dot{x})}{\partial \dot{x}}\ddot{x}}_{reducable\ terms},$$

where, $f(x,\dot{x}) = \frac{a}{2}x^2 + bx\dot{x} + \frac{c}{2}\dot{x}^2$. As it is mentioned above, the effect of reducible quadratic terms has been studied in [14]. In what follows, we will study the effect of irreducible quadratic terms and consider the equation

$$\dddot{x} + \omega^2 \dot{x} = Ax^2 + B\dot{x}^2 + C\ddot{x}^2. \tag{3}$$

Because of the form of the above equation, many of analytical methods (such as, center manifold, normal form, averaging methods and functional analysis methods) are not suitable for investigating periodic solutions of the system. Therefore, we will apply the Poincare method to find periodic solutions. However, because of the complexity of the formula, computing the derivatives of the Poincare map is a very long process, such that we can say, manual computation and simplification of the formula are almost impossible. So, constructing of the displacement function and the further computations and simplifications in Sec. 3 (and some parts of Sec. 2) are done by using algebraic methods and computer softwares. In Sec. 2, we will re-scale (3) and obtain an irreducible small perturbed system. Then, we will study the structure of the Poincare map and introduce the corresponding main distance variation function. Sec. 3 is devoted to the periodic solutions. We will investigate conditions guaranteeing existence (lack) of periodic solutions. We will see that iterative derivatives of the Poincare map play important role in existence and stability of the periodic solutions. Also, they may cause global bifurcation for the system. Finally, in Sec. 4, we will re-scale the small perturbed equation and derive the results obtained in Sec. 3, for the irreducible quadratic system (3).

2. Construction of Poincare Map

Consider the equation (3) and put $\mu a = A, \mu b = B$, $\mu c = C$, then we can write

$$\dddot{x} + \omega^2 \dot{x} = \mu[ax^2 + b\dot{x}^2 + c\ddot{x}^2].$$

One more time, putting $\bar{x}(t) = \frac{1}{\omega}x(\frac{t}{\omega})$, $\bar{a} = \frac{a}{\omega^2}$, $\bar{b} = b$, $\bar{c} = \omega^2 c$, after dropping the bars, we obtain

$$\dddot{x} + \dot{x} = \mu[ax^2 + b\dot{x}^2 + c\ddot{x}^2].$$

The above equation can be written by the vector form

$$X^{'} = \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}}_{A} X + \mu \underbrace{\begin{pmatrix} 0 \\ 0 \\ ax^2 + by^2 + cz^2 \end{pmatrix}}_{F(x,y,z)}. \tag{4}$$

Let $\Phi(t, \zeta, \mu) = (\phi_1, \phi_2, \phi_3)$ be the flow of (4) such that $\Phi(0, \zeta, \mu) = \zeta$. If $\Phi(T, \zeta_0, \mu) - \zeta_0 = 0$ then (4) has periodic solutions, indeed, $\Phi(t, \zeta_0, \mu)$ is the periodic solution with period T. We can expand Φ for $\mu = 0$ and obtain

$$\Phi(t, \zeta, \mu) = \Phi(t, \zeta, 0) + \mu\Phi_\mu(t, \zeta, 0) + \frac{\mu^2}{2}\Phi_{\mu^2}(t, \zeta, 0) + \cdots + \frac{\mu^n}{n!}\Phi_{\mu^n}(t, \zeta, 0) + o(\mu^{n+1}).$$

Therefore, finding periodic solutions for (4) turns to the problem

$$0 = \Phi(T, \zeta, 0) - \zeta + \mu\Phi_\mu(T, \zeta, 0) + \frac{\mu^2}{2}\Phi_{\mu^2}(T, \zeta, 0) + \cdots + \frac{\mu^n}{n!}\Phi_{\mu^n}(T, \zeta, 0) + o(\mu^{n+1}).$$

In the above formula, the subscripts denote partial derivatives. Because we naturally consider $|\mu|$ small, so the period T must be such that $\Phi(T, \zeta, 0) - \zeta = 0$. On the other hand, the map $\Phi(t, X, 0)$ is the flow of the linear oscillator $\dot{X} = AX$ with the corresponding fundamental matrix

$$\chi(t) = \begin{pmatrix} 1 & \sin t & (1 - \cos t) \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{pmatrix}.$$

Hence, $\Phi(t, \zeta, 0) = \chi(t)\zeta$. This implies $T = 2N\pi$. Therefore, we can introduce the (first) displacement function (for $N = 1$) as below

$$d(\zeta, \mu) = \Phi_\mu(2\pi, \zeta, 0) + \frac{\mu}{2}\Phi_{\mu^2}(2\pi, \zeta, 0) + \cdots + \frac{\mu^{n-1}}{n!}\Phi_{\mu^n}(2\pi, \zeta, 0) + o(\mu^n). \tag{5}$$

The map $\zeta \mapsto \Phi_\mu(2\pi, \zeta, 0)$ is called the main distance variation function. This is because of the fact that, for $|\mu|$ small, the values of displacement function $d(\zeta, \mu)$ is well near to the values of $\Phi_\mu(2\pi, \zeta, 0)$ i.e. $d(\zeta, \mu) = \Phi_\mu(2\pi, \zeta, 0) + o(\mu)$. The function $\Phi_{\mu^k}(t, X, 0)$, $k = 1, 2, ...$, is the solution of the $k-$th variational equation. Differentiating (4) with respect to μ we obtain

$$\Phi_\mu^{'} = A\,\Phi_\mu + F(\Phi) + \mu DF(\Phi)\Phi_\mu \;=\; A\,\Phi_\mu + F(\Phi) + 2\mu \begin{pmatrix} 0 \\ 0 \\ <\bar{\Phi}, \Phi_\mu> \end{pmatrix}, \quad (6)$$

where $\bar{\Phi} = (a\phi_1, b\phi_2, c\phi_3)$, and $< .,. >$ denotes the inner product of vectors. Computing the $n-$th derivative of Φ with respect to μ, for $n \geq 2$, we obtain

$$\begin{aligned} \Phi_{\mu^n}^{'} &= [A + \mu DF(\Phi)]\,\Phi_{\mu^n} + \sum_{k=0}^{n-2} \Big[n \begin{pmatrix} cn-2 \\ k \end{pmatrix} DF(\Phi_{\mu^k}) \\ &\quad + \mu \begin{pmatrix} cn-1 \\ k+1 \end{pmatrix} DF(\Phi_{\mu^{k+1}}) \Big] \Phi_{\mu^{n-k-1}}. \end{aligned}$$

This implies

$$\begin{aligned} \Phi_{\mu^n}^{'}(t, \zeta, 0) &= A\Phi_{\mu^n}, (t, \zeta, 0) \\ &\quad + \sum_{k=0}^{n-2} n \begin{pmatrix} cn-2 \\ k \end{pmatrix} DF(\Phi_{\mu^k}(t, \zeta, 0))\Phi_{\mu^{n-k-1}}(t, \zeta, 0), \quad n \geq 2. \end{aligned}$$

Therefore, by the constant formula, we have

$$\Phi_{\mu^n}(t, \zeta, 0) = \chi(t) \int_0^t \chi^{-1}(s) \sum_{k=0}^{n-2} n \begin{pmatrix} cn-2 \\ k \end{pmatrix} DF(\Phi_{\mu^k}(s, \zeta, 0))\Phi_{\mu^{n-k-1}}(s, \zeta, 0)\, ds \quad (7)$$

$$= \sum_{k=0}^{n-2} 2n \begin{pmatrix} cn-2 \\ k \end{pmatrix} \chi(t) \int_0^t <\bar{\Phi}_{\mu^k}, \Phi_{\mu^{n-k-1}}> \begin{pmatrix} c(1-\cos s) \\ -\sin s \\ \cos s \end{pmatrix} ds. \quad (8)$$

The above equation helps us to compute iterative derivatives of Φ with respect to μ. For the first step, let $\Phi_\mu(2\pi, \zeta, 0) = (f_1, f_2, f_3)$. It can be checked from (6) that

$$\begin{aligned} f_1 &= \pi[(b+c)(y^2 + z^2) + a(2x^2 + y^2 + 5z^2 + 6xy)], \\ f_2 &= -\pi[2ay(x+z)], \\ f_3 &= -2a\pi[z(x+z)], \end{aligned} \quad (9)$$

where, $\zeta = (x, y, z) \in R^3$. The solutions of the equation $\Phi_\mu(2\pi, \zeta, 0) = 0$ can be find with respect to the parameters a, b, c and variables x, y, z. As we will see later, under some conditions on the parameters a, b, c, the solutions of

$\Phi_\mu(2\pi, \zeta, 0) = 0$ are simple. The next theorem is probably well known, but we write it to have continuation of theory.

Theorem 2.1. *If ζ_0 is a simple zero of the main distance variation function, then for $|\mu|$ small enough, the displacement function $d(\zeta, \mu)$ has a simple zero $\zeta_\mu = \zeta_0 + o(\mu)$.*

Proof. It is a direct application of the implicit function theorem. ∎

Now, we need to show some facts about the eigenvalues of the displacement function. Let $P(\zeta, \mu)$ be the Poincare map of (4) based on the 2π time flow and let $\lambda \in R$ be a real constant. By equation (5), we have $P(\zeta, \mu) - \zeta = \mu d(\zeta, \mu)$. Therefore, we can write

$$(DP(\zeta, \mu) - I) - \mu\lambda I = \mu(Dd(\zeta, \mu) - \lambda I),$$

where, D is the differential operator with respect to ζ. This shows that, for $\mu \neq 0$, λ is an eigenvalue for $Dd(\zeta, \mu)$, if and only if, $1 + \mu\lambda$ is an eigenvalue for $DP(\zeta, \mu)$. On the other hand, if λ_0 is an eigenvalue for $D\Phi_\mu(\zeta, 0)$, such that $\frac{\partial}{\partial\lambda}\det\,(D\Phi_\mu(\zeta, 0) - \lambda I)|_{\lambda=\lambda_0} \neq 0$, then for $|\mu|$ small enough, $d(\zeta, \mu)$ has an eigenvalue $\lambda = \lambda_0 + o(\mu)$. The next lemma shows the relationship between the eigenvalues for the main distance variation function and the displacement function.

Lemma 2.2. *Suppose that $\lambda_i(\mu)$ ($i = 1, 2, 3$) is an eigenvalues for $Dd(\zeta, \mu)$. Then, $\lambda_i(\mu)$ is smooth; furthermore, if $\lambda_i(\mu) = \lambda_{0i} + o(\mu)$ is the Taylor expansion of $\lambda_i(\mu)$, then, λ_{0i} is an eigenvalue for $D\Phi_\mu(\zeta, 0)$. Moreover, if λ_0 is an eigenvalue for $D\Phi_\mu(\zeta, 0)$, then, there exists an eigenvalue $\lambda_i(\mu)$ for $Dd(\zeta, \mu)$ such that $\lambda_i(0) = \lambda_0$.*

Proof. Because of the smoothness of the determinant function, the smoothness of $\lambda_i(\mu)$ is obvious. Now let $\lambda_i(\mu)$, $i = 1, 2, 3$, be the eigenvalues for $Dd(\zeta, \mu)$, then

$$\det\,(Dd(\zeta, \mu) - \lambda I) = (\lambda - \lambda_1(\mu))(\lambda - \lambda_2(\mu))(\lambda - \lambda_3(\mu)).$$

Therefore,

$$\det\,(D\Phi_\mu(\zeta, 0) - \lambda I) + o(\mu) = (\lambda - \lambda_{01})(\lambda - \lambda_{02})(\lambda - \lambda_{03}) + o(\mu).$$

This shows that λ_0 is an eigenvalue for $D\Phi_\mu(\zeta, 0)$ if and only if, for some $0 \leq i \leq 3$, $\lambda_0 = \lambda_{i0}$. This completes the proof. ∎

Corollary 2.3. *By the above lemma, λ_0 is an eigenvalue for $D\Phi_\mu(\zeta, 0)$, if and only if, there exists an eigenvalue $\lambda(\mu)$ for $DP(\zeta, \mu)$, such that $\lambda(\mu) = 1 + \mu\lambda_0 + o(\mu^2)$. Also, similar method yields the same relation between the iterative derivatives of the eigenvalue $\lambda(\mu)$ and the eigenvalues for the iterative derivatives of the displacement function.*

3. Periodic Solutions

Let us consider the equation (4); we recall from [15] that, if a, b, c are non-positive (non-negative), then the equation does not have any periodic solution or homoclinic orbit. The equation (4) shows that any periodic solution of the system intersects the plane $y = 0$ for at least two different points. This is because of the fact that, if a nontrivial periodic solution $\gamma(t) = (x(t), y(t), z(t))$ with period T, intersects the plane for at most one point $\gamma(t_0) = (x_0, 0, z_0)$, then for $t \geq t_0$, we have $y(t) > 0$ (or $y(t) < 0$, depending on the sign of z_0). Hence, $0 = x(T + t_0) - x(t_0) = \int_{t_0}^{T+t_0} y(t)dt > 0$, which is a contradiction. This shows that the plane $y = 0$ is an appropriate cross section for the Poincare map. Similar proof indicates that any periodic solution intersects the plane $z = 0$ for at least two different points. Let us consider the time flow for $t = 2\pi$ and define the Poincare map $P(\xi, \mu) = \Phi(2\pi, \zeta, \mu)$, where $\xi = (x, z)$ is a point of the plane $y = 0$. Then, we expand $P(\xi, \mu)$ for $\mu = 0$ and obtain the displacement function

$$d(\xi, \mu) = \Phi_\mu(2\pi, \zeta, 0) + \frac{\mu}{2}\Phi_{\mu^2}(2\pi, \zeta, 0) + \cdots + \frac{\mu^{(n-1)}}{n!}\Phi_{\mu^n}(2\pi, \zeta, 0) + o(\mu^n).$$

If, for ζ_0, $\Phi_\mu(2\pi, \zeta_0, 0) \neq 0$, then, for all $0 < |\mu|$ small enough, $d(\xi, \mu) \neq 0$. This means that the solution through ζ_0 is not periodic. The next theorem present conditions guaranteeing lack of periodic solutions. The theorem shows that, if $a + b + c \neq 0$, then, as $0 < |\mu|$ tends to zero, the periodic solutions of the system vanish; however, for $\mu = 0$, the system has only periodic solutions.

Theorem 3.1. *Consider the equation* (4) *and suppose that $\mathcal{A} \subset R^2$ is a compact subset of the plane $y = 0$ such that $\mathcal{A}$ does not contain any fixed points of the system. If $a+b+c \neq 0$ or $a = 0$ then, there exists $\mu_0 \geq 0$ such that, for $|\mu| < \mu_0$, the equation* (4) *does not have any periodic solution through $\mathcal{A}$.*

Proof. We consider the system in two cases.

Case one, $a \neq 0$ **and** $a + b + c \neq 0$**:** In this case, the only fixed point of the system is the origin, so $\mathcal{A}$ is a compact set which does not contain the origin. If $f_3(x, 0, z) = 0$ then, $z = 0$ or $z = -x$. In the first case (i.e. $z = 0$), $f_1(x, 0, 0) = 2a\pi x^2$. In the second case (i.e. $z = -x$), $f_1(x, 0, -x) = (a + b + c)\pi x^2$. This implies that the only solution of $\Phi_\mu(2\pi, \zeta, 0) = 0$ is $\zeta = 0$. Hence, there exists $M > 0$, such that, for any $\zeta \in \mathcal{A}$, $\|\Phi_\mu(2\pi, \zeta, 0)\| > M$. Moreover, for $\zeta \in \mathcal{A}$, there exists an open neighborhood V_ζ containing ζ and $0 < \mu_\zeta$ such that for each $p \in V_\zeta$ and $0 < |\mu| < \mu_\zeta$, we have $d(p, \mu) \neq 0$. Therefore, the orbit of p is not a periodic orbit. Suppose that $\mathcal{A} \subset V_{\zeta_1} \cup ... \cup V_{\zeta_k}$ and $0 < \mu_0 = \min\{\mu_{\zeta_1}, ..., \mu_{\zeta_k}\}$, then, for $0 < |\mu| < \mu_0$ and $\zeta \in \mathcal{A}$, $d(\xi, \mu) \neq 0$, hence, the orbit of ζ is not a periodic orbit.

Case two, $a = 0$**:** In this case, any point on the $x-$axis is a fixed point for the system, so $\mathcal{A}$ does not have any intersection with the $x-$axis. First let $b + c \neq 0$. It is easy to see from (9) that for $\zeta \in \mathcal{A}$, $\phi_{1\mu}(2\pi, \zeta, 0) = 2(b + c)z^2$ and $\phi_{2\mu}(2\pi, \zeta, 0) = \phi_{3\mu}(2\pi, \zeta, 0) = 0$. This means that, there exists $M > 0$ such that for any $\zeta \in \mathcal{A}$, $||\Phi_\mu(2\pi, \zeta, 0)|| > M$. Similar method, like the case one, proves the existence of μ_0. Now, suppose that $b + c = 0$ (obviously $b, c \neq 0$), then $\Phi_\mu(2\pi, \zeta, 0) = 0$, however, $\Phi_{\mu^2}(2\pi, \zeta, 0) = (0, -\frac{2}{3}\pi c^2 z^3, 0)$, which shows that restriction of $\Phi_{\mu^2}(t, \zeta, 0)$ to $\mathcal{A}$ is always nonzero. The same proof, like what we did for the case one, completes the proof. ■

Now, this question arises that what happens, if the condition of Theorem 3.1 is reversed or if $|\mu|$ is not small enough in the sense of Theorem 3.1. Note that Theorem 3.1 shows that, if $|\mu|$ is arbitrarily small, the necessary condition for existence of periodic solution is $a \neq 0$ and $a + b + c = 0$. So to find periodic solutions for the system, we have to consider $a = -(b + c) \neq 0$. Before that, assume that $h : R^3 \to R$ is a function and $X \in R^3$ is a point such that $\Phi_\mu(t, X, 0) = h(a, b, c)V_1(t, X)$, where, $V_1 : R \times R^3 \to R^3$ is a smooth map. Using (7), for any $k \geq 2$, we have $\Phi_{\mu^k}(t, X, 0) = h(a, b, c)V_k(t, X)$, where $V_k : R \times R^3 \to R^3$ is a smooth map. We will use this property in the following theorem and prove the existence of periodic solutions for the system.

Theorem 3.2. *Consider the equation* (4) *with* $b = a + c = 0$. *Also, suppose that* $\mathcal{U} \subset R^3$ *is an open set containing the origin. Let* $0 < \mu_0$ *be such that for* $|\mu| < \mu_0$ *and* $\zeta \in \mathcal{U}$, $\Phi(2\pi, \zeta, \mu)$ *is an analytic function with respect to the parameter* μ. *Then, any solution through* $(x, y, -x) \in \mathcal{U}$ *is periodic with period* 2π.

Proof. If $a = 0$ then $c = 0$; therefore, the equation (4) has only periodic solutions. Assume that $a \neq 0$. If $\zeta = (x, y, -x) \in \mathcal{U}$ and $b = 0$, then equation (7) yields that

$$\Phi_\mu(t, \zeta, 0) = (a + c)V_1(\zeta, t),$$

where,

$$V_1(\zeta, t) =$$
$$= \begin{pmatrix} -\frac{1}{12}\left[x^2(6t-4\sin t-\sin 2t)+y^2(6t-8\sin t+\sin 2t)+xy(6-8\cos t+2\cos 2t)\right] \\ \frac{2}{3}\left[x^2(2+\cos t)+y^2(1-\cos T)+2xy\sin t\right]\sin^2\frac{t}{2} \\ \frac{1}{3}\left[x^2(2\cos\frac{t}{2}+\cos\frac{3t}{2})+y^2(2\cos\frac{t}{2}-\cos\frac{3t}{2}) = 2xy\sin\frac{3t}{2}\right]\sin\frac{t}{2} \end{pmatrix}.$$

This implies that, for any integer $k \geq 0$,

$$\Phi_{\mu^k}(t, \zeta, 0) = (a + c)V_k(x, t).$$

Since for $|\mu| < \mu_0$, $\Phi(2\pi, \zeta, \mu)$ is analytic, so we can expand it for $\mu = 0$ and obtain

$$d(\zeta, \mu) = (a + c)\Big[V_1(\zeta, 2\pi) + \frac{\mu}{2}V_2(\zeta, 2\pi) + \cdots + \frac{\mu^n}{(n+1)!}V_{n+1}(\zeta, 2\pi) + \cdots\Big].$$

Therefore, if $a + c = 0$, then $d(\zeta, \mu) \equiv 0$, i.e. any solution of (4) through $\zeta = (x, y, -x) \in \mathcal{U}$ is a 2π periodic solution. ■

The proof of the above theorem shows that if $\Phi(2\pi, \zeta, \mu)$ is analytic, then the value of μ does not effect on the periodic solutions, i.e. even for big values of $|\mu|$, the system has periodic solutions through the plane $x + z = 0$.

Now, we turn to the case that $|\mu|$ is not small. If $\mu = 0$, then any solution through $\zeta = (x, 0, z)$ returns to the plane $y = 0$ for $t = 2\pi$. In fact, the first return time to the plane is $T = 2\pi$. Let $\mathcal{A} \subset R^2$ be a bounded subset of the plane $y = 0$. If $0 < |\mu|$ is sufficiently small, then by the continuity theorem of solutions [13, 11], any solution through $\mathcal{A}$ intersects the plane $y = 0$ for a return time $T(\mu, x, z) = 2\pi + o(\mu)$, however, as we will see later, this return time may be non-equal to 2π. Let, for $\xi = (x, z)$, $T(\xi, \mu)$ denote the first return time. Then, the Poincare map has the form.

$$
\begin{aligned}
&P(\xi,\mu) = \Phi(T(\xi,\mu),(x,0,z),\mu)\\
=&(x,0,z) + \mu\big(T_\mu(\xi,0)\dot{\Phi}(2\pi,(x,0,z),0) + \Phi_\mu(2\pi,(x,0,z),0)\big) + \cdots + o(\mu^{n+1}).
\end{aligned}
$$

Therefore, we have the displacement function defined on the plane $y = 0$ with the form

$$
d(\xi,\mu) = \begin{pmatrix} \phi_{1\mu}(2\pi,\xi,0) \\ zT_\mu(\xi,0) \\ \phi_{3\mu}(2\pi,\xi,0) \end{pmatrix} + \cdots + o(\mu^n),
$$

where the maps $\phi_{i\mu}(2\pi, \xi, 0) = f_i(x, 0, z)$, $i = 1, 2, 3$, are given by (9), furthermore, $T_\mu(\xi, 0) = 0$. Using the fact that $\phi_2(T(\mu, \xi), \xi, \mu) = 0$, we can compute the iterative derivatives of $T(\mu)$ and find

$$
T(\mu) = 2\pi + \frac{\mu^2}{2}\Big(\frac{-\phi_{2\mu^2}}{z}\Big) + \frac{\mu^3}{6}\Big(\frac{-3\phi_{3\mu}\phi_{2\mu^2} + z\phi_{2\mu^3}}{z^2}\Big) + o(\mu^4), \tag{10}
$$

where, all partial derivatives are computed for $\mu = 0$, $t = 2\pi$ and $\zeta = (x, 0, z)$. It is easy to check from (7) that if $b = 0$ then, for $\mu = 0$, all higher order derivatives of $T(\mu)$ are equal to zero. This means that, if T is analytic, then for $b = 0$, the first return time from the plane $y = 0$ to itself is $T = 2\pi$. Hence again, we have the situation of Theorem 3.2. However by assuming $b \neq 0$, then, for $|\mu| \neq 0$ small enough, we have $T(\mu) \neq 2\pi$. In this case, using (10), we can find the second and third terms of the displacement function $d(\xi, \mu)$. After computation and simplification, we obtain

$$
d(\xi,\mu) = \begin{pmatrix} d_1(\xi,\mu) \\ d_2(\xi,\mu) \end{pmatrix} \tag{11}
$$

$$
= \underbrace{\begin{pmatrix} \phi_{1\mu} \\ \phi_{3\mu} \end{pmatrix}}_{L(\xi)} + \frac{\mu}{2}\begin{pmatrix} \phi_{1\mu^2} \\ \phi_{3\mu^2} \end{pmatrix} + \frac{\mu^2}{6}\begin{pmatrix} -\dfrac{-2\phi_{2\mu}\phi_{2\mu^2}}{z} + \phi_{1\mu^3} \\ \phi_{3\mu^3} - 3cz\phi_{2\mu^2} + \dfrac{3(-ax^2+\phi_{2\mu})\phi_{2\mu^2}}{z} \end{pmatrix} + o(\mu^3), \tag{12}
$$

also, because it is always equal to zero, the second row is omitted by the definition of the Poincare map. If $L(\xi_0) \neq 0$, then for $|\mu|$ small enough, $d(\xi_0, \mu) \neq 0$. The next proposition proves the existence of periodic solutions for the system.

Proposition 3.3. *Consider the equation* (4) *with* $b(b + c) \neq 0$. *Then, for any* $|\mu|$ *sufficiently small, there exist* $a = -(b + c) + o(\mu)$ *and* $z = -x + o(\mu)$ *such that the solution through* $\zeta = (x, 0, z)$ *is periodic.*

Proof. Suppose that $b+c, x \neq 0$. We define the map

$$d^x : R^3 \to R^2$$
$$(z,a,\mu) \mapsto d(x,z,\mu) = L(x,z) + o(\mu).$$

Then, $d^x(-x,-(b+c),0) = 0$, but

$$\det \left[D_{(z,a)} d^x(-x,-(b+c),0)\right] = 2(b+c)\pi^2 x^3 \neq 0.$$

Therefore, by the implicit function theorem, for $|\mu| > 0$ small enough, there exist $z = -x + o(\mu)$ and $a = -(b+c) + o(\mu)$ such that $d(x, z(\mu), \mu) = 0$. This completes the proof. ∎

It is easy to check from (5) that $a(\mu) = -(b+c)+o(\mu)$ and $z(\mu) = -x+o(\mu)$, $d(x, z(\mu), \mu) = 0$, hence,

$$D_{(z,a)} d(\xi,\mu) \begin{pmatrix} z_\mu(\mu) \\ a_\mu(\mu) \end{pmatrix} + \frac{\partial d}{\partial \mu}(\xi,\mu) = 0. \tag{13}$$

Therefore,

$$\begin{aligned} \begin{pmatrix} z_\mu(0) \\ a_\mu(0) \end{pmatrix} &= -\left[D_{(z,a)} d(\xi,0)\right]^{-1} \frac{\partial d}{\partial \mu}(x,-x,0) \\ &= \left[D_{(z,a)} L(x,z)\right]^{-1} \begin{pmatrix} \phi_{1\mu^2} \\ \phi_{3\mu^2} \end{pmatrix}, \end{aligned}$$

where, the functions are computed for $\zeta = (x, 0-x)$, $t = 2\pi$, $a = -(b+c)$ and $\mu = 0$. After simplification, we find that $z_\mu(0) = a_\mu(0) = 0$. Again, differentiating (13), for $\mu = 0$, we find that

$$\begin{pmatrix} z_{\mu^2}(0) \\ a_{\mu^2}(0) \end{pmatrix} = -\frac{1}{3}[D_{(z,a)} L(x,-x)]^{-1} \begin{pmatrix} -\frac{-2\phi_{2\mu}\phi_{2\mu^2}}{z} + \phi_{1\mu^3} \\ \phi_{3\mu^3} - 3cz\phi_{2\mu^2} + \frac{3(-ax^2+\phi_{2\mu})\phi_{2\mu^2}}{z} \end{pmatrix}.$$

After simplification, finally we have,

$$z(\mu) = -x - b(b+2c)\frac{x^2\mu^2}{3} + o(\mu^3), \quad a(\mu) = -(b+c) - b(b^2+5bc+4c^2)\frac{x^2\mu^2}{4} + o(\mu^3). \tag{14}$$

We need the above equations to compute the eigenvalues for the corresponding fixed point. The eigenvalues are computed by

$$\begin{aligned} &\det\left[Dd(x,z(\mu),\mu) - \lambda(\mu) I_{2\times 2}\right] \\ &= \det \begin{pmatrix} d_{1x}(x,z(\mu),\mu) - \lambda(\mu) & d_{1z}(x,z(\mu),\mu) \\ d_{2x}(x,z(\mu),\mu) & d_{2z}(x,z(\mu),\mu) - \lambda(\mu) \end{pmatrix} = 0, \end{aligned}$$

or equivalently,

$$\lambda^2(\mu) - \left[d_{1x}(x,z(\mu),\mu) + d_{2z}(x,z(\mu),\mu)\right]\lambda(\mu) + \det\left[Dd(x,z(\mu),\mu)\right] = 0.$$

We define

$$U(\lambda,\mu) = \lambda^2(\mu) - \left[d_{1x}(x,z(\mu),\mu) + d_{2z}(x,z(\mu),\mu)\right]\lambda(\mu) + \det\left[Dd(x,z(\mu),\mu)\right],$$

then, we have $U(\lambda(\mu),\mu)=0$. Expanding the map $U(\lambda(\mu),\mu)$ for $\mu=0$, we get
$U(\lambda(0),0)+\mu\big[U_\lambda\lambda^{'}(0)+U_\mu\big]+\frac{\mu^2}{2}\big[U_{\lambda^2}\lambda^{'^2}(0)+2U_{\lambda\mu}\lambda^{'}(0)+U_{\mu^2}+U_\lambda\lambda^{''}(0)\big]+$
$o(\mu^3)$.

Using (5) and (13), finally we obtain

$$\lambda(0)+\frac{\mu^2}{2}\big[2\lambda^{'^2}(0)-2b(b+c)^2(b+4c)\pi^2x^4\big]+o(\mu^3)=0.$$

Therefore, by Lemma 2.2, the eigenvalues for the corresponding fixed point are given by

$$\lambda_{1,2}=1\pm\mu^2\left[\pi x^2(b+c)\sqrt{b(b+4c)}\right]+o(\mu^3).$$

This implies that, for $b(b+4c)<0$ and $|\mu|$ small enough, the eigenvalues have nonzero imaginary part and their modules are greater that 1. Hence, the periodic solution is a hyperbolic repelling orbit; but, for $b(b+4c)>0$ and $|\mu|$ small enough, the periodic solution is a hyperbolic saddle orbit. This means that for $b=-4c$ a global bifurcation occurs for the system. The next theorem summarizes the results

Theorem 3.4. *Consider the equation* (4) *with* $b(b+c)\neq 0$. *Then, for each* $x\neq 0$ *and* $|\mu|$ *small enough, there exist* $z(\mu)$ *and* $a(\mu)$ *(given by* (14)*) such that the orbit through* $\zeta(\mu)=(x,0,z(\mu))$ *is periodic. Furthermore, if* $b(b+4c)<0$ *then, the periodic orbit is a hyperbolic repelling orbit, and if* $b(b+4c)<0$ *then, the orbit is a hyperbolic saddle orbit. This means that, for* $b=-4c$ *a global bifurcation occurs for the system.*

Proof. It is obvious by Proposition 3.3 and the above discussion. ∎

4. Conclusion

In this paper we considered the effect of the irreducible quadratic terms on the three dimensional linear oscillator, i.e.

$$\dddot{x}+\omega^2\dot{x}=Ax^2+B\dot{x}^2+C\ddot{x}^2, \tag{15}$$

and obtained conditions guaranteeing the existence (lack) of periodic solutions. At first, we changed the system to the simple equation $\dddot{x}+\dot{x}=\mu[ax^2+b\dot{x}^2+c\ddot{x}^2]$ and constructed the corresponding Poincare map. Then, we derived the results explained in Sec. 3. Now, we re-scale the equation to the first case and rewrite the results that we obtained. We have

$$\mu a=\frac{A}{\omega^2},\ \ \mu b=B,\ \ \mu c=\omega^2 C.$$

- Let $\mathcal{A}\subset R^2$ be a compact set in the plane $y=0$ which does not contain any fixed point of the system. By Theorem 3.1, for $|A|$, $0<|B|$ and $0<\omega^2|C|$ sufficiently small, if $\frac{A}{\omega^2}+B+\omega^2C\neq 0$ or $A\neq 0$, then the equation has no periodic solution through $\mathcal{A}$.

- If $B = 0$ and the flow of the system (15) is analytic, then, by Theorem 3.2, for $\frac{|A|}{\omega^2}$ and $\omega^2|C|$ small enough such that $A+\omega^4 C = 0$, any solution through $(\frac{x}{\omega}, \frac{y}{\omega^2}, -\omega^2 x)$ is $\frac{2\pi}{\omega^2}$ periodic.
- If $B(B + \omega^2 C) \neq 0$, then by Theorem 3.4, for any $x \neq 0$ and for $|B|$ and $\omega^2|C|$ small enough, there exist A near to $-\omega^2 B(B + \omega^2 C)$ and z near to $-\omega^2 x$ such that the orbit through $(\frac{x}{\omega}, 0, -\omega^2 x)$ is periodic. In this case, if $B(B + 4\omega^2 C) < 0$, then, the periodic orbit is a hyperbolic repelling orbit, and if $B(B+4\omega^2 C) > 0$, then, the periodic orbit is a hyperbolic saddle orbit. This means that for $B = -4\omega^2 C$ a global bifurcation occurs for the periodic solutions of the system.
- According to the previous item, if $BC < 0$, then, increasing the frequency of oscillation i.e. ω, global bifurcation occurs for the system at $\omega = \sqrt{-\frac{B}{4C}}$.

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