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Applying Fixed Point Theory to the Initial Value Problem for the Functional Differential Equations with Finite Delay

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Abstract. This paper is devoted to the study of the existence and uniqueness of strong solutions for the functional differential equations with finite delay. We also study the asymptotic stability of solutions and the existence of periodic solutions. Furthermore, under some suitable assumptions on the given functions, we prove that the solution set of the problem is nonempty, compact and connected. Our approach is based on the fixed point theory and the topological degree theory of compact vector fields.

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1. Introduction

In this paper, we consider the initial value problem for the following functional differential equations with finite delay

$$x'(t) + A(t)x(t) = g(t, x(t), x_t), \; t \geq 0, \tag{1.1}$$

$$x_0 = \varphi, \tag{1.2}$$

in which $A(t) = \text{diag}[a_1(t), a_2(t), ..., a_n(t)]$, $a_i \in BC[0, \infty)$ for all $i = 1, ..., n$, where $BC[0,\infty)$ denotes the Banach space of bounded continuous fuctions $x : [0,\infty) \to \mathbb{R}^n$.

The equation of the form (1.1) with finite or infinite delay has been studied by many authors using various techniques. There are many important results about the existence and uniqueness of solutions, the existence periodic solutions and the asymptotic behavior of the solutions; for example, we refer to the $[1, 3, 5, 7, 9, 10, 11]$ and references therein.

In $[1, 3]$, the authors used the notion of fundamental solutions to study the stability of the semi-linear retarded equation

$$x'(t) = A(t)x(t) + B(t)x_t + F(t, x_t)\,,\ t \geq s \geq 0,$$
$$x_s = \varphi \in C([-r,0],E) \text{ or } x_s = \varphi \in C_0((-\infty,0],E),$$

where $(A(t), D(A(t)))_{t\geq 0}$ generates the strongly continuous evolution family $(V(t,s))_{t\geq s\geq 0}$ on a Banach space E, and $(B(t))_{t\geq 0}$ is a family of bounded linear operators from $C([-r,0],E)$ or $C_0((-\infty,0],E)$ into E.

In $[9, 10]$, the authors studied the relationship between the bounded solutions and the periodic solutions of finite (or infinite) delay evolution equation in a general Banach space as follows

$$u'(t) + A(t)u(t) = f(t, u(t), u_t)\,,\ t > 0,$$
$$u(s) = \varphi(s), s \in [-r,0] \text{ (or } s \leq 0),$$

where $A(t)$ is a unbounded operator, f is a continuous function and $A(t), f(t,x,y)$ are T-periodic in t such that there exists a unique evolution system $U(t,s)$, $0 \leq s \leq t \leq T$, for the equation as above.

In [5], by using a Massera type criterion, the author proved the existence of a periodic solution for the partial neutral functional differential equation

$$\frac{d}{dt}(x(t) + G(t, x_t)) = Ax(t) + F(t, x_t),\ \ t > 0,$$
$$x_0 = \varphi \in D,$$

where A is the infinitesimal generator of a compact analytic semigroup of linear operators, $(T(t))_{t\geq 0}$, on a Banach space E. The history x_t, $x_t(\theta) = x(t+\theta)$, belongs to an appropriate phase D and $G, F : \mathbb{R} \times D \to E$ are continuous functions.

In [7], the existence of positive periodic solutions of the system of functional differential equations

$$x'(t) = A(t)x(t) + f(t, x_t),\ \ t \geq s \geq 0,$$

was established, in which $A(t) = \text{diag}[a_1(t), a_2(t), ..., a_n(t)]$, $a_j \in C(\mathbb{R},\mathbb{R})$ is ω-periodic. And recently in [11], the authors studied the existence and uniqueness of periodic solutions and the stability of the zero solution of the nonlinear neutral differential equation with functional delay

$$\frac{d}{dt}x(t) = -a(t)x(t) + \frac{d}{dt}Q(t, x(t-g(t))) + G(t, x(t), x(t-g(t))),$$

where $a(t)$ is a continuous real-valued function, the functions $Q : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ and $G : \mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are continuous. In the process, the authors used integrating factors and converted the given neutral differential equation into an equivalent integral equation. Then appropriate mappings were constructed and the Krasnoselskii's fixed point theorem was employed to show the existence of a periodic solution of that neutral differential equation.

In this paper, let us consider (1.1)–(1.2) with $A(t) = \text{diag}[a_1(t), a_2(t), ..., a_n(t)]$, $a_i \in BC[0, \infty)$, $i = 1, ..., n$. Then for each bounded continuous function f on $[0, \infty)$, there exists a unique strong solution $x(t)$ of the equation $Lx(t) := x'(t) + A(t)x(t) = f(t)$, with $x(0) = 0$. Here, the solution is differentiable and $x' \in BC[0, \infty)$. This implies that the problem (1.1)–(1.2) can be reduced to a fixed point problem, and hence, we can give the suitable assumptions in order to obtain the existence of strong solutions, periodic solutions... The paper has five sections. In Sec. 2, at first we present preliminaries. These results allow us to reduce (1.1)–(1.2) to the problem $x = Tx$, where T is completely continuous operator. It follows that under the other suitable assumptions, we get the existence and uniqueness of strong solutions. With the same conditions as in the previnus section, in Sec. 3, we show that the solutions are asymptotically stable and in Sec. 4, if in addition $g(t, ., .), a_i(t)$, $i = 1, ..., n$, are ω-periodic then basing on the paper [5], we get the existence of periodic solutions. Finally, in the Sec. 5, we shall consider the compactness and connectivity of the soltution set for the problem (1.1)–(1.2) corresponding to $g(t, \xi, \eta) = g_1(t)g_2(\xi, \eta)$.

Our results can not be deduced from previous works (to our knowledge) and our approach is based on the Schauder's fixed point theorem, the contraction mapping, and the following fixed point theorems.

Theorem 1.1. ([4]) *Let Y be a Banach space and $\Gamma = \Gamma_1 + y$, where $\Gamma_1 : Y \to Y$ is a bounded linear operator and $y \in Y$. If there exists $x_0 \in Y$ such that the set $\{\Gamma^n(x_0) : n \in N\}$ is relatively compact in Y, then Γ has a fixed point in Y.*

Theorem 1.2. ([5]) *Let X be a Banach space and M be a nonempty convex subset of X. If $\Gamma : M \to 2^X$ is a multivalued map such that*

(i) *For every $x \in M$, the set $\Gamma(x)$ is nonempty, convex and closed,*

(ii) *The set $\Gamma(M) = \bigcup_{x \in M} \Gamma(x)$ is relatively compact,*

(iii) *Γ is upper semi-continuous,*

then Γ has a fixed point in M.

Theorem 1.3. ([8]) *Let $(E, |.|)$ be a real Banach space, D be a bounded open set of E with boundary ∂D and closure $\overline{D}$ and $T : \overline{D} \to E$ be a completely continuous operator. Assume that T satisfies the following conditions:*

(i) *T has no fixed point on ∂D and $\gamma(I - T, D) \neq 0$.*

(ii) *For each $\varepsilon > 0$, there is a completely continuous operator T_ε such that*

$$||T(x) - T_\varepsilon(x)|| < \varepsilon, \forall x \in \overline{D},$$

and such that for each h with $||h|| < \varepsilon$ the equation $x = T_\varepsilon(x) + h$ has at most one solution in $\overline{D}$. Then the set $N(I - T, D)$ of all solutions to the equation $x = T(x)$ in D is nonempty, connected, and compact.

The proof of Theorem 1.3 is given in [8, Theorem 48.2]. We note more that, the condition (i) is equivalent to the following condition:
$(\tilde{i})$ *T has no fixed point on* ∂D *and* $\deg(I - T, D, 0) \neq 0$,
because, if a completely continuous operator T is defined on $\overline{D}$ and has no fixed point on ∂D, then the rotation $\gamma(I - T, D)$ coincides with the Leray - Schauder degree of $I - T$ on D with respect to the origin, (see [8, Sec. 20.2]).

We need more the theorem in Sec. 5. The proof of the following theorem, needed in Sec. 5, can be found [2, Ch. 2].

Theorem 1.4. ([2]) (The locally Lipschitz approximation) *Let* E, F *be Banach space,* D *be an open subset of* E *and* $f : D \to F$ *be continuous. Then for each* $\varepsilon > 0$, *there is a mapping* $f_\varepsilon : D \to F$ *that is locally Lischitz such that* $|f(x) - f_\varepsilon(x)| < \varepsilon$, *for all* $x \in D$ *and* $f_\varepsilon(D) \subset cof(D)$, *where* $cof(D)$ *is the convex hull of* $f(D)$.

2. The Existence and the Uniqueness of Strong Solutions

Let $r > 0$ be a given real number. Denote by $\mathbb{R}^n$ the ordinary n-dimensional Euclidean with norm $|.|$ and let $C = C\big([-r, 0], \mathbb{R}^n\big)$ be the Banach space of all continuous functions on $[-r, 0]$ to $\mathbb{R}^n$ with the usual norm $||.||$. In what follows, for an interval $I \subset \mathbb{R}$, we will use $BC(I)$ to denote the Banach space of bounded continuous functions $x : I \to \mathbb{R}^n$, equipped with the norm

$$||x|| = \sup_{t \in I} \{|x(t)|\} = \sup_{t \in I} \{\sum_{i=1}^{n} |x_i(t)|\},$$

$BC^1[0, \infty)$ denote the space of functions $x \in BC[0, \infty)$ such that x is differentiable and $x' \in BC[0, \infty)$.

Let X_0 be the space of functions $x \in BC[-r, \infty)$ such that $x(t) = 0$ for all $t \in [-r, 0]$. This is a closed subspace of $BC[-r, \infty)$ and hence is a Banach space.

Finally, let X^1 denote the space of functions $x \in X_0$ such that the restriction of x to $[0, \infty)$ is in $BC^1[0, \infty)$. If $x \in BC[-r, \infty)$ then $x_t \in BC[-r, 0]$ for any $t \in [0, \infty)$ is defined by $x_t(\theta) = x(t + \theta),\ \theta \in [-r, 0]$.

We make the following assumptions.

(I.1) $A(t) = \mathrm{diag}[a_1(t), a_2(t), ..., a_n(t)], t \in [0, \infty)$ where $a_i \in BC[0, \infty)$, for all $i = 1, ..., n$.

(I.2) For all $i = 1, ..., n$, there exist constants $\widetilde{a}_i > 0$ such that $a_i(t) \geq \widetilde{a}_i$, $\forall t \in [0, \infty)$.

Now we define the operator $L : X^1 \to BC[0, \infty)$ by

$$Lx(t) = x'(t) + A(t)x(t),\ t \in [0, \infty).$$

Clearly, the operator L is bounded linear. On the other hand, we have the following lemma.

Lemma 2.1. *For each* $f \in BC[0, \infty)$, *the equation* $Lx = f$ *has a unique solution.*

Proof. For each $f \in BC[0,\infty)$, $f(t) = (f_1(t), f_2(t), ..., f_n(t))$, the equation $Lx = f$ is rewritten as follows:

$$\begin{cases} x_i(t) = 0, & t \in [-r, 0], \\ \dot{x}_i(t)) + a_i(t)x_i(t) = f_i(t), & t \in [0,\infty), \end{cases} \tag{2.1}$$

where $i = 1, 2, ..., n$.

Multiply both the sides of the equation (2.1) by $e^{\int_0^t a_i(\tau)d(\tau)}$ and then integrate from 0 to t we have:

$$x_i(t) = \int_0^t f_i(s) e^{-\int_s^t a_i(\tau)d(\tau)} ds, \ i = 1, 2, ..., n.$$

Clearly, for all $i = 1, 2, ..., n$, for all $t \geq 0$,

$$|x_i(t)| \leq \sup_{t\in[0,\infty)} |f_i(t)| \frac{1}{\widetilde{a}_i}(1 - e^{-\widetilde{a}_i t}) \leq \frac{1}{\widetilde{a}_i} \sup_{t\in[0,\infty)} |f_i(t)|. \tag{2.2}$$

This implies that the equation $Lx = f$ has a unique solution $x(.) = (x_1(.), x_2(.), ..., x_n(.)) \in X^1$, where for $i = 1, 2, ..., n$,

$$x_i(t) = \begin{cases} 0, & t \in [-r, 0], \\ \int_0^t f_i(s) e^{-\int_s^t a_i(\tau)d(\tau)} ds, & t \in [0,\infty), \end{cases} \tag{2.3}$$

The Lemma 2.1 holds. ∎

Then L is invertible, with the inverse given by $L^{-1}f(t) = x(t) = (x_1(t), x_2(t), ..., x_n(t))$ as (2.3). Put

$$\widetilde{a} = \max\{\frac{1}{\widetilde{a}_1}, \frac{1}{\widetilde{a}_2}, ..., \frac{1}{\widetilde{a}_n}\}.$$

From (2.2), we get

$$||L^{-1}f|| \leq \widetilde{a}||f||.$$

We have the following theorem about the existence of solution. It is often called "the strong solution" or "the classical solution" of (1.1)–(1.2) (i.e. have derivatives).

Theorem 2.2. *Let* (I.1)–(I.2) *hold and* $g : [0,\infty) \times \mathbb{R}^n \times C \to \mathbb{R}^n$ *satisfy the following conditions:*

(G_1) *g is continuous,*

(G_2) *For each v_0 belonging to any bounded subset Ω of $BC[-r,\infty)$, for all $\epsilon > 0$, there exists $\delta = \delta(\epsilon, v_0) > 0$ such that for all $v \in \Omega$*

$$||v - v_0|| < \delta \Rightarrow |g(t, v(t), v_t) - g(t, v_0(t), (v_0)_t)| < \epsilon,$$

for all $t \in [0,\infty)$,

(G_3) *There exist positive constants $\widetilde{C}_1, \widetilde{C}_2$ with $\widetilde{C}_1 < \frac{1}{2\widetilde{a}}$, such that*

$$|g(t,\xi,\eta)| \leq \widetilde{C}_1\big(|\xi| + ||\eta||\big) + \widetilde{C}_2, \ \forall (t,\xi,\eta) \in [0,\infty) \times \mathbb{R}^n \times C.$$

Then, for every $\varphi \in C$, the problem (1.1)–(1.2) has a solution $x \in BC[-r, \infty)$ and the restriction of x to $[0, \infty)$ belongs to $BC^1[0, \infty)$.
If, in addition that g is locally Lipschitzian in the second and the third variables, then the solution is unique.

Proof. The existence.

Step 1. Consider first the case $\varphi = 0$.

For each $v \in X_0$, put $f(t) = g(t, v(t), v_t), \forall t \in [0, \infty)$, then we have $f \in BC[0, \infty)$. Hence, the following operator is defined

$$\begin{aligned} &F : X_0 \to BC[0, \infty) \\ &v \in X_0 \mapsto F(v)(.) = f(.) \in BC[0, \infty). \end{aligned}$$

Consider the operator $T = L^{-1}F$. We note that $x \in X_0$ is a solution of the problem (1.1)–(1.2) if only if x is a fixed point of T in $X_1 \subset X_0$. Suppose $x \in X_0$ is a solution of the problem (1.1)–(1.2). Then for $t \geq 0$,

$$x'(t) + A(t)x(t) = g(t, x(t), x_t) \Leftrightarrow Lx(t) = Fx(t) \Leftrightarrow x(t) = L^{-1}F(t)x(t).$$

It means $x = Tx$. Conversely, if $x \in X_0$ and $x = Tx = L^{-1}F(x)$, then $x \in X^1$ and $x'(t) + A(t)x(t) = g(t, x(t), x_t)$. So, we shall show that T has a fixed point $v \in X_1 \subset X^0$.

Choose

$$M_2 > \frac{\widetilde{a}\widetilde{C}_2}{1 - 2\widetilde{a}\widetilde{C}_1}, \tag{2.4}$$

and put

$$D = \{v \in X_0 : ||v|| < M_2\}. \tag{2.5}$$

It is obvious that D is a bounded open, convex subset of X_0 and

$$\overline{D} = D \cup \partial D = \{v \in X_0 : ||v|| \leq M_2\}. \tag{2.6}$$

At first, we see that $T = L^{-1}F : \overline{D} \to X^1 \subset X_0$ is continuous and $T(\overline{D}) \subset D$. Indeed, For each $v_0 \in \overline{D}$, for all $\epsilon > 0$, it follows from (G_2) that there exists $\delta > 0$ such that for all $v \in \overline{D}$,

$$||v - v_0|| < \delta \Rightarrow |(F(v) - F(v_0))(t)| = |g(t, v(t), v_t) - g(t, v_0(t), (v_0)_t)| < \frac{\epsilon}{\widetilde{a}},$$

for all $t \in [0, \infty)$. Then $||F(v) - F(v_0)|| \leq \frac{\epsilon}{\widetilde{a}}$, and so

$$||T(v) - T(v_0)|| = ||L^{-1}(F(v) - F(v_0))|| \leq \widetilde{a}||F(v) - F(v_0)|| < \epsilon.$$

For any $v \in \overline{D}$, for all $t \in [0, \infty)$ we have:

$$|Fv(t)| \leq \widetilde{C}_1\big(|v(t)| + ||v_t||\big) + \widetilde{C}_2 \leq 2\widetilde{C}_1 M_2 + \widetilde{C}_2,$$

so

$$||Tv|| = ||L^{-1}(Fv)|| \leq \widetilde{a}||Fv|| \leq \widetilde{a}(2\widetilde{C}_1 M_2 + \widetilde{C}_2) < M_2.$$

Next, we show that $T(\overline{D})$ is relatively compact.

Since $T(\overline{D}) \subset D$, we only need show that $T(\overline{D})$ is equicontinuous. For all $\epsilon > 0$, for any $x \in T(\overline{D})$, for all $t_1, t_2 \in \mathbb{R}$, we consider 3 cases.

The case 1: $t_1, t_2 \in [0, \infty)$.

Since $x \in X^1$ and hence the restriction of x to $[0, \infty)$ is in $BC^1[0, \infty)$, it implies that

$$|x(t_1) - x(t_2)| = |x'(t)||t_1 - t_2|,$$

where $t \in (t_1, t_2)$ *or* $t \in (t_2, t_1)$.

On the other hand, $x \in T(\overline{D})$, ie., $x = Tv = L^{-1}(F(v)) \Leftrightarrow Lx = Fv$ for some $v \in \overline{D}$, it implies that

$$|x'(t)| = |-A(t)x(t) + Fv(t)| \leq aM_2 + 2\widetilde{C}_1 M_2 + \widetilde{C}_2,$$

where $a = \max\{||a_1||, ||a_2||, ..., ||a_n||\}$.

If we choose δ' such that $0 < \delta' < \dfrac{\epsilon}{M_2(a + 2\tilde{C}_1) + \tilde{C}_2}$ then

$$|t_1 - t_2| < \delta' \Rightarrow |x(t_1) - x(t_2)| = |x'(t)||t_1 - t_2| < \epsilon.$$

The case 2: $t_1, t_2 \in [-r, 0]$. It follows from x(t) $= 0$ for all $t \in [-r, 0]$, that

$$|x(t_1) - x(t_2)| < \epsilon.$$

The case 3: $t_1 \in [-r, 0), t_2 \in [0, \infty)$. By

$$|x(t_1) - x(t_2)| \leq |x(t_1) - x(0)| + |x(0) - x(t_2)| \leq |x(0) - x(t_2)|,$$

the case 3 is reduced to the case 1. We conclude that $T(\overline{D})$ is equicontinuous and then is relatively compact by the Arzela–Ascoli theorem.

By applying the Schauder theorem, T has a fixed point $v \in \overline{D}$ (not in ∂D), that is also a solution of the problem (1.1)–(1.2) on $[-r, \infty)$. Clearly, the restriction of v to $[0, \infty)$ belongs to $BC^1[0, \infty)$.

Step 2. Consider the case $\varphi \neq 0$.

We define the function $\overline{\varphi} : [-r, \infty) \to \mathbb{R}^n$, that is an extension of φ, as follows.

$$\overline{\varphi}(t) = \begin{cases} \overline{\varphi}(t), & t \in [-r, 0], \\ \overline{\varphi}(0), & t \in [0, \infty). \end{cases} \tag{2.7}$$

Then $\overline{\varphi} \in BC[-r, \infty)$. We note that, for each $x \in BC[-r, \infty)$, for any $t \geq 0$,

$$\big(x - \overline{\varphi}\big)_t(\theta) = x_t(\theta) - \overline{\varphi}_t(\theta), \forall \theta \in [-r, 0],$$

it means that, $\big(x - \overline{\varphi}\big)_t = x_t - \overline{\varphi}_t$. So, by the transformation $y = x - \overline{\varphi}$, the problem (1.1)–(1.2) is rewritten as follows

$$\begin{cases} y'(t) + A(t)y(t) = g\big(t, y(t) + \varphi(0), y_t + \overline{\varphi}_t\big) - A(t)\varphi(0)\,, \ t \geq 0, \\ y_0 = 0, \end{cases} \tag{2.8}$$

Thus, if we define $h : [0,\infty) \times \mathbb{R}^n \times C \to \mathbb{R}^n$ by

$$h(t,\xi,\eta) = g\big(t, \xi + \varphi(0), \eta + \overline{\varphi}_t\big) - A(t)\varphi(0), \tag{2.9}$$

then we can also rewrite (2.8) as

$$\begin{cases} y'(t) + A(t)y(t) = h\big(t, y(t), y_t\big)\,,\ t \geq 0, \\ y_0 = 0. \end{cases} \tag{2.10}$$

We shall consider the properties of h. It is obvious that h is continuous. For each v_0 in any bounded subset Ω of $BC[-r,\infty)$, then $(v_0 + \overline{\varphi})$ also belongs to a bounded subset of $BC[-r,\infty)$. It implies that for all $\epsilon > 0$, there exists $\delta = \delta(\epsilon, v_0, \overline{\varphi}) > 0$ such that for all $v \in \Omega$, if

$$||v - v_0|| = ||(v + \overline{\varphi}) - (v_0 + \overline{\varphi})|| < \delta,$$

then we have

$$|g(t, (v + \overline{\varphi})(t), v_t + \overline{\varphi}_t) - g(t, (v_0 + \overline{\varphi})(t), (v_0)_t + \overline{\varphi}_t)| < \epsilon,$$

or

$$|h(t, v(t), v_t) - h(t, v_0(t), (v_0)_t)| < \epsilon.$$

For all $(t,\xi,\eta) \in [0,\infty) \times \mathbb{R}^n \times C$, we have

$$\begin{aligned} |h(t,\xi,\eta)| &\leq \widetilde{C}_1\big(|\xi| + |\varphi(0)| + ||\eta|| + ||\overline{\varphi}_t||\big) + \widetilde{C}_2 + |A(t)\varphi(0)| \\ &\leq \widetilde{C}_1\big(|\xi| + ||\eta||\big) + \widetilde{C}_1|\varphi(0)| + \widetilde{C}_1||\overline{\varphi}_t|| + \widetilde{C}_2 + |A(t)\varphi(0)| \\ &\leq \widetilde{C}_1\big(|\xi| + ||\eta||\big) + \widetilde{C}_3, \end{aligned}$$

where $\widetilde{C}_3 = \widetilde{C}_3(\varphi) = \widetilde{C}_1|\varphi(0)| + \widetilde{C}_1||\varphi|| + \widetilde{C}_2 + a|\varphi(0)|$ is a positive constant. By the step 1, we obtain that the problem (2.10) has a solution y on $[-r,\infty)$. This implies that the problem (1.1)–(1.2) has a solution $x = y + \overline{\varphi}$ on $[-r,\infty)$ and the restriction of x to $[0,\infty)$ also belongs to $BC^1[0,\infty)$.

Thus the existence part is proved.

The uniqueness. Now, let $g : [0,\infty) \times C \times \mathbb{R}^n \to \mathbb{R}^n$ be locally Lipschitzian with respect to the second and the third variables, we show that the solution is unique.

Indeed, Suppose that $\bar{x}, \bar{y}$ are the solutions of the problem (1.1)–(1.2). We have to prove that for all $n \in \mathbb{N}$,

$$\bar{x}(t) = \bar{y}(t),\ \forall t \in [-r, n]. \tag{2.11}$$

Clearly

$$\bar{x}(t) = \bar{y}(t) = \varphi(t), \forall t \in [-r, 0].$$

Let

$$b = \max\left\{\alpha \in \mathbb{R} : \bar{x}(t) = \bar{y}(t), t \in [-r, \alpha]\right\}. \tag{2.12}$$

Clearly, $0 \leq b \leq n$. We need to show that $b = n$.

We suppose by contradiction that $b < n$. Since g is locally Lipschitzian in the second and the third variables, there exists $\rho > 0$ such that g is Lipschitzian with lipschitzian constant m in $[0,n] \times B_{1,\rho} \times B_{2,\rho}$, where

$$B_{1,\rho} = \{w \in \mathbb{R}^n : |w - \bar{x}(b)| < \rho\},$$

$$B_{2,\rho} = \{z \in C : ||z - \bar{x}_b|| < \rho\}.$$

Since $\bar{x}, \bar{y}$ are continuous, there exists $\sigma_1 > 0$ such that $b+\sigma_1 \leq n$ and $\bar{x}(s), \bar{y}(s) \in B_{1,\rho}$ for all $s \in [b, b+\sigma_1]$.

We note that, for each fixed $\bar{u} \in C([-r,n],\mathbb{R}^n)$, the mapping is defined by

$$s \in [0,n] \mapsto \bar{u}_s \in C, \text{ where } \bar{u}_s(\theta) = \bar{u}(s+\theta),\ \theta \in [-r,0],$$

is continuous.
Indeed, Since $\bar{u} \in C([-r,n],\mathbb{R}^n)$, $\bar{u}$ is uniformly continuous on $[-r,n]$. This implies that, for all $\varepsilon > 0$, there exists $\hat{\delta} > 0$ such that for each $\hat{s}_1, \hat{s}_2 \in [-r,n]$,

$$|\hat{s}_1 - \hat{s}_2| < \hat{\delta} \Rightarrow |\bar{u}(\hat{s}_1) - \bar{u}(\hat{s}_2)| < \varepsilon.$$

Consequently, for all $s_1, s_2 \in [0,n]$, for all $\theta \in [-r,0]$, we have

$$|s_1 - s_2| < \hat{\delta} \Rightarrow |(s_1+\theta) - (s_2+\theta)| < \hat{\delta} \Rightarrow |\bar{u}(s_1+\theta) - \bar{u}(s_2+\theta)| < \varepsilon.$$

It means that for all $\varepsilon > 0$, there exists $\hat{\delta} > 0$ such that for each $s_1, s_2 \in [0,n]$,

$$|s_1 - s_2| < \hat{\delta} \Rightarrow ||\bar{u}_{s_1} - \bar{u}_{s_2}|| < \varepsilon.$$

The continuity of the above mapping follows. On the other hand, $\bar{x}_b = \bar{y}_b$. So, there exists a constant $\sigma_2 > 0$ such that $b + \sigma_2 \leq n$ and $\bar{x}_s, \bar{y}_s \in B_{2,\rho}$, for all $s \in [b,\ b+\sigma_2]$.
Choose $\sigma = \min\left\{\sigma_1, \sigma_2,\ \dfrac{1}{4(a+2m)}\right\}$. We note that $[b,\ b+\sigma] \subset [0,n]$.
Let $X_b = C([b, b+\sigma],\mathbb{R}^n)$ be the Banach space of all continuous functions on $[b, b+\sigma]$ to $\mathbb{R}^n$, with the usual norm also denoted by $||.||$. For each $u \in X_b$, we define the operator $\tilde{u} : [b-r,\ \ b+\sigma] \to \mathbb{R}^n$ as follows :

$$\widetilde{u}(s) = \begin{cases} u(s) + \bar{x}(b) - u(b), & \text{if } s \in [b, b+\sigma], \\ \bar{x}(s) & \text{if } t \in [b-r,\ \ b]. \end{cases}$$

We consider the equation:

$$u(t) = \overline{x}(b) + \int_b^t [-A(s)\tilde{u}(s) + g(s, \tilde{u}(s), \tilde{u}_s)]ds\,,\ t \in [b,\ \ b+\sigma]. \tag{2.13}$$

Put

$$\Omega_b = \{u \in X_b : \tilde{u}_s \in B_{2,\rho},\ s \in [b,\ b+\sigma]\},$$

and consider the operator $H : \Omega_b \to X_b$, be defined as follows:

$$H(x)(t) = \overline{x}(b) + \int_b^t [-A(s)\tilde{u}(s) + g(s, \tilde{u}(s), \tilde{u}_s)]ds\,,\ t \in [b,\ \ b+\sigma].$$

It is easy to see that u is a fixed point of H if and only if u is a solution of (2.13). For $u, v \in \Omega_b$, for all $s \in [b, \ b+\sigma]$, since $\tilde{u}_s, \tilde{v}_s \in B_{2,\rho}$ and then $\tilde{u}(s), \tilde{v}(s)$ also belong to $B_{1,\rho}$, we have:

$$\begin{aligned}
|H(u)(t) - H(v)(t)| &\leq \int_b^t \left[a \ |\tilde{u}(s) - \tilde{v}(s)| + |g(s, \tilde{u}(s), \tilde{u}_s) - g(s, \tilde{v}(s), \tilde{v}_s)|\right] ds \\
&\leq \int_b^t \left[a \ |\tilde{u}(s) - \tilde{v}(s)| + m|\tilde{u}(s) - \tilde{v}(s)| + m||\tilde{u}_s - \tilde{v}_s)||\right] ds \\
&\leq (a + 2m) \int_b^t ||\tilde{u}_s - \tilde{v}_s)|| ds \\
&\leq 2(a+2m)\sigma \ ||u - v||.
\end{aligned}$$

Therefore

$$||H(u) - H(v)|| \leq \frac{1}{2}||u - v||. \tag{2.14}$$

Since $\bar{x}, \bar{y}$ are the solutions of (1.1)–(1.2), the restrictions $\bar{x}_{|[b,b+\sigma]}, \bar{y}_{|[b,b+\sigma]}$ are the solutions of (2.13).
By (2.14), we have:

$$x_{|[b,b+\sigma]} = y_{|[b,b+\sigma]} .$$

It follows that

$$x(t) = y(t), \forall t \in [-r, b+\sigma]. \tag{2.15}$$

From (2.12) and (2.15), we get a contradiction. Then (2.11) holds. The proof is complete. ∎

3. Asymptotic Stability

In the sequel, for an interval $I \subset \mathbb{R}$, we will use $BC_0(I)$ to denote the space of continuous functions $x \in BC(I)$ vanishing at infinity.

Theorem 3.1. *Let* (I.1)–(I.2) *hold. Let* $g : [0, \infty) \times \mathbb{R}^n \times C \to \mathbb{R}^n$ *be locally Lipschitzian in the second and the third variables satisfying the conditions* (G_1)–(G_3). *Assume that* x_1 *and* x_2 *are solutions of* (1.1)–(1.2) *for different initial conditions* $\varphi = \varphi_1$ *and* $\varphi = \varphi_2$ *respectively. Then,*

$$\lim_{t \to \infty} |x_1(t) - x_2(t)| = 0.$$

Proof.

Step 1. At first, suppose that for each $v \in BC_0[-r, \infty)$, we have $g(t, v(t), v_t) \to 0$ as $t \to \infty$, then we can show that if x is a solution of (1.1)–(1.2) for an initial condition $\varphi \in C$ then $x \in BC_0[-r, \infty)$, i.e. $x(t) \to 0$ as $t \to \infty$.
The case $\varphi = 0$. Consider the operator T and the set $\overline{D}$ as in Theorem 2.2. Put

$$\overline{\Omega} = \{v \in \overline{D} : v(t) \to 0 \ \text{as} \ t \to \infty\}.$$

It is obvious that $\overline{\Omega}$ is a closed convex subset of X_0. On the other hand, for all $v \in \overline{\Omega}$, it follows from (2.2) that

$$|Tv(t)| \leq \widetilde{a} \sup_{t\in[0,\infty)} |g(t, v(t), v_t)|,$$

so $Tv(t) \to 0$ as $t \to \infty$. Therefore $T(\overline{\Omega}) \subset \overline{\Omega}$. By applying Schauder theorem, T has a fixed point $x_0 \in \overline{\Omega}$, that is also a solution of the problem (1.1)–(1.2) on $[-r, \infty)$. Thus, if x is a solution of (1.1)–(1.2) with the initial condition $\varphi = 0$ then by the uniqueness, $x = x_0$, so

$$\lim_{t\to\infty} x(t) = 0.$$

The case $\varphi \neq 0$. Similarly, we also consider $\overline{\varphi} : [-r, \infty) \to \mathbb{R}^n$ being an extension of φ. Here, we choose $\overline{\varphi}$ such that it is continuously differentiable on $[0, \infty)$ and $\overline{\varphi}(t) \to 0$ as $t \to \infty$. Then, as above, the problem (1.1)–(1.2) has a unique solution $y + \overline{\varphi}$ on $[-r, \infty)$, where y is a unique solution of the problem:

$$\begin{cases} y'(t) + A(t)y(t) = h\big(t, y(t), y_t\big), \ t \geq 0, \\ y_0 = 0, \end{cases}$$

in which

$$h(t, \xi, \eta) = g\big(t, \xi + \overline{\varphi}(t), \eta + \overline{\varphi}_t\big) - A(t)\overline{\varphi}(t).$$

Clearly, for each $v \in BC_0[-r, \infty)$, by $\overline{\varphi}(t) \to 0$ as $t \to \infty$, we have

$$h(t, v(t), v_t) = g(t, v(t) + \overline{\varphi}(t), v_t + \overline{\varphi}_t) - A(t)\overline{\varphi}(t) \to 0 \ \text{ as } \ t \to \infty.$$

This implies that $y(t) \to 0$ as $t \to \infty$. So $x = y + \overline{\varphi} \to 0$ as $t \to \infty$.

Step 2. Let x_1 and x_2 be solutions of (1.1)–(1.2) for different initial conditions $\varphi = \varphi_1$ and $\varphi = \varphi_2$ respectively. We put $z = x_2 - x_1$. Then z is a solution of the following problem

$$\begin{cases} z'(t) + A(t)z(t) = g\big(t, z(t) + x_1(t), z_t + x_{1t}\big) - g\big(t, x_1(t), x_{1t}\big), & t \geq 0, \\ z(t) = \varphi_2(t) - \varphi_1(t), & t \in [-r, 0]. \end{cases} \tag{3.1}$$

As above, if we also define $\psi = \varphi_2 - \varphi_1$ and $\widetilde{h} : [0, \infty) \times \mathbb{R}^n \times C \to \mathbb{R}^n$ by

$$\widetilde{h}(t, \xi, \eta) = g\big(t, \xi + x_1(t), \eta + x_{1t}\big) - g\big(t, x_1(t), x_{1t}\big), \tag{3.2}$$

then we can rewrite (3.1) as

$$\begin{cases} z'(t) + A(t)z(t) = \widetilde{h}\big(t, z(t), z_t\big), & t \geq 0, \\ z(t) = \psi(t), & t \in [-r, 0]. \end{cases} \tag{3.3}$$

It is easy to see that $\widetilde{h} : [0, \infty) \times \mathbb{R}^n \times C \to \mathbb{R}^n$ satisfies (G_1)–(G_3). Further, g is locally Lipschitzian in the second and the third variables, so is $\widetilde{h}$. On the

other hand, $\widetilde{h}(t,0,0)=0$, for each $v\in BC_0[-r,\infty)$, we have $\widetilde{h}(t,v(t),v_t)\to 0$ as $t\to\infty$.
By the step 1, $z(t)\to 0$ as $t\to\infty$. This implies that

$$\lim_{t\to\infty}|x_1(t)-x_2(t)|=0.$$

Theorem 3.1 is proved. ■

4. Periodic Solution

In this section, we study the existence of ω-periodic solutions ($\omega>r$) for the prolem (1.1)–(1.2).

Definition. *A function $x:[-r,\infty)\to\mathbb{R}^n$ is an ω-periodic solution of the prolem* (1.1)–(1.2) *if $x(.)$ is a solution of* (1.1)–(1.2) *and $x(t+\omega)=x(t)$, $\forall t\in[0,\infty)$.*

We make the following assumptions.

Assumption 4.1. (I.1), (I.2) *hold and $g:[0,\infty)\times\mathbb{R}^n\times C\to\mathbb{R}^n$ is locally Lipschitzian in the second and the third variables satisfying the conditions* (G_1)–(G_3).

Assumption 4.2. *For a constant $\omega>r$,*

$$A(t+\omega)=A(t),\;\; g(t,\xi,\eta)=g(t+\omega,\xi,\eta),\;\; t\geq 0.$$

At first, we note that for given $\varphi\in C$, there is a unique strong solution $x(t,\varphi)$ of (1.1)–(1.2). If we put

$$P(t)\varphi=x_t(.,\varphi),\;\text{ for all }\; t\geq 0,$$

then the mapping $P(t):C\to C$ is defined for all $t\geq 0$.
To prove the main result of this section, we need the following lemma.

Lemma 4.1. *For a constant $T>r$, for all $t\in[r,T]$, the mapping $P(t)$ maps bounded subsets of C into relatively compact sets.*

Proof. Let Ω_2 be a bounded subset in C. We show that $P(t)\Omega_2$, for all $t\in[r,T]$, is relatively compact in C. This fact is proved as follows. Put

$$m_1=\max\{\,\|\varphi\|\,,\varphi\in\Omega_2\,\}\,.$$

For all $t\in[0,T]$, we have

$$x(t,\varphi)=\varphi(0)+\int_0^t\big[-A(s)x(s,\varphi)+g(s,x(s,\varphi),x_s(.,\varphi))\big]ds.$$

It implies that

$$|x(t,\varphi)| \leq m_1 + \int_0^t [a|x(s,\varphi)| + 2\widetilde{C}_1||x_s(.,\varphi)|| + \widetilde{C}_2]ds$$
$$\leq m_1 + \widetilde{C}_2 T + (a + 2\widetilde{C}_1)\int_0^t ||x_s(.,\varphi)||ds,$$

and clearly for all $t \in [-r, 0]$,

$$|x(t,\varphi)| = |\varphi(t)| \leqslant m_1.$$

So, for all $\varphi \in \Omega_2$, for all $t \in [0, T]$,

$$|x_t(.,\varphi)| \leq m_1 + \widetilde{C}_2 T + (a + 2\widetilde{C}_1)\int_0^t ||x_s(.,\varphi)||ds,$$

by using Gronwall's lemma, we get

$$|x_t(.,\varphi)| \leq (m_1 + \widetilde{C}_2 T) \ \exp(a + 2\widetilde{C}_1).$$

Therefore $P(t)\Omega_2$ is uniformly bounded, for all $t \in [0, T]$. Then, there exists a constant $K > 0$ such that for all $\varphi \in \Omega_2$, for all $t \in [0, T]$,

$$|x'(t,\varphi)| \leq (a + 2\widetilde{C}_1)||x_t(.,\varphi)|| + \widetilde{C}_2 \leq K.$$

Hence, for all $\varphi \in \Omega_2$, for all $t \in [r, T]$,

$$|x_t(.,\varphi)(\theta_1) - x_t(.,\varphi)(\theta_2)| = |x(t+\theta_1,\varphi) - x(t+\theta_2,\varphi)| \leq K|\theta_1 - \theta_2|,$$

for all $\theta_1, \theta_2 \in [-r, 0]$.
Thus, $P(t)\Omega_2$ is equi-continuous, for all $t \in [r, T]$. Applying the Arzela–Ascoli theorem, $P(t)\Omega_2$, is relatively compact in C, for all $t \in [r, T]$.

Next, the following theorem is a preliminary result for the main result.

In the sequel, let CP denote the Banach space of functions $x \in BC[-r, \infty)$ such that $x(t + \omega) = x(t)$ for all $t \geq 0$, with norm

$$||x|| = \sup_{t\in[-r,\infty)} |x(t)| = \sup_{t\in[-r,\omega]} |x(t)|,$$

and let $B_{\tilde{\rho}}$ be the closed ball, with center at 0 and radius $\widetilde{\rho}$, in the Banach space CP.

Theorem 4.2. *Let the Assumptions* 4.1, 4.2 *be satisfied. Then for every* $\widetilde{\rho} > 0$, *for each* v *belongs to* $B_{\tilde{\rho}}$, *there exists an* ω*-periodic solution of the equation*

$$x'(t) + A(t)x(t) = g\big(t, v(t), v_t\big), \ \ t \geq 0, \tag{4.1}$$

Proof. For a solution $x(.) = x(., \varphi)$, with a given $\varphi \in C$, we have the decomposition

$$x(., \varphi) = v(., \varphi) + z(., 0),$$

where $v(., \varphi)$ is a solution of

$$\begin{cases} x'(t) + A(t)x(t) = 0, \ t \geq 0, \\ x_0 = \varphi, \end{cases}$$

and $z(., 0)$is a solution of

$$\begin{cases} x'(t) + A(t)x(t) = g(t, v(t), v_t), \ t \geq 0, \\ x_0 = 0, \end{cases}$$

Fix $\varphi_0 \in C$. By Theorem 2.2, the problem

$$\begin{cases} x'(t) + A(t)x(t) = g(t, v(t), v_t), \ t \geq 0, \\ x_0 = \varphi_0, \end{cases} \tag{4.2}$$

has a solution $y : [-r, \infty) \to \mathbb{R}^n$. Furthermore, it follows from $v \in B_{\widetilde{\rho}}$ that

$$|y(t)| \leq \widetilde{a}(2\widetilde{\rho}\widetilde{C}_1 + \widetilde{C}_3) + ||\varphi_0||$$

(see the proof of Theorem 2.2 in Step 2), i.e., y is bounded.

Define the mappings $\Gamma, \ \Gamma_1 : C \to C$ as follows:

$$\Gamma(\varphi) = \Gamma_1(\varphi) + z_\omega = v_\omega + z_\omega.$$

Then $\Gamma_1 : C \to C$ is a bounded linear operator and

$$\bigcup_{n \geq 0} \Gamma^n(y_0) = \{y_{n\omega} : n \in \mathbb{N}\}.$$

Since $\{y_{n\omega}(., \varphi), n \in \mathbb{N}\}$ is bounded, by Lemma 4.1, the following set is relatively compact in C:

$$P(\omega)\{y_{n\omega}(., \varphi), n \in \mathbb{N}\} = \{x_\omega(., y_{n\omega}(., \varphi)), n \in \mathbb{N}\}.$$

This implies that $\{y_{n\omega}, n \in \mathbb{N}\}$ is relatively compact in C. It follows from Theorem 1.1 that Γ has a fixed point $\widetilde{\varphi} \in C$. This fixed point gives an $\omega-$periodic solution $x(\widetilde{\rho}, v) = x(., \widetilde{\varphi})$ of (4.1). ∎

Remark 1. From the proof of Theorem 1.1 in [4, Theorem 2.6.8], Γ has a fixed point $\widetilde{\varphi} \in \mathrm{Cl}D$, with $D = \mathrm{co}\ \{y_0, \Gamma y_0, \Gamma^2 y_0, ...,\}$. Here, we have the subset Cl D is bounded, so there exists a constant $\widetilde{K} > 0$ such that for all $\varphi \in \mathrm{Cl}D$, $||\varphi|| \leq \widetilde{K}$. On the other hand, for all $t \in [-r, \infty)$, we have

$$|x(\widetilde{\rho}, v)(t)| = |x(t, \widetilde{\varphi})| \leq \widetilde{a}(2\widetilde{\rho}\widetilde{C}_1 + \widetilde{C}_3) + ||\widetilde{\varphi}||,$$

where $\widetilde{C}_3 = \widetilde{C}_3(\widetilde{\varphi})$. Combining these, there exists a constant $\widetilde{C} > 0$ independent of $\widetilde{\rho}, \widetilde{\varphi}$ such that for all $v \in B_{\widetilde{\rho}}$, for all $t \in [-r, \infty)$,

$$|x(\widetilde{\rho}, v)(t)| = |x(t, \widetilde{\varphi})| \leq 2\widetilde{a}\widetilde{\rho}\widetilde{C}_1 + \widetilde{C}.$$

If we choose $\widetilde{\rho} \geq \widetilde{C}/(1 - 2\widetilde{C}_1\widetilde{a})$ then $|||x(\widetilde{\rho}, v)|| \leq \widetilde{\rho}$. We conclude that there exists $\widetilde{\rho} > 0$ such that ω-periodic solution $x(\widetilde{\rho}, v)$ of (4.1) as above belongs to $B_{\widetilde{\rho}}$. Now, we state our main result as follows.

Theorem 4.3. *Let the Assumptions* 4.1, 4.2 *be satisfied. Then there exists an ω-periodic solution of the problem* (1.1)–(1.2).

Proof. On $B_{\widetilde{\rho}}$, with $\widetilde{\rho}$ is chosen as in remark 1, we define the multivalued map $\widehat{\Gamma} : B_{\widetilde{\rho}} \to 2^{CP}$ by : $x \in \widehat{\Gamma}(v)$ if and only if

$$x(t) = x(0) + \int_0^t \big[-A(s)x(s) + g(s, v(s), v_s)\big]ds,\ t > 0.$$

We shall prove that $\widehat{\Gamma}$ satisfies the conditions (i)–(iii) of Theorem 1.2.

For every $v \in B_{\tilde{\rho}}$, by Remark 1, $\widehat{\Gamma}(v)$ is nonempty. It is easy to prove that $\widehat{\Gamma}(v)$ is convex and closed. The condition (i) holds.

The same arguments as used in the proof of Lemma 4.1 imply that $\widehat{\Gamma}(B_{\tilde{\rho}})$ is uniformly bounded and equi-continuous. Hence the Ascoli–Arzela Theorem can be applied to deduce that $\widehat{\Gamma}(B_{\tilde{\rho}})$ is relatively compact. The condition (ii) holds. Finally, we show that $\widehat{\Gamma}$ closed. Let $(v_n), (x_n)$ are convergent sequences to v, x, respectively as $n \to \infty$ and $x_n \in \widehat{\Gamma}(v_n)$, then for all $t > 0$,

$$\int_0^t \big[-A(s)x_n(s) + g(s, v_n(s), (v_n)_s)\big]ds \to \int_0^t \big[-A(s)x(s) + g(s, v(s), v_s)\big]ds,$$

as $n \to \infty$, so

$$x(t) = x(0) + \int_0^t \big[-A(s)x(s) + g(s, (v)(s), v_s)\big]ds.$$

We get $x \in \widehat{\Gamma}(v)$. This implies that the condition (iii) holds. Applying Theorem 1.2, the operator $\widehat{\Gamma}$ has a fixed point. This fixed point is an $\omega-$periodic of (1.1)–(1.2). Theorem 4.3 is proved completely. ■

5. The Connectivity and Compactness of Solution Set

In this section, applying Theorem 1.3 and Theorem 1.4, we prove the set of solutions of the problem (1.1)–(1.2) corresponding to $g = g_1(t)g_2(\xi, \eta)$ is nonempty, compact and connected. This result is based on the ideas and techniques in [6].

We make the following assumptions.

Assumption 5.1. (I.1), (I.2) hold and $g = g_1(t)g_2(\xi, \eta)$.

Assumption 5.2. $g_1 \in BC[0, \infty)$ and $g_2 : \mathbb{R}^n \times C \to \mathbb{R}^n$ is continuous with the following properties:

(G_4) For each v_0 belongs to any bounded subset Ω of $BC[-r,\infty)$, for all $\epsilon > 0$, there exists $\delta = \delta(\epsilon, v_0) > 0$ such that for all $v \in \Omega$

$$||v - v_0|| < \delta \Rightarrow |g_2(v(t), v_t) - g_2(v_0(t), (v_0)_t)| < \epsilon,$$

for all $t \in [0, \infty)$,

(G_5) There exist positive constants $\overline{C}_1, \overline{C}_2$ with $C_1\overline{C}_1 < \dfrac{1}{2\widetilde{a}}$, such that

$$|g_2(\xi,\eta)| \le \overline{C}_1\big(|\xi| + ||\eta||\big) + \overline{C}_2, \;\; \forall(\xi,\eta) \in \mathbb{R}^n \times C,$$

where $C_1 = \sup\limits_{t\in[0,\infty)} |g_1(t)|$.

Theorem 5.1. *Let the assumptions* 5.1 *and* 5.2 *be satisfied. Then, for every* $\varphi \in C$*, the solution set of the problem* (1.1)–(1.2) *in* D *is nonempty, compact and connected, where* D *is defined as in Theorem* 2.2.

Proof.

Step 1. Consider first the case $\varphi = 0$.

Obviously, g satisfies the conditions (G_1)–(G_3). We again consider the operator T, defined in Theorem 2.2 and the following subset (as (2.5))

$$D = \{v \in X_0 : ||v||_1 < M_2\}. \tag{5.1}$$

Note that the set of all solutions to the problem (1.1)–(1.2) in D is the set of fixed points of the operator $T = L^{-1}F : \overline{D} \to X^1 \subset X_0$.
We have that T is continuous. Furthermore, since $T(\overline{D})$ is relatively compact, T maps bounded subsets of $\overline{D}$ into relatively compact sets. Hence, T is completely continuous.
Since $T(\overline{D}) \subset D$, T has no fixed point in ∂D. On the other hand, D is convex, so we have

$$\deg(I - T, D, 0) = 1. \tag{5.2}$$

For all $\epsilon > 0$, since $g_2 : \mathbb{R}^n \times C \to \mathbb{R}^n$ is continuous, by Theorem 1.4, there exists a locally Lipschitzian mapping $g_{2\epsilon}$ such that for all $(\xi, \eta) \in \mathbb{R}^n \times C$

$$|g_{2\epsilon}(\xi,\eta) - g_2(\xi,\eta)| < \frac{\epsilon}{2C_1\widetilde{a}}. \tag{5.3}$$

For each $v \in BC[-r,\infty)$, put $f_\epsilon(t) = g_1(t)g_{2\epsilon}(v(t), v_t), \forall t \in [0,\infty)$, then we have $f_\epsilon \in BC[0,\infty)$. Consider the operator $T_\epsilon = L^{-1}F_\epsilon,$ where

$$\begin{aligned}
&F_\epsilon : X_0 \to BC[0,\infty)\\
&v \in X_0 \mapsto F_\epsilon(v)(.) = f_\epsilon(.) \in BC[0,\infty).
\end{aligned}$$

Similarly, we get the completely continuous operator $T_\epsilon : \overline{D} \to X_0$.
For all $v \in \overline{D}$, it follows from (5.3) that:

$$||T(v) - T_\epsilon(v)|| \le C_1\widetilde{a} \sup_{t\in[0,\infty)} |g_{2\epsilon}(v(t), v_t) - g_2(v(t), v_t)| < \epsilon. \tag{5.4}$$

Finally, we need only to prove that for each $\widehat{h}$ with $||\widehat{h}|| < \epsilon$, the following equation has at most one solution in $\overline{D}$:

$$x = T_\epsilon(x) + \widehat{h}. \tag{5.5}$$

The equation 5.5 is equivalent to the equation:

$$Lx = F_\epsilon(x) + L\widehat{h}.$$

Then for $t \geq 0$, we have the equation:

$$\begin{aligned} x'(t) + A(t)x(t) &= g_1(t)g_{2\epsilon}(x(t), x_t) + L\widehat{h}(t), \\ &\equiv g_3(t, x(t), x_t). \end{aligned} \tag{5.6}$$

It is easy to see that $g_3 : [0, \infty) \times \mathbb{R}^n \times C \to \mathbb{R}^n$ is locally Lipschitzian in the second and the third variables, hence, by Theorem 2.2, (5.6) has at most one solution in $\overline{D}$. This implies that (5.5) has at most one solution in $\overline{D}$.
By applying Theorem 1.3 the set of solutions of the problem (1.1)–(1.2) in D is nonempty, compact and connected. The proof of step 1 is completed.

Step 2. Consider first the case $\varphi \neq 0$.

As in the proof of Theorem 2.2, by the transformation $y = x - \overline{\varphi}$, the problem (1.1)–(1.2) reduces to the problem (2.10). By the step 1, the set of solutions of (2.10) in D_1 is nonempty, compact and connected, where D_1 is defined corresponding to h as follows

$$D_1 = \{v \in X_0 : ||v|| < \hat{M}_2\},$$

where $\hat{M}_2 > \dfrac{\widetilde{a}\widetilde{C}_3}{1 - 2\widetilde{a}\widetilde{C}_1}$, in which $\widetilde{C}_1, \widetilde{C}_3$ are defined as in Theorem 2.2.

We deduce that the set of solutions of (1.1)–(1.2) in D_2 is nonempty, compact and connected, where $D_2 = \{y + \overline{\varphi}, y \in D_1\}$. Theorem 5.1 is proved completely.

■

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