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Outer γ-Convexity and Inner γ-Convexity of Disturbed Functions

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Dedicated to Professor Hoang Tuy on the occasion of his 80^{th} birthday

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Abstract. The most kinds of generalized convexities cannot resist perturbations, even linear ones, while real application problems are often affected by disturbances, both linear and nonlinear ones. For instance, we showed earlier that quasiconvexity, explicit quasiconvexity, and pseudoconvexity cannot withstand arbitrarily small linear disturbances to keep their characteristic properties, and convex functions are the only ones which can resist every linear disturbance to preserve property "each local minimizer is a global minimizer", but it fails if perturbation is nonlinear, even with arbitrarily small supremum norm. In this paper, we present some sufficient conditions for the outer γ-convexity and the inner γ-convexity of disturbed functions, for instance, when convex functions are added with arbitrarily wild but accordingly bounded functions. That means, in spite of such nonlinear disturbances, some weakened properties can be saved, namely the properties of outer γ-convex functions and inner γ-convex ones. For instance, each γ-minimizer of an outer γ-convex function $f : D \to \mathbb{R}$ defined by $f(x^*) = \inf_{x \in \bar{B}(x^*, \gamma) \cap D} f(x)$ is a global minimizer, or if an inner γ-convex function $f : D \to \mathbb{R}$ defined on some bounded convex subset D of an inner product space attains its supremum, then it does so at least at some strictly γ-extreme point of D, which cannot be represented as midpoint of some segment $[z', z''] \subset D$ with $\|z' - z''\| \geq 2\,\gamma$, etc.

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1. Introduction

As ideal mathematical object, convex functions have several particular properties. Two of them are:

(α) each local minimizer is a global minimizer,

(β) if a convex function defined on a finite-dimensional compact set D attains its supremum, then it does so at least at some extreme point of D

(see, e.g., [17, 18],...). These properties are useful for optimization. (α) serves as a sufficient condition for global minimum and justifies local search. Due to (β), in order to seek a global maximizer, one can restrict himself to investigating extreme points, as done by simplex method.

A generalization trend to get similar properties for wider function classes consists of different kinds of rough convexity, where some characteristics are required to be satisfied at some certain places between points whose distance is greater than given roughness degree $\gamma > 0$. Some representatives are global δ-convexity ([3]), rough ρ-convexity ([2, 19]), γ-convexity ([4, 6]), and symmetrical γ-convexity ([1]). All mentioned kinds of roughly convex functions have two properties similar to (α) and (β), namely:

(α_γ) each γ-minimizer of $f : D \to \mathbb{R}$ defined by $f(x^*) = \inf_{x \in \bar{B}(x^*, \gamma) \cap D} f(x)$ is a global minimizer,

(β_γ) under some suitable additional hypothesis, if $f : D \to \mathbb{R}$ attains its supremum, then it does so at least at some strictly γ-extreme point of D, which cannot be represented as midpoint of some segment $[z', z''] \subset D$ with $\|z' - z''\| \geq 2\,\gamma$

(see [8]). But they are by far not general enough in order to model a lot of important practical problems. To get a function class which is as wide as possible and has such properties, we choose two separate ways for generalization, because essentially different natures hide behind minimum and maximum. Outer γ-convexity is introduced in [10] and [15] to get (α_γ) and other properties similar to those of convex functions relative to their infimum. Inner γ-convexity is defined in [11] and [12] to obtain (β_γ) and other similar properties relative to supremum.

In the present paper, we show the outer γ-convexity and the inner γ-convexity of some classes of disturbed functions. As consequence, these disturbed functions inherit the mentioned optimization properties of roughly convex functions.

Such a research is of practical importance because real application problems are almost always affected by disturbances, while the most kinds of generalized convexities cannot resist perturbations. We showed in [13] that known kinds of generalized convexities like quasiconvexity, explicit quasiconvexity, and pseudoconvexity cannot withstand arbitrarily small linear disturbances to keep their characteristic properties. Due to [14], convex functions are the only ones which can resist every linear disturbance to preserve property (α), i.e. concretely, if the sum of some certain lower semicontinuous function $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ and an arbitrary linear function always has property (α), then f must be convex. Similarly, if the sum of some certain lower semicontinuous function $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ and an arbitrary linear function always has property (α_γ), then f must be outer

γ-convex, i.e., only outer γ-convex functions withstand all linear disturbances to hold (α_γ) (see [14] and [15]).

How about nonlinear disturbances? In general, convex functions cannot tolerate relatively wild disturbances without losing their characteristic properties, even if the supremum norm of disturbances is arbitrarily small. But we will present in Sec. 1 and Sec. 2 some classes of convex functions which remain to be outer γ-convex and/or inner γ-convex if they are disturbed by arbitrarily wild but accordingly bounded disturbances, i.e., some weakened properties can be saved in spite of such wild disturbances, namely properties of outer γ-convex and inner γ-convex functions.

Throughout this paper, X is a normed linear space over the field of real numbers, D is a convex subset of X, and γ is a positive real number. For any x_0 and x_1 in X, let us denote

$$x_\lambda := (1-\lambda)x_0 + \lambda x_1. \tag{1}$$

Moreover, the following notations are used

$$B(x,r) := \{x' \in X \mid \|x - x'\| < r\},$$
$$\bar{B}(x,r) := \{x' \in X \mid \|x - x'\| \le r\}.$$

2. Outer γ-Convexity of Disturbed Functions

A real-valued function $f : D \to \mathbb{R}$ is said to be *outer γ-convex* or *strictly outer γ-convex* with respect to (w.r.t. for short) roughness degree $\gamma > 0$ if for all $x_0, x_1 \in D$ there exists $\Lambda \subset [0,1]$ such that

$$[x_0, x_1] \subset \{x_\lambda \mid \lambda \in \Lambda\} + \bar{B}(0, \gamma/2) \tag{2}$$

and

$$\forall \lambda \in \Lambda : f(x_\lambda) \le (1-\lambda)f(x_0) + \lambda f(x_1), \tag{3}$$

or

$$\forall \lambda \in \Lambda : f(x_\lambda) < (1-\lambda)f(x_0) + \lambda f(x_1), \tag{4}$$

respectively.

(2) holds if and only if there exist $k \in \mathbb{N}$ and $\lambda_i \in \Lambda \subset [0,1]$, $i = 0, 1, \ldots, k$ such that

$$\lambda_0 = 0,\ \lambda_k = 1,\ 0 \le \lambda_{i+1} - \lambda_i \le \frac{\gamma}{\|x_0 - x_1\|} \ \text{ for } \ i = 0, 1, \ldots, k-1, \tag{5}$$

since it follows from (1) that (5) just means $x_{\lambda_0} = x_0$, $x_{\lambda_k} = x_1$, and

$$\|x_{\lambda_i} - x_{\lambda_{i+1}}\| = (\lambda_{i+1} - \lambda_i)\, \|x_0 - x_1\| \le \gamma \ \text{ for } \ i = 0, 1, \ldots, k-1.$$

Note that conditions (2)–(3) are proper only when $\|x_0 - x_1\| > \gamma$, because if $\|x_0 - x_1\| \le \gamma$ then these conditions are always fulfilled by choosing $\Lambda = \{0, 1\}$.

The relation between convexity and outer γ-convexity is given by the following.

Proposition 1.
(a) *Every convex function is outer γ-convex w.r.t. any $\gamma > 0$.*
(b) *$f + g$ is outer γ-convex if f is outer γ-convex and g is convex.*
(c) *$f+g$ is strictly outer γ-convex if f is strictly outer γ-convex and g is convex, or if f is outer γ-convex and g is strictly convex.*

The above assertions follow directly from definition, so their proof are omitted.

The concrete form of property (α_γ) of outer γ-convex functions is as follows.

Theorem 2. ([10, 15]) *Let $f : D \to \mathbb{R}$ be outer γ-convex and let $x^* \in D$.*
(a) *If $f(x^*) = \inf_{x\in\bar{B}(x^*,\gamma)\cap D} f(x)$ then $f(x^*) = \inf_{x\in D} f(x)$, i.e., a γ-minimizer is a global minimizer.*
(b) *If there exists an $\epsilon > 0$ such that $\liminf_{x\to x^*} f(x) = \inf_{x\in B(x^*,\gamma+\epsilon)\cap D} f(x)$ then $\liminf_{x\to x^*} f(x) = \inf_{x\in D} f(x)$, i.e., a local γ-infimizer is a global infimizer.*

An important property of strictly convex functions is that they have at most one minimizer. This uniqueness is crucial for proving the continuity of optimal solutions or of optimal control functions. A roughly generalized version of strictly convex functions was investigated in [9], whose result was applied in [16] to show the rough continuity of the optimal control of a transportation problem. Since a strictly outer γ-convex function is strictly r-convexlike w.r.t. $r = \gamma$, Proposition 2.2 in [9] yields immediately the following.

Proposition 3. *If $f : D \to \mathbb{R}$ is strictly outer γ-convex, then the diameter of the set of its global minimizers (if any) is not greater than γ.*

A remarkable property of convex functions is concerned with the existence of a subgradient $\xi \in X^*$ at some $x^* \in D$ defined by

$$\forall z \in D: \; f(z) \geq f(x^*) + \langle \xi, z - x^* \rangle$$

(see [18]). Outer γ-convex functions have a similar property as follows.

Theorem 4. ([10]) *Let $X = \mathbb{R}^n$ be some n-dimensional normed vector space, and $D \subset X$ be compact and convex. Let $f : D \to \mathbb{R}$ be outer γ-convex, bounded below, and lower semicontinuous. Then for all $z^* \in \operatorname{ri} D$, there is $\xi \in \mathbb{R}^n$ such that*

$$\exists \tilde{z} \in \bar{B}(z^*,\, J_s(X)\,\gamma/2) \quad \forall z \in D: \; f(z) \geq f(\tilde{z}) + \langle \xi, z - \tilde{z} \rangle,$$

where

$$J_s(X) := \sup \left\{ \frac{2r_{\operatorname{conv} S}(S)}{\operatorname{diam} S} \,\middle|\, S \subset X \text{ bounded, non-empty, non-singleton} \right\},$$

with $r_{\operatorname{conv} S}(S) = \inf\limits_{x\in\operatorname{conv} S} \sup\limits_{y\in S} \|x - y\|$, $\operatorname{diam} S = \sup\limits_{x,y\in S} \|x - y\|$, *is the so-called self-Jung constant.*

Let us now come to the outer γ-convexity of disturbed functions. The next three propositions deal with disturbances which are already outer γ-convex, therefore, due to Proposition 1, if we add it to any convex function, the sum is obviously outer γ-convex, too.

Proposition 5. (Insistent disturbance) *Suppose*

$$z^j \in \mathbb{R},\ \ 0 < z^{j+1} - z^j \le \gamma \ \ \textit{for all } j \in \mathbb{Z}. \tag{6}$$

Let $D \subset \mathbb{R}$ be any interval and $g : D \to \mathbb{R}$ be any function satisfying

$$g(z^j) = \inf_{x \in D} g(x) \ \ \textit{for all } z^j \in D. \tag{7}$$

Then g is outer γ-convex. Hence, $f+g$ is outer γ-convex if $f : D \to \mathbb{R}$ is convex.

Proof. Consider arbitrary $x_0, x_1 \in D$ with $x_1 - x_0 > \gamma$. By choosing

$$\mu^j = (x_0 - z^i)/(x_0 - x_1),\ \ j \in \mathbb{Z},$$

we have

$$0 < \mu^{j+1} - \mu^j = \frac{-z^{j+1} + z^j}{x_0 - x_1} \le \frac{\gamma}{|x_0 - x_1|},\ \ j \in \mathbb{Z}, \tag{8}$$

and

$$x_{\mu^j} = (1 - \mu^j)x_0 + \mu^j x_1 = z^j, \quad j \in \mathbb{Z}. \tag{9}$$

Let

$$j^* := \min\{j \mid \mu^{j+1} > 0\},\ \ k := \max\{j - j^* \mid \mu^{j-1} < 1\},$$
$$\lambda_0 = 0,\ \ \lambda_k = 1,\ \ \lambda_i = \mu^{i+j^*} \ \text{ for } i = 1, \dots, k-1.$$

Then (8)–(9) imply

$$0 \le \lambda_{i+1} - \lambda_i \le \mu^{i+1+j^*} - \mu^{i+j^*} \le \frac{\gamma}{\|x_0 - x_1\|} \ \text{ for } i = 0, 1, \dots, k-1,$$

and

$$g(x_{\lambda_i}) = g(z^{i+j^*}) = \inf_{x \in D} g(x) \le (1 - \lambda_i)g(x_0) + \lambda_i g(x_1) \ \text{ for } i = 1, 2, \dots, k-1,$$

i.e., (3) and (5) hold for $\Lambda = \{\lambda_i \mid 0 \le i \le k\}$. By definition, g is outer γ-convex. Due to Proposition 1, if $f : D \to \mathbb{R}$ is convex then $f + g$ is outer γ-convex. ■

In particular, if $\inf_{x \in D} g(x) = 0$, then (6)–(7) describe an one-sided non-negative disturbance function, which vanishes at least once in every arbitrary interval $[x, x + \gamma] \subset D$.

Proposition 6. (γ-homogenous disturbance) *Let $D \subset \mathbb{R}$ be any interval and $g : D \to \mathbb{R}$ be any function satisfying*

$$[x, x + \gamma] \subset D \Longrightarrow g([x, x + \gamma]) = g(D) \tag{10}$$

Then g is outer γ-convex. Hence, $f+g$ is outer γ-convex if $f : D \to \mathbb{R}$ is convex.

Obviously, (10) yields (11). Therefore, Proposition 6 follows directly from the next one.

Proposition 7. *Let $D \subset \mathbb{R}$ be any interval and $g : D \to \mathbb{R}$ be any function satisfying*

$$\big([x, x+\gamma] \subset D,\ y \in g(D)\big) \Longrightarrow \big(\exists x' \in [x, x+\gamma] : g(x') \le y\big). \tag{11}$$

Then g is outer γ-convex. Hence, $f+g$ is outer γ-convex if $f : D \to \mathbb{R}$ is convex.

Proof. Consider arbitrary $x_0, x_1 \in D$ with $x_1 - x_0 > \gamma$. Let

$$\Lambda := \{\lambda \in [0,1] \mid g(x_\lambda) \le \min\{g(x_0), g(x_1)\},$$

then g satisfies (3) and $\{0,1\} \subset \Lambda$. If (2) is not fulfilled, then there are λ' and λ'' such that

$$0 < \lambda' < \lambda'' < 1, \quad [\lambda', \lambda''] \cap \Lambda = \emptyset, \quad x_{\lambda''} - x_{\lambda'} > \gamma.$$

This means that

$$g(x) > \min\{g(x_0), g(x_1)\} \text{ for all } x \in [x_{\lambda'}, x_{\lambda''}],$$

a contradiction to (11). Therefore, (2) is fulfilled, too. By definition, g is outer γ-convex. Due to Proposition 1, if $f : D \to \mathbb{R}$ is convex then $f + g$ is outer γ-convex. ■

In the following, we consider bounded disturbances, which may be arbitrarily wild from the analytical point of view, nevertheless, the disturbed function is outer γ-convex.

Proposition 8. (Bounded disturbance) *Let $f : D \subset X \to \mathbb{R}$ be convex and*

$$h_1(\gamma) := \inf_{x_0,\, x_1 \in D,\, \|x_0 - x_1\| = \gamma} \left(\frac{1}{2}\big(f(x_0) + f(x_1)\big) - f\big(\frac{1}{2}(x_0 + x_1)\big) \right) > 0 \tag{12}$$

and $\gamma > 0$. Then the disturbed function $\tilde{f} = f + g$ is outer γ-convex if the disturbance function satisfies

$$|g(x)| \le h_1(\gamma)/2 \text{ for all } x \in D. \tag{13}$$

Proof. Consider arbitrary $x_0, x_1 \in D$ and $x_\lambda = (1-\lambda)x_0 + \lambda x_1 \in [x_0, x_1]$ satisfying

$$\|x_0 - x_1\| \ge \gamma, \quad \|x_0 - x_\lambda\| \ge \gamma/2, \quad \|x_1 - x_\lambda\| \ge \gamma/2. \tag{14}$$

Let

$$\lambda' = \lambda - \frac{\gamma}{2\,\|x_0 - x_1\|}, \quad \lambda'' = \lambda + \frac{\gamma}{2\,\|x_0 - x_1\|}.$$

Then we have

$$x_{\lambda'} = (1-\lambda')x_0 + \lambda' x_1 \in [x_0, x_\lambda], \quad x_{\lambda''} = (1-\lambda'')x_0 + \lambda'' x_1 \in [x_\lambda, x_1]$$

and

$$\lambda = \frac{1}{2}(\lambda' + \lambda''), \quad x_\lambda = \frac{1}{2}(x_{\lambda'} + x_{\lambda''}), \quad \|x_{\lambda'} - x_{\lambda''}\| = \gamma. \tag{15}$$

Since f is convex, there holds

$$\begin{aligned}(1-\lambda)f(x_0) + \lambda f(x_1) &= \left(1 - \frac{\lambda' + \lambda''}{2}\right) f(x_0) + \frac{\lambda' + \lambda''}{2} f(x_1) \\ &= \frac{1}{2}\big((1-\lambda')f(x_0) + \lambda' f(x_1) + (1-\lambda'')f(x_0) + \lambda'' f(x_1)\big) \\ &\geq \frac{1}{2}\big(f(x_{\lambda'}) + f(x_{\lambda''})\big).\end{aligned}$$

Hence, (12) and (15) imply

$$\begin{aligned}(1-\lambda)f(x_0) + \lambda f(x_1) - f(x_\lambda) &\geq \frac{1}{2}\big(f(x_{\lambda'}) + f(x_{\lambda''})\big) - f\big(\frac{1}{2}(x_{\lambda'} + x_{\lambda''})\big) \\ &\geq h_1(\gamma).\end{aligned}$$

This inequality and (13) yield

$$\begin{aligned}&(1-\lambda)\tilde{f}(x_0) + \lambda \tilde{f}(x_1) - \tilde{f}(x_\lambda) \\ &= (1-\lambda)(f(x_0) + g(x_0)) + \lambda(f(x_1) + g(x_1)) - (f(x_\lambda) + g(x_\lambda)) \\ &\geq (1-\lambda)(f(x_0) - h_1(\gamma)/2) + \lambda(f(x_1) - h_1(\gamma)/2) - f(x_\lambda) - h_1(\gamma)/2 \\ &= (1-\lambda)f(x_0) + \lambda f(x_1) - f(x_\lambda) - h_1(\gamma) \\ &\geq 0.\end{aligned} \tag{16}$$

That means

$$(1-\lambda)\tilde{f}(x_0) + \lambda \tilde{f}(x_1) \geq \tilde{f}(x_\lambda) \tag{17}$$

holds for all $x_0, x_1 \in D$ and $x_\lambda \in [x_0, x_1]$ satisfying (14). Obviously, (2) holds then for Λ which contains all λ satisfying (14). Thus, by definition, $\tilde{f} = f + g$ is outer γ-convex. ■

Proposition 9. (Bounded disturbance) *Let $f : D \subset X \to \mathbb{R}$ be convex and fulfil* (12)*, and let $\gamma > 0$. Then the disturbed function $\tilde{f} = f + g$ is strictly outer γ-convex if the disturbance function satisfies*

$$|g(x)| < h_1(\gamma)/2 \quad \text{for all} \quad x \in D. \tag{18}$$

Proof. Since the only difference between the assumptions of Proposition 8 and of Proposition 9 is the substitution of (13) by (18), almost all the proof of Proposition 8 can be taken over, where only the first greater or equal sign ($\geq$)

in (16) and in (17) must be changed to the greater sign (>). Finally, we obtain that (4) holds for Λ which contains all λ satisfying (14). ∎

Example 1. Let X be the n-dimensional Euclidian space and $f : X \to \mathbb{R}$ be defined by

$$f(x) = \|x\|^2 = \sum_{i=1}^{n} \xi_i^2, \quad x = (\xi_1, ..., \xi_n) \in X. \tag{19}$$

Then, for all $x_0 = (\xi_{01}, ..., \xi_{0n}) \in X$ and $x_1 = (\xi_{11}, ..., \xi_{1n}) \in X$ satisfying $\|x_0 - x_1\| = \gamma$, we have

$$\begin{aligned}
&\frac{1}{2}\left(f(x_0) + f(x_1)\right) - f(\frac{1}{2}(x_0 + x_1)) \\
&= \frac{1}{4}\sum_{i=1}^{n}\left(2\xi_{0i}{}^2 + 2\xi_{1i}{}^2 - (\xi_{0i}{}^2 + 2\,\xi_{0i}\xi_{1i} + \xi_{1i}{}^2)\right) \\
&= \frac{1}{4}\sum_{i=1}^{n}\left(\xi_{0i}{}^2 + \xi_{1i}{}^2 - 2\,\xi_{0i}\xi_{1i}\right) \\
&= \frac{1}{4}\|x_0 - x_1\|^2 \\
&= \frac{1}{4}\gamma^2.
\end{aligned}$$

Following, (12) implies $h_1(\gamma) = \gamma^2/4$. Hence, by Proposition 8, the disturbed function $\tilde{f} = f + g$ is outer γ-convex if the disturbance function $g : X \to \mathbb{R}$ satisfies

$$|g(x)| \le h_1(\gamma)/2 = \gamma^2/8 \quad \text{for all} \quad x \in X, \tag{20}$$

and, due to Proposition 9, $\tilde{f} = f + g$ is strictly outer γ-convex if g fulfils

$$|g(x)| < h_1(\gamma)/2 = \gamma^2/8 \ \text{ for all } x \in X.$$

Remark 1. Actually, in the proof of Proposition 8, we have proven that if f and g satisfy (12)–(13) then $f + g$ is globally δ-convex w.r.t. $\delta = \gamma$. Hence, for f defined by (19) and g satisfying (20), $f + g$ is globally δ-convex w.r.t. $\delta = \gamma$. Thus, Example 1 shows that in general a globally δ-convex function may be nowhere continuous and therefore also nowhere differentiable. This fact was shown in [7], but only for functions defined on some interval of $\mathbb{R}^1$, while Example 1 gives us an example for $D = X = \mathbb{R}^n$, $n > 1$.

3. Inner γ-Convexity of Disturbed Functions

A real-valued function $f : D \to \mathbb{R}$ is said to be *inner γ-convex* or *strictly inner γ-convex* w.r.t. roughness degree $\gamma > 0$ if there is a fixed *refinement rate* $\nu \in]0, 1]$ such that

$$\begin{aligned}
&\text{for all } x_0, x_1 \in D \text{ satisfying } \|x_0 - x_1\| = \nu\gamma \\
&\text{and } \ x_{1+1/\nu} = -(1/\nu)x_0 + (1 + 1/\nu)x_1 \in D
\end{aligned} \tag{21}$$

there holds

$$\sup_{\lambda\in[2,1+1/\nu]} \big(f((1-\lambda)x_0+\lambda x_1)-(1-\lambda)f(x_0)-\lambda f(x_1)\big) \geq 0, \qquad (22)$$

or

$$\exists\lambda\in[2,1+1/\nu] : f((1-\lambda)x_0+\lambda x_1)-(1-\lambda)f(x_0)-\lambda f(x_1) > 0, \qquad (23)$$

respectively.

Note that the corresponding positions of x_λ for $\lambda = 2$ and $\lambda = 1 + 1/\nu$ are characterized by

$$\|x_1 - x_2\| = \|x_1-(-x_0+2x_1)\| = \|x_0-x_1\| = \nu\gamma,$$
$$\|x_1 - x_{1+1/\nu}\| = \|x_1-(-(1/\nu)x_0+(1+1/\nu)x_1)\| = (1/\nu)\|x_0-x_1\| = \gamma.$$

The next sufficient condition (24) is easier to check than (22), and it becomes necessary if the considered function is upper semicontinuous. We will use it for proving Proposition 15.

Proposition 10. ([11])

(a) *$f : D \to \mathbb{R}$ is inner γ-convex if there is $\nu \in]0,1]$ such that for all $x_0, x_1 \in D$ satisfying* (21) *there holds*

$$\exists\lambda\in[2,1+1/\nu] : f((1-\lambda)x_0+\lambda x_1) \geq (1-\lambda)f(x_0)+\lambda f(x_1). \qquad (24)$$

(b) *Let $f : D \to \mathbb{R}$ be upper semicontinuous. Then it is inner γ-convex if and only if there is $\nu \in]0,1]$ such that* (24) *holds for all $x_0, x_1 \in D$ satisfying* (21)*.*

Let us collect some assertions describing the relation between convexity and inner γ-convexity.

Proposition 11. ([11])

(a) *Each convex function is inner γ-convex and each strictly convex function is strictly inner γ-convex w.r.t. any $\gamma > 0$.*

(b) *If f is convex and g is inner γ-convex, then $f + g$ is inner γ-convex w.r.t. the same roughness degree γ.*

(c) *If f is strictly convex and g is inner γ-convex, or if f is convex and g is strictly inner γ-convex, then $f+g$ is strictly inner γ-convex w.r.t. the same roughness degree γ.*

To characterize the location of maximizers and supremizers of inner γ-convex functions, we need two generalizations of extreme points defined as follows. $z \in D$ is said to be a *γ-extreme point* (or *strictly γ-extreme point*) of D if a representation $z = 0.5(z' + z'')$ by $z', z'' \in D$ is only possible when $\|z' - z''\| \leq 2\,\gamma$ (or

$\|z' - z''\| < 2\gamma$, respectively). One of these notions was introduced in [5] for representing finite-dimensional convex sets which are bounded but not necessarily closed.

For inner γ-convex functions, property (β_γ) appears as follows.

Theorem 12. ([11]) *Let X be an inner product space and D be a bounded convex subset of X and $f : D \to \mathbb{R}$ be inner γ-convex. If f attains its supremum, then it does so at some strictly γ-extreme point of D.*

When introducing Proposition 3, we already mentioned an important property of strictly convex functions w.r.t. their minimizers. The second important property of strictly convex functions is concerned with their maximizers, namely: a strictly convex function is only able to have maximizers at extreme points of its domain. For strictly inner γ-convex functions, we also have a similar property.

Theorem 13. ([11]) *A strictly inner γ-convex function $f : D \to \mathbb{R}$ can only have maximizers at strictly γ-extreme points of D.*

Due to the generality of inner γ-convexity, the existence of maximizers is not always guaranteed, even for inner γ-convex functions defined on compact sets. Therefore, we consider, in addition, the so-called *supremizers* $x^* \in D$ of $f : D \to \mathbb{R}$ defined by

$$\limsup_{x \to x^*} f(x) = \sup_{x \in D} f(x),$$

where x belongs to D while converging to x^* and it may equal x^*. A version of (β_γ) for supremizers of inner γ-convex functions is the following.

Theorem 14. [12] *Let X be an inner product space and $D \subset X$ be bounded. Let $f : D \to \mathbb{R}$ be inner γ-convex and bounded above and possess supremizers on D. Then there is at leat a supremizer on the boundary of D relative to* aff$D$ *or at a γ-extreme point of D. If, in addition, D is open relative to* aff$D$ *or* $\dim D \le 2$, *then there is certainly a supremizer at a γ-extreme point of D.*

Let us come to two sufficient conditions for the inner γ-convexity and the strict inner γ-convexity of disturbed functions when disturbances may behave very wildly and have only to be bounded by some corresponding quantity.

Proposition 15. *Let $\gamma > 0$ and let $f : D \subset X \to \mathbb{R}$ fulfil*

$$h_2(\gamma) := \inf_{x_0, x_1 \in D,\, \|x_0 - x_1\| = \gamma,\, -x_0 + 2x_1 \in D} \left(f(x_0) - 2f(x_1) + f(-x_0 + 2x_1) \right) > 0. \tag{25}$$

Then the disturbed function $\tilde{f} = f + g$ is inner γ-convex (with $\nu = 1$) if the disturbance function satisfies

$$|g(x)| \le h_2(\gamma)/4 \quad \text{for all } x \in D. \tag{26}$$

Proof. For all $x_0, x_1 \in D$ satisfying $\|x_0 - x_1\| = \gamma$ and $-x_0 + 2x_1 \in D$, (25) and (26) imply

$$\begin{aligned}
&\tilde{f}(x_0) - 2\,\tilde{f}(x_1) + \tilde{f}(-x_0 + 2x_1) \\
&= f(x_0) + g(x_0) - 2\,f(x_1) - 2\,g(x_1) + f(-x_0 + 2x_1) + g(-x_0 + 2x_1) \\
&\geq f(x_0) - h_2(\gamma)/4 - 2\,f(x_1) - 2\,h_2(\gamma)/4 + f(-x_0 + 2x_1) - h_2(\gamma)/4 \\
&= f(x_0) - 2f(x_1) + f(-x_0 + 2x_1)) - h_2(\gamma) \\
&\geq 0.
\end{aligned} \tag{27}$$

Hence, for $\lambda = 2$,

$$\begin{aligned}
\tilde{f}((1-\lambda)x_0 + \lambda x_1) &= \tilde{f}(-x_0 + 2x_1) \\
&\geq -\tilde{f}(x_0) + 2\,\tilde{f}(x_1) \\
&= (1-\lambda)\tilde{f}(x_0) + \lambda \tilde{f}(x_1),
\end{aligned}$$

i.e., (24) holds for $\nu = 1$ and $\lambda = 2$. Due to Proposition 10, $\tilde{f}$ is inner γ-convex. ∎

It is worth emphasizing that in Proposition 15 function f is not required to be convex. Condition (25) means only a concrete demand to the γ-midpoint convexity.

Proposition 16. *Let $\gamma > 0$ and let $f : D \subset X \to \mathbb{R}$ fulfil (25). Then the disturbed function $\tilde{f} = f + g$ is strictly inner γ-convex (with $\nu = 1$) if the disturbance function satisfies*

$$|g(x)| < h_2(\gamma)/4 \quad \text{for all} \quad x \in D. \tag{28}$$

Proof. Since "$\leq$" in (26) is replaced by "$<$" in (28), "$\leq$" in (27) must also be replaced by "$<$" accordingly. Following, for $\lambda = 2$, we have

$$\begin{aligned}
\tilde{f}((1-\lambda)x_0 + \lambda x_1) &= \tilde{f}(-x_0 + 2x_1) \\
&> -\tilde{f}(x_0) + 2\,\tilde{f}(x_1) \\
&= (1-\lambda)\tilde{f}(x_0) + \lambda \tilde{f}(x_1),
\end{aligned}$$

i.e., (23) holds for $\nu = 1$ and $\lambda = 2$. By definition, $\tilde{f}$ is strictly inner γ-convex. ∎

Example 2. Let X be the n-dimensional Euclidian space and $f : D \to \mathbb{R}$ be defined by

$$f(x) = \|x\|^2 = \sum_{i=1}^{n} \xi_i^2, \quad x = (\xi_1, ..., \xi_n) \in X.$$

Then, for all $x_0 = (\xi_{01}, ..., \xi_{0n}) \in X$ and $x_1 = (\xi_{11}, ..., \xi_{1n}) \in X$ satisfying $\|x_0 - x_1\| = \gamma$, we have

$$\begin{aligned} f(x_0) - 2f(x_1) + f(-x_0 + 2x_1) &= \|x_0\|^2 - 2\,\|x_1\|^2 + \| - x_0 + 2\,x_1\|^2 \\ &= \sum_{i=1}^{n} \Big(\xi_{0i}^2 - 2\,\xi_{1i}^2 + (-\xi_{0i} + 2\,\xi_{1i})^2 \Big) \\ &= \sum_{i=1}^{n} 2\Big(\xi_{0i}^2 - 2\,\xi_{0i}\xi_{1i} + \xi_{1i}^2 \Big) \\ &= 2\,\|x_0 - x_1\|^2 \\ &= 2\,\gamma^2, \end{aligned}$$

which yields by (25) that $h_2(\gamma) = 2\,\gamma^2$. Therefore, due to Proposition 15, the disturbed function $\tilde{f} = f + g$ is inner γ-convex if the disturbance function $g : X \to \mathbb{R}$ satisfies

$$|g(x)| \le h_2(\gamma)/4 = \gamma^2/2 \quad \text{for all} \quad x \in X,$$

and by Proposition 16, $\tilde{f} = f + g$ is strictly inner γ-convex if g fulfils

$$|g(x)| < h_2(\gamma)/4 = \gamma^2/2 \quad \text{for all} \quad x \in X.$$

4. Concluding Remarks

If $f : D \subset X \to \mathbb{R}$ is convex and if disturbance function $g : D \to \mathbb{R}$ fulfils both conditions (13) and (26), i.e.,

$$|g(x)| \le \min\{h_1(\gamma)/2, h_2(\gamma)/4\} \ \text{ for all } x \in D,$$

then the disturbed function $\tilde{f} = f + g$ is both outer γ-convex and inner γ-convex. For instance, due to Example 1 and Example 2, if f is defined by (19) and if g satisfies

$$|g(x)| \le \gamma^2/8 \ \text{ for all } x \in X,$$

then $f + g$ is both outer γ-convex and inner γ-convex. Following, $f + g$ inherits all properties of outer γ-convex functions and inner γ-convex ones.

In this paper, only some properties of outer γ-convex functions and inner γ-convex functions are mentioned. Other properties can be found in [10 - 12], and [15].

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