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On Hopfian and Co-Hopfian Modules*

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Abstract. A R-module M is said to be Hopfian(respectively Co-Hopfian) in case any surjective(respectively injective) R-homomorphism is automatically an isomorphism. In this paper we study sufficient and necessary conditions of Hopfian and Co-Hopfian modules. In particular, we show that the weakly Co-Hopfian regular module ${}_RR$ is Hopfian, and the left R-module M is Co-Hopfian if and only if the left $R[x]/(x^{n+1})$-module $M[x]/(x^{n+1})$ is Co-Hopfian, where n is a positive integer.

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1. Introduction

Throughout this paper, unless stated otherwise, ring R is associative and has an identity, M is a left R-module. An essential submodule K of M is denoted by $K \leq_e M$, and a superfluous submodule L of M is denoted by $L \ll M$.

In 1986, Hiremath introduced the concept of the Hopfian module [1]. Lately, the dual of Hopfian, i.e., the concept of Co-Hopfian was given, and such modules

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have been investigated by many authors, e.g. [1-8]. In [9], it is proved that if ${}_RR$ is Artinian then ${}_RR$ is Noetherian. In the second section, we introduce the concept of generalized Artinian and generalized Noetherian, which are Co-Hopfian and Hopfian, respectively, and prove that if ${}_RR$ is generalized Artinian then ${}_RR$ is generalized Noetherian. Varadarajan [2] showed that if ${}_RR$ is Co-Hopfian then ${}_RR$ is Hopfian, and we considerably strengthen this result by proving that ${}_RR$ is Hopfian under the condition of weak Co-Hopficity. So we get the following relationships for the regular module ${}_RR$:

$$\begin{array}{ccccccc} Artinian & \Rightarrow & generalized\ Artinian & \Rightarrow Co-Hopfian & \Rightarrow & weakly\ Co-Hopfian \\ \Downarrow & & \Downarrow & \Downarrow & \swarrow & \Downarrow \\ Noetherian & \Rightarrow & generalized\ Noetherian & \Rightarrow \quad Hopfian & \Rightarrow & generalized\ Hopfian \end{array}$$

Varadarajan [2, 3] showed that the left R-module M is Hopfian if and only if the left $R[x]$-module $M[x]$ is Hopfian if and only if the left $R[x]/(x^{n+1})$-module $M[x]/(x^{n+1})$ is Hopfian, lately, Liu extended the result to the module of generalized inverse polynomials [8]. But for any $0 \neq M$, the $R[x]$-module $M[x]$ is never Co-Hopfian. In fact, the map "multiplication by x" is an injective non-surjective map, where x is a commuting indeterminate over R. In the third part of the paper, the Co-Hopficity of the polynomial module $M[x]/(x^{n+1})$ is considered. We showed that the R-module M is Co-Hopfian if and only if the $R[x]/(x^{n+1})$-module $M[x]/(x^{n+1})$ is Co-Hopfian, where n is any positive integer. The following are several conceptions we will use in this paper.

Definition 1.1. [2] *Let M be a left R-module,*

(1) *M is called Hopfian, if any surjective R-homomorphism $f : M \longrightarrow M$ is an isomorphism.*

(2) *M is called Co-Hopfian, if any injective R-homomorphism $f : M \longrightarrow M$ is an isomorphism.*

Definition 1.2. [12] *A left R-module M is said to be weakly Co-Hopfian if every injective R-endomorphism $f : M \to M$ is essential, i.e., $f(M) \leq_e M$.*

Definition 1.3. ([13]) *A left R-module M is said to be generalized Hopfian if every surjective R-endomorphism f of M is superfluous, i.e., $Ker(f) \ll M$.*

2. Hopfian and Co-Hopfian Modules

Definition 2.1. *Let M be a left R-module,*

(1) *M is called generalized Noetherian, if for any R-homomorphism $f : M \longrightarrow M$, there exists $n \geq 1$ such that $Ker(f^n) = Ker(f^{n+i})$ for $i = 1, 2, \cdots$.*

(2) *M is called generalized Artinian, if for any R-homomorphism $f : M \longrightarrow M$, there exists $n \geq 1$ such that $Im(f^n) = Im(f^{n+i})$ for $i = 1, 2, \cdots$.*

Obviously, any Noetherian (resp. Artinian) module is generalized Notherian (resp. Artinian), but the converses are not true.

Example 2.1. The Z-module $M = \bigoplus_{p \in \mathcal{P}} Z_p$ is both generalized Noetherian and generalized Artinian, but it is neither Noetherian nor Artinian, where $\mathcal{P}$ is the set of all primes.

Proof. Using the fact that $Hom_Z(Z_p, Z_q) = 0$ if p and q are distinct primes we see that any Z-endomorphism of M has the form of $f = \bigoplus_{p \in \mathcal{P}} f_p$, where every $f_p : Z_p \longrightarrow Z_p$ ($p \in \mathcal{P}$ is prime) is a Z-endomorphism, therefore, there are $Im(f^n) = Im(f^{n+i})$ and $\text{Ker}\,(f^n) = \text{Ker}\,(f^{n+i})$ for any positive integer n and i. It is easy to prove that M is neither Noetherian nor Artinian. ∎

Thus Noetherian and Artinian modules are properly contained in generalized Noetherian and generalized modules respectively. It is also obvious that ${}_RR$ is generalized Artinian if and only if there exists $n \geq 1$ such that $Rr^n = Rr^{n+1}$ for any $r \in R$ and ${}_RR$ is generalized Noetherian if and only if there exists $n \geq 1$ such that $\ell_R(r^n) = \{x \in R | xr^n = 0\} = \{x \in R | xr^{n+1} = 0\} = \ell_R(r^{n+1})$ for any $r \in R$. A ring R is called left π-regular if there are $n \geq 1$ and $s \in R$ such that $r^n = sr^{n+1}$ for any $r \in R$. By [10], R is left π-regular if and only if R is right π-regular. It is well known that if ${}_RR$ is Artinian then it is Noetherian, the following extends this result to generalized Artinian and generalized Noetherian.

Theorem 2.1. *Let R be a ring, if ${}_RR$ is generalized Artinian then ${}_RR$ is generalized Noetherian.*

Proof. Let $f : R \to R$ be any R-endomorphism and $r \in R$ satisfy $r = f(1)$, then there exists a positive integer n such that $Rr^n = \Im(f^n) = Im(f^{n+i}) = Rr^{n+i}$ for $i = 1, 2, \cdots$. It is clear that R is left π-regular, so R is right π-regular by [10], which means that there are $m \geq 1$ and $s \in R$ such that $r^m = r^{m+1}s$. Let $k = \max\{n, m\}$, then we have that $Rr^k = Rr^{k+1}$ and $r^k = r^{k+1}t$, where $t = r^{k-m}s$. Since $\text{Ker}\,(f^k) = \{x \in R | xr^k = 0\} = \ell_R(r^k)$, we only have to show $\ell_R(r^k) = \ell_R(r^{k+1})$. It is obvious that $\ell_R(r^k) \subseteq \ell_R(r^{k+1})$. Let $x \in \ell_R(r^{k+1})$, then $xr^k = x(r^{k+1}t) = (xr^{k+1})t = 0$, so $x \in \ell_R(r^k)$, thus we get $\ell_R(r^{k+1}) \subseteq \ell_R(r^k)$. ∎

It is proved in [11, Prop.1.14] that Noetherian (resp. Artinian) modules are Hopfian (resp. Co-Hopfian). In fact, the results can be extended to the following, and the proof is the same, so we omit it.

Theorem 2.2. *Let M be a left R-module.*
(1) *If M is generalized Noetherian then M is Hopfian,*
(2) *If M is generalized Artinian then M is Co-Hopfian.*

Question 2.1 Is any Hopfian module M generalized Noetherian?

Question 2.2. Is any Co-Hopfian module M generalized Artinian?

We have not an answer to Question 2.1, but we have a negative answer to 2.2 by the following example.

Example 2.2. Let the ring

$$R = \begin{pmatrix} Z/2Z & Z/2Z \\ 0 & Z_{(2)} \end{pmatrix},$$

where $Z_{(2)}$ is 2-localization of Z, namely $Z_{(2)} = \{\frac{m}{n} | (n,2) = 1\}$. Then R is Co-Hopfian as R-module and not generalized Artinian.

Proof. From [2, Ex.1.5], ${}_RR$ is Co-Hopfian. It is easy to check that

$$R\begin{pmatrix} \alpha & \beta \\ 0 & 2b \end{pmatrix}^n \supset R\begin{pmatrix} \alpha & \beta \\ 0 & 2b \end{pmatrix}^{n+1}$$

for any positive integer n. ■

Recall that an element a of a ring R is called a left (resp. right) unit if there exists $b \in R$ such that $ba = 1$ $(ab = 1)$. We call $c \in R$ left (resp. right) regular if $\ell_R(c) = \{r \in R | rc = 0\} = 0$ (resp. $\gamma_R(c) = \{r \in R | cr = 0\} = 0$). It is clear that ${}_RR$ is Co-Hopfian if and only if there exists $a \in R$ such that $ac = 1$ for any left regular element c of R, ${}_RR$ is weakly Co-Hopfian if and only if there is $Rc \leq_e {}_RR$ for any left regular element $c \in R$, ${}_RR$ is Hopfian if and only if $\ell_R(a) = 0$ for any left unit $a \in R$. Varadarajan [2] proved that if ${}_RR$ is Co-Hopfian then ${}_RR$ is Hopfian. We weaken the condition of this result as follow.

Theorem 2.3. *Let R be a ring, if ${}_RR$ is weakly Co-Hopfian then ${}_RR$ is Hopfian.*

Proof. Suppose that ${}_RR$ is not Hopfian, then there is a left unit $a \in R$ such that $0 \neq \ell_R(a) \leq {}_RR$. By the condition of the weak Co-Hopficity of ${}_RR$, we have $Rc \leq_e {}_RR$, where $c \in R$ satisfies $ca = 1$, so $Rc \bigcap \ell_R(a) \neq 0$. For any $x \in Rc \bigcap \ell_R(a)$, we have that $x = rc$ for some element $r \in R$, therefore $r = r(ca) = (rc)a = xa = 0$ and $x = 0$, this contradicts $Rc \bigcap \ell_R(a) \neq 0$. Thus the result is proved. ■

It is well known that Hopfian modules are generalized Hopfian, but the converse is not true. So we easily get the following result.

Corollary 2.1. *If ${}_RR$ is weakly Co-Hopfian then ${}_RR$ is generalized Hopfian.*

Let $\{S_\lambda\}_{\lambda \in \Lambda}$ be a family of rings indexed by a set Λ, $\prod_\Lambda S_\lambda = S$ be the Cartesian product of $\{S_\lambda\}_{\lambda \in \Lambda}$. A ring R is called the subdirect product of the rings $\{S_\lambda\}_{\lambda \in \Lambda}$, if there exists an injective ring homomorphism $\phi : R \to S = \prod_\Lambda S_\lambda$ such that $\pi_\lambda \phi$ is an surjective ring homomorphism for any $\lambda \in \Lambda$, where each $\pi_\lambda : S = \prod_\Lambda S_\lambda \to S_\lambda$ is the projection onto the λth components[14]. It is easy to show that R is the subdirect product of a family rings if and only if there exists a family of ideals of $\{I_\lambda\}_{\lambda \in \Lambda}$ of R such that R is the subdirect product of $\{R/I_\lambda\}_{\lambda \in \Lambda}$, where $\{I_\lambda\}_{\lambda \in \Lambda}$ satisfy $\bigcap_\Lambda I_\lambda = 0$.

Proposition 2.1. *Let a ring R be the subdirect product of a family of rings $\{S_\lambda\}_{\lambda \in \Lambda}$, if each S_λ is Hopfian as an S_λ-module then ${}_RR$ is Hopfian.*

Proof. Let $\{I_\lambda\}_{\lambda\in\Lambda}$ be a family of ideals of R such that R is the subdirect product of $\{R/I_\lambda\}_{\lambda\in\Lambda}$, where $\{I_\lambda\}_{\lambda\in\Lambda}$ satisfy $\bigcap_\Lambda I_\lambda = 0$. For any surjective R-homomorphism $f : R \to R$, define $f_i : R/I_i \longrightarrow R/I_i, r + I_i \longmapsto rf(1) + I_i$. If $r_1 - r_2 \in I_i$, then $f_i(r_1 + I_i) - f_i(r_2 + I_i) = r_1f(1) - r_2f(1) + I_i = (r_1 - r_2)f(1) + I_i = \overline{0}$, thus each f_i is well defined. Clearly, each f_i is a surjective R/I_i-homomorphism, also each f_i is a surjective R-homomorphism, therefore $Ker(f_i) = \overline{0}$ since $R/I_i, i \in \Lambda$ are Hopfian, we get that $\{r \in R | f(r) \in I_i\} = I_i$, thus $Ker(f) \subseteq I_i, i \in \Lambda$. So $Ker(f) \subseteq \bigcap_{i\in\Lambda} I_i = 0$. It follows that f is an injective R-homomorphism. ∎

Recall that $M^* = Hom_R(M, R)$ is said to be the R-dual of ${}_RM$, also M^{**} is called the double dual of ${}_RM$. If the evaluation map $\sigma : M \longrightarrow M^{**}$ defined by $[\sigma_M(m)](\alpha) = \alpha(m)$ is injective, where $m \in M$ and $\alpha \in M^*$, then M is called torsionless. M is torsionless if and only if $Rej(M, R) = \bigcap_{f\in Hom_R(M,R)} Ker(f) = 0$. It is shown in [15, Prop.3.1] that if the R-dual M^* is weakly Co-Hopfian and M is torsionless, then M is generalized Hopfian. Similarly, we have the following.

Proposition 2.2. *Let the R-dual M^* of ${}_RM$ be Co-Hopfian, if M is torsionless, then M is Hopfian.*

Proof. Let $\phi : M \longrightarrow M$ be any surjective R-homomorphism, then $\overline{\phi} : M^* \longrightarrow M^*$ defined by $\overline{\phi}(f) = f\phi$ for any $f \in M^*$ is an injective R-homomorphism. Since M^* is Co-Hopfian, we get that $\Im(\overline{\phi}) = M^*$, which means that there exists $f \in M^*$ such that $g = f\phi$ for every $g \in M^*$, by $g(\operatorname{Ker}(\phi)) = f\phi(\operatorname{Ker}(\phi)) = 0$ and $\Re j(M, R) = 0$, we have that $\operatorname{Ker}(\phi) = 0$. ∎

Corollary 2.2. *Let ${}_RM_S$ be a bimodule, $E = Hom_R(M, M)$, if the right S-module E is Co-Hopfian, then the left R-module M is Hopfian.*

Proof. By Proposition 2.2, it is clear since $Rej(M, M) = 0$. ∎

Corollary 2.3. *Let ${}_RM_S$ be a bimodule, $S = End_R(M)$, ${}_RM$ is quasi-injective. If ${}_SS$ is Hopfian, then the left R-module M is Co-Hopfian.*

Proof. Let $\phi : M \longrightarrow M$ be an injective R-homomorphism, then we have that $\overline{\phi} : S \longrightarrow M$ defined by $\overline{\phi}(f) = f\phi$ for any $f \in S$ is a surjective S-homomorphism. Since ${}_RM$ is quasi-injective, there is $f : M \longrightarrow M$ such that $f\phi = 1_M$, so $\phi f\phi = \phi$, i.e., $\overline{\phi}(\phi f - 1_M) = 0$, by the Hopficity of ${}_SS$, we get that $\overline{\phi}$ is an injective S-homomorphism, so $\phi f = 1_M$, which implies that ϕ is a surjective R-homomorphism. ∎

3. The Co-Hopficity of $M[x]/(x^{n+1})$

Let x be a commuting indeterminate over R, M be a left R-module. Set

$M[x]/(x^{n+1}) = \{\sum_{i=0}^{n} m_i x^i + (x^{n+1}) | m_i \in M, i = 0, 1, \ldots n.\}$, $R[x]/(x^{n+1}) = \{\sum_{i=0}^{n} r_i x^i + (x^{n+1}) | r_i \in R, i = 0, 1, \ldots, n.\}$. The addition in $R[x]/(x^{n+1})$ and $M[x]/(x^{n+1})$ are given componently, and the $R[x]/(x^{n+1})$-module structure is defined by

$$(\sum_{i=0}^{n} r_i x^i + (x^{n+1}))(\sum_{i=0}^{n} m_j x^j + (x^{n+1})) = \sum_{t=0}^{n} m'_t x^t + (x^{n+1}),$$

where each $m'_t = \sum_{i+j=t} r_i m_j$ for any $r_i \in R$ and $m_j \in M$.
The left $R[x_1, x_2, \ldots, x_k]/(x_1^{n_1+1}, x_2^{n_2+1}, \ldots, x_k^{n_k+1})$-module $M[x_1, x_2, \ldots, x_k]/(x_1^{n_1+1}, x_2^{n_2+1}, \ldots, x_k^{n_k+1})$ is defined similarly.

Lemma 3.1. *Let x be a commuting indeterminate over R, $\alpha \in End_{R[x]/(x^{n+1})}(M[x]/(x^{n+1}))$, where n is a positive integer. If $\alpha(f(x)) \neq 0$ for some element $f(x) = \sum_{j=0}^{n} m_j x^j + (x^{n+1}) \in M[x]/(x^{n+1})$, then $\partial(f(x)) \leq \partial(\alpha(f(x)))$, where $\partial(f(x))$ denotes the smallest index number of x of the polynomial $f(x)$. In particular, we have that $\partial(f(x)) = \partial(\alpha(f(x)))$ when α is injective.*

Proof. We denote $x + (x^{n+1})$ by u. So $M[x]/(x^{n+1}) = M + Mu + \cdots + Mu^n$, where u is a commuting indeterminate over R and $u^{n+1} = 0$(the following is the same). We have that $\alpha(m) = \sum_{j=0}^{n} m_j u^j$ for any element $m \in M$, therefore $\alpha(mu^k) = u^k(\sum_{j=0}^{n} m_j u^j) = \sum_{j=0}^{n-k} m_j u^{j+k}$ since α is injective, where $0 \leq k \leq n$. Obviously, $\partial(\sum_{j=0}^{n} m_j u^j) \leq \partial(\alpha(\sum_{j=0}^{n} m_j u^j))$, that is $\partial(f(x)) \leq \partial(\alpha(f(x)))$. When α is injective, suppose $\alpha(mu^k) = \sum_{j=k+1}^{n} m_j u^j$, then we get that $\alpha(mu^n) = \alpha(u^{n-k}(mu^k)) = u^{n-k} \sum_{j=k+1}^{n} m_j u^j = 0$, thus $m = 0$. So $\partial(f(x)) = \partial(\alpha(f(x)))$. ∎

Theorem 3.1. *Let x be a commuting indeterminate over R, M a left R-module and n a positive integer. Then the left R-module M is Co-Hopfian if and only if the left $R[x]/(x^{n+1})$-module $M[x]/(x^{n+1})$ is Co-Hopfian.*

Proof. ($\Rightarrow$)Let $\alpha : M[u] \longrightarrow M[u]$ be any injective $R[u]$-module homomorphism. Define the R-module homomorphisms $\tau : M \longrightarrow M[u]$ via $\tau(m) = m$ for any $m \in M$ and $p_i : M[u] \longrightarrow M$ via $p_i(\sum_{j=0}^{n} m_j u^j) = m_i, i = 0, 1, \ldots, n$ for any $\sum_{j=0}^{n} m_j u^j \in M[u]$, then τ is injective and each p_i is surjective. Since for some element $m \in M$ satisfying $p_0 \alpha \tau(m) = 0$, we obtain that $p_0 \alpha(m) = p_0(\sum_{j=0}^{n} m_j u^j) = m_0 = 0$, by Lemma 3.1, $m = 0$. Hence, $p_0 \alpha \tau : M \longrightarrow M$ is an injective R-homomorphism, so $p_0 \alpha \tau$ is an isomorphism by the Co-Hopficity of M.

For any $\sum_{j=0}^{n} m_j u^j \in M[u]$, there is $m'_0 \in M$ such that $p_0 \alpha \tau(m'_0) = m_0$. Assume $\alpha(m'_0) = m_0 + a_1^{(0)} u + \cdots + a_n^{(0)} u^n \in M[u]$, if $m_1 \neq a_1^{(0)}$, then there is $m'_1 \in M$ such that $p_0 \alpha \tau(m'_1) = m_1 - a_1^{(0)}$, where $\alpha(m'_1) = (m_1 - a_1^{(0)}) + a_1 u + \cdots + a_n u^n \in M[u]$. Thus we have that $\alpha(m'_0 + m'_1 u) = \alpha(m'_0) + u\alpha(m'_1) = m_0 + m_1 u + a_2^{(1)} u^2 + \cdots + a_n^{(1)} u^n$. If $m_2 \neq a_2^{(1)}$, continue the above process at most $n + 1$ times, we will obtain that $f(u) = \sum_{j=0}^{n} m'_j u^j \in M[u]$ satisfies

$\alpha(f(u)) = \sum_{j=0}^{n} m_j u^j$. So α is surjective and the left $R[u]$-module $M[u]$ is Co-Hopfian.

($\Leftarrow$) Let $g : M \longrightarrow M$ be any injective R-homomorphism. Define $\alpha : M[u] \longrightarrow M[u]$, $\sum_{j=0}^{n} m_j u^j \longmapsto \sum_{j=0}^{n} g(m_j)u^j$, it is easy to prove that α is an injective $R[u]$-homomorphism. Therefore α is an isomorphism by the Co-Hopficity of the left $R[u]$-module $M[u]$. So there exists $\sum_{j=0}^{n} m_j u^j \in M[u]$ such that $\alpha(\sum_{j=0}^{n} m_j u^j) = m$ for any $m \in M$, i.e., $g(m_0) = m$, now we obtain that g is surjective. ∎

Theorem 3.2. *Let M be a left R-module, $x_1, x_2, \ldots, x_k$ k commuting indeterminates over R, then the left R-module M is Co-Hopfian if and only if the left $R[x_1, x_2, \ldots, x_k]/(x_1^{n_1+1}, x_2^{n_2+1}, \ldots, x_k^{n_k+1})$-module $M[x_1, x_2, \ldots, x_k]/(x_1^{n_1+1}, x_2^{n_2+1}, \ldots, x_k^{n_k+1})$ is Co-Hopfian for any positive integers $n_1, n_2, \ldots, n_k$.*

Proof. Notice that the left $(R[x_1, \ldots, x_{k-1}]/(x_1^{n_1+1}, \ldots, x_{k-1}^{n_{k-1}+1}))[x_k]/(x_k^{n_k+1})$-module isomorphism $(M[x_1, \ldots, x_{k-1}]/(x_1^{n_1+1}, \ldots, x_{k-1}^{n_{k-1}+1}))[x_k]/(x_k^{n_k+1}) \simeq M[x_1, \ldots, x_k]/(x_1^{n_1+1}, \ldots, x_k^{n_k+1})$ and ring isomorphism $(R[x_1, \ldots, x_{k-1}]/(x_1^{n_1+1}, \ldots, x_{k-1}^{n_{k-1}+1}))[x_k]/(x_k^{n_k+1}) \simeq R[x_1, \ldots, x_k]/(x_1^{n_1+1}, \ldots, x_k^{n_k+1})$. By induction, it is easy to prove. ∎

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