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Some Examples of ACS-Rings

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Abstract. A ring R is called a right ACS-ring if the annihilator of any element in R is essential in a direct summand of R. In this note we will exhibit some elementary but important examples of ACS-rings. Let R be a reduced ring, then R is a right ACS-ring if and only if $R[x]$ is a right ACS-ring. Let R be an α-rigid ring. Then R is a right ACS-ring if and only if the Ore extension $R[x;\alpha]$ is a right ACS-ring. A counterexample is given to show that the upper matrix ring $T_n(R)$ over a right ACS-ring R need not be a right ACS-ring.

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1. Introduction and Preliminaries

Throughout this paper, unless otherwise stated, all rings are associative rings with identity and all modules are unitary right R-modules.

In [1] a submodule N of M is called an essential submodule, denoted by $N \leq_e M$, if for any nonzero submodule L of M, $L \cap N \neq 0$. (Note that we are employing the convention that $0 \leq_e 0$.) Let M be a module and N a submodule of M. Then $N \leq_e M$ if and only if for any $0 \neq m \in M$, there is $r \in R$ such that $0 \neq mr \in N$.

From [2] a ring R is called a right ACS-ring if the right annihilator of every element of R is essential in a direct summand of R_R; or equivalently, R is a right ACS-ring if, for any $a \in R$, $aR = P \oplus S$ where P_R is a projective right ideal and S_R is a singular right ideal of R. A ring R is called a right p.p.-ring if every principal ideal of R is projective; or equivalently, the right annihilator of every element of R is generated by an idempotent of R. It is known that for a right nonsingular ring R, R is a right ACS-ring if and only if R is a right p.p.-ring. Also it is shown in [4] that polynomial rings over right p.p.-rings need not be right p.p.-rings.

From [5] a ring R is called right p.q-Baer if the right annihilator of right principal ideal of R is generalized by an idempotent of R. A ring R is called reduced if it has no nonzero nilpotent. In a reduced ring R, all idempotents are central in R and $r_R(X) = l_R(X)$ for any subset X of R. A ring R is called abelian if all idempotents of R are central. Reduced rings are abelian.

A ring R is called *Armendariz* if whenever polynomials $f(x) = a_0+a_1x+\cdots+a_mx^m,\ g(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$ for each i, j(see [7]). Reduced rings are Armendariz rings and Armendariz rings are abelian (see [7, Lemma 7]).

In Sec. 2, we first characterize reduced right ACS-ring and then show that R is a right ACS-ring if and only if S is a right ACS-ring, where $S = R * \mathbb{Z}$ is the Dorroh extension of R by $\mathbb{Z}$.

In Sec. 3, it is shown that, for a reduced ring R, R is a right ACS-ring if and only if $R[x]$ is a right ACS-ring. Let R be an α-rigid ring. Then R is a right ACS-ring if and only if the Ore extension $R[x;\alpha]$ is a right ACS-ring.

In Sec. 4, a counterexample is given to show that the upper matrix ring $T_n(R)$ over a right ACS-ring R need not be a right ACS-ring.

Let R be a ring and $a \in R$, we denote by $r_R(a) = \{r \in R \mid ar = 0\}$(resp.$l_R(a) = \{r \in R \mid ra = 0\}$) the right (resp.left) annihilator of a.

2. Some Results and the Dorroh Extension of ACS-Rings

In this section we will first characterize reduced ACS-rings and then investigate the Dorroh extension of ACS-rings. Firstly, it is easy to see:

Lemma 2.1. *Let R be a right nonsingular ring. Then the following are equivalent for any $a \in R$ and a right ideal I of R:*
(1) $r_R(a) \leq_e I$;
(2) $r_R(a) = I$.

Theorem 2.1. *Let R be a reduced ring. Then the following are equivalent:*
(1) *R is a right ACS-ring;*
(2) *The right annihilator of every finitely generated right ideal is essential (as right ideal) in a direct summand;*
(3) *The right annihilator of every principal right ideal is essential (as right ideal) in a direct summand;*
(4) *The right annihilator of every principal ideal is essential (as right ideal) in a direct summand;*
(5) *The left annihilator of every principal ideal is essential (as a left ideal) in a direct summand;*
(6) *The left annihilator of every finitely generated left ideal is essential (as a left ideal) in a direct summand;*

(7) *The left annihilator of every principal left ideal is essential (as left ideal) in a direct summand;*
(8) *R is a left ACS-ring.*

Proof.
(1) ⇒ (2). Let $X = \sum_{i=1}^{n} x_i R$ be any finitely generated right ideal of R. Then $r_R(X) = \cap_{i=1}^{n} r_R(x_i R)$. Since R is a reduced right ACS-ring, then there are $e_i^2 = e_i \in R$ such that $r_R(x_i R) = r_R(x_i) \leq_e e_i R$ for $1 \leq i \leq n$. Set $e = e_1 e_2 \cdots e_n \in R$, then, since R is reduced, we have $e^2 = e$ and $\cap_{i=1}^{n} e_i R = eR$. Thus we have $r_R(X) \leq_e eR$.

(2) ⇒ (1). This is obvious.

(1) ⇔ (3). Trivially.

(3) ⇔ (4). Note that $r_R(aR) = r_R(RaR)$ for any $a \in R$.

(4) ⇔ (5). Note that in a reduced ring R $r_R(X) = l_R(X)$ for any subset X of R and that any idempotent of R is central.

(5) ⇔ (7). Note that $l_R(aR) = l_R(RaR)$ for any $a \in R$.

(5) ⇔ (6). The proof is similar to that of (2) ⇔ (3).

(7) ⇔ (8). Trivially. ■

Recall that a commutative ring R is nonsingular if and only if R is reduced; and that a right nonsingular ring R is a right ACS-ring if and only if R is a right p.p.-ring. Thus as an immediate consequence of the theorem and lemma above, we have:

Corollary 2.1. *Let R be a commutative reduced ring. Then the following are equivalent:*
(1) *R is a right ACS-ring;*
(2) *The right annihilator of every finitely generated right ideal is essential (as right ideal) in a direct summand;*
(3) *The right annihilator of every principal right ideal is essential (as right ideal) in a direct summand;*
(4) *The right annihilator of every principal ideal is essential (as right ideal) in a direct summand;*
(5) *R is a right p.p.-ring;*
(6) *R is a right p.q-Baer ring;*
(7) *The left annihilator of every finitely generated left ideal is essential (as left ideal) in a direct summand;*
(8) *The left annihilator of every principal left ideal is essential (as left ideal) in a direct summand;*
(9) *The left annihilator of every principal left ideal is essential (as left ideal) in a direct summand;*
(10) *R is a left ACS-ring;*
(11) *R is a left p.p.-ring;*
(12) *R is a left p.q-Baer ring.*

Secondly, we consider the Dorroh extension of ring R by $\mathbb{Z}$. Let R be a ring and $\mathbb{Z}$ the ring of all integers. Let $S = R * \mathbb{Z}$ be the Dorroh extension of R by $\mathbb{Z}$. As sets, $S = R \times \mathbb{Z}$, the Cartesian product of R and $\mathbb{Z}$. The addition and multiplication of S are defined as follows: for all $(r_i, n_i) \in S,\ i = 1, 2$

$$(r_1, n_1) + (r_2, n_2) = (r_1 + r_2, n_1 + n_2),$$
$$(r_1, n_1)(r_2, n_2) = (r_1 r_2 + n_1 r_2 + n_2 r_1, n_1 n_2),$$

S is an associative ring with identity $(0, 1)$.

Lemma 2.2. *Let R be a ring and $S = R * \mathbb{Z}$ the Dorroh extension of R by $\mathbb{Z}$. If S is a right ACS-ring, then so is R.*

Proof. Let $a \in R$, then $(a, 0) \in S$. Since S is a right ACS-ring, then there is an idempotent $s = (r, n) \in S$ such that $r_S((a, 0)) \leq_e sS$. Since $s^2 = s$, we have that either $n = 0$ or $n = 1$.

Case 1. If $n = 0$, then $r^2 = r \in R$. We now show that $r_R(a) \leq_e rR$. For any $x \in r_R(a)$, we have $0 = (a, 0)(x, 0)$ and $(x, 0) = (r, 0)(b, m) = (rb + mr, 0)$ for some $(b, m) \in S$. Thus $x \in rR$.

For $0 \neq rb \in rR$, $(0, 0) \neq (rb, 0) = (r, 0)(b, 0) \in (r, 0)S$. Thus there is $(c, m) \in S$ such that $0 \neq (rb, 0)(c, m) = (rbc + mrb, 0) \in r_S((a, 0))$. Obviously $0 \neq rb(c + m1_R) \in r_R(a)$. Therefore R is a right ACS-ring.

Case 2. If $n = 1$, then $t = 1 + r$ is an idempotent of R. We will show that $r_R(a) \leq_e tR$. Let $x \in r_R(a)$, then $(a, 0)(x, 0) = (0, 0)$ and $(x, 0) = (r, 1)(b, m) = (rb + b + mr, m)$ for some $(b, m) \in S$. So $m = 0$ and $x = (r + 1)b = tb \in tR$. Thus $r_R(a) \leq tR$.

Let $0 \neq tc \in tR$, then $(0, 0) \neq (tc, 0) = (r, 1)(c, 0) \in (r, 1)S$. Thus there is $(b, m) \in S$ such that $(0, 0) \neq (tc, 0)(b, m) = (tcb + mtc, 0) \in r_S((a, 0))$. Obviously $b + m1_R \neq 0$ and $tc(b + m1_R) \in r_R(a)$. Thus R is a right PCS-ring. ■

Lemma 2.3. *Let R be a right ACS-ring. Then $S = R * \mathbb{Z}$ is a right ACS-ring.*

Proof.
Case 1. Let $(a, m) \in S$ and $m \neq 0$. Then there is $e^2 = e \in R$ such that $r_R((a + m1_R)R) \leq_e eR$. We now show that $r_S((a, m)) \leq_e (e, 0)S$.

For any $(b, n) \in r_S((a, m))$, we have $(a, m)(b, n) = (ab + mb + na, mn) = (0, 0)$. Thus $n = 0$ and $b \in r_R((a + m1_R)) \leq eR$. Hence $b = er$ for some $r \in R$ and therefore $(b, 0) = (e, 0)(r, 0) \in (e, 0)S$.

For any $(0, 0) \neq (e, 0)(b, n) \in (e, 0)S$, we have $0 \neq e(b + n1_R)$. So there is $r \in R$ such that $0 \neq e(b + n1_R)r \in r_R((a + m1_R))$. Hence we have

$$(a, m)(e(b + n1_R), 0)(r, 0) = ((a + m1_R)e(b + n1_R)r, 0) = (0, 0).$$

Thus $r_S((a, m)) \leq_e (e, 0)S$.

Case 2. Let $(a, 0) \in S$, then there is $e^2 = e \in R$ such that $r_R(a) \leq_e eR$. We now show that $r_S((a, 0)) \leq_e (e - 1, 1)S$. It is easy to see that $r_S((a, 0)) \leq (e - 1, 1)S$.

For any $(0,0) \neq (e-1,1)(b,n) = (eb+ne-n1_R, n) \in (e-1,1)S$.

Subcase 1. If $n = 0$, then $eb \neq 0$ and there is $r \in R$ such that $0 \neq ebr \in r_R(a)$. Thus we have

$$(a,0)(e-1,1)(b,0)(r,0) = (aebr,0) = (0,0).$$

So $r_S((a,0)) \leq_e (e-1,1)S$.

Subcase 2. If $n \neq 0$ and $e(b+n1_R) = 0$, then we have $(a,0)(-n1_R, n) = (0,0)$. So $r_S((a,0)) \leq_e (e-1,1)S$.

Subcase 3. If $n \neq 0$ and $e(b+n1_R) \neq 0$, then there is $r \in R$ such that $0 \neq e(b+n1_R)r \in r_R(a)$. Thus we have

$$(a,0)(e-1,1)(b,n)(r,0) = (a,0)(e(b+n1_R)r,0) = (0,0).$$

So $r_S((a,0)) \leq_e (e-1,1)S$.

Therefore S is a right ACS-ring. ∎

As a consequence of these two lemmas, we have:

Theorem 2.2. *Let R be a ring and $S = R * \mathbb{Z}$ the Dorroh extension of R by $\mathbb{Z}$. Then R is a right ACS-ring if and only if S is a right ACS-ring.*

Now we investigate the trivial extension of R. Let R be a commutative ring and M an R-module. Denote by $S = R \propto M$ the trivial extension of R by M with pairwise addition and multiplication given by: $(a,m)(a',m') = (aa', am' + a'm)$. Note that any idempotent of S is of form $(e,0)$, where $e^2 = e \in R$.

Proposition 2.1. *Let R be a commutative ring and I an ideal of R. Let $S = R \propto I$ be the trivial extension of R by I. If S is an ACS-ring, so is R.*

Proof. Let $a \in R$, then $r_S((a,0)) \leq_e (e,0)S$ for some idempotent $(e,0) \in S$. It is easy to see that $r_R(a) \leq_e eR$ and that R is an ACS-ring. ∎

3. (SKEW) Polynomial Rings of ACS-Rings

As we know, polynomial rings over right p.p.-rings need not be right p.p.-rings. In this section we first investigate the relation between the ACS-property of ring R and that of the ring of all polynomials over ring R in indeterminant x.

Lemma 3.1. *Let R be any reduced ring and $S = R[x]$ the ring of all polynomials over R in indeterminant x. If S is a right ACS-ring, then so is R.*

Proof. Suppose that S is a right ACS-ring. Let $a \in R$, then there is an idempotent $e(x)$ of S such that $r_S(a) \leq_e e(x)S$. Let e_0 be the constant of $e(x)$, then, since R is reduced, we have $e(x) = e_0 \in R$. We now show that $r_R(a) \leq_e e_0R$.

It is easy to see that $r_R(a) \leq e_0R$. For any $0 \neq e_0r \in e_0R$, then there is $0 \neq g(x) \in S$ such that $0 \neq e_0rg(x) \in r_S(a)$. Thus $ae_0rg(x) = 0$. Let

$g(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0$ and $b_n \neq 0$. Then we have that $ae_0 r b_n = 0$ and that $r_R(a) \leq_e e_0 R$. Thus R is a right ACS-ring. ■

Remark 3.1. If R is not reduced and $S = R[x]$ is an ACS-ring, R may be an ACS-ring. For example, set $R = Z_4$. Then it is easy to see that $R[x]$ is an ACS-ring.

Let R be a right ACS-ring. When is $S = R[x]$ a right ACS-ring?

Lemma 3.2. *Let R be an Armendariz ACS-ring and $S = R[x]$. Then $S = R[x]$ is a right ACS-ring.*

Proof. Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be any nonzero polynomial of S. Since R is an Armendariz ACS-ring, then $r_R(a_i) \leq_e e_i R$ for some $e_i^2 = e_i \in R,\ 0 \leq i \leq n$. Set $e = e_0 e_1 \cdots e_n \in R$, then $e^2 = e$ and $\cap_{i=0}^n r_R(a_i) \leq_e \cap_{i=0}^n e_i R = eR$. Let $h(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0 \in r_S(f(x))$, then $f(x)h(x) = 0$ and $a_i b_j = 0$ for all $0 \leq i \leq n,\ 0 \leq j \leq m$. Thus $h(x) \in eS$ and $r_S(f(x)) \leq eS$.

Let $0 \neq ek(x) = ec_m x^m + ec_{m-1}x^{m-1} + \cdots + ec_1 x + ec_0 \in eS$. Since $ec_t \in eR$, we may find $r \in R$ such that $ec_t r \in \cap_{i=0}^n r_R(a_i)$ for all $0 \leq t \leq m$ and $ec_k r \neq 0$ for some $0 \leq k \leq m$. Thus $ek(x)r \neq 0$ and $f(x)ek(x)r = 0$, which means that $r_S(f(x)) \leq_e eS$. So S is a right ACS-ring. ■

Theorem 3.1. *Let R be a reduced ring. Then R is a right ACS-ring if and only if $R[x]$ is a right ACS-ring.*

Proof. This is an immediate consequence of the two lemmas above and of the fact that any reduced ring is an Armendariz ring. ■

Since R is reduced if and only if $R[x]$ is reduced, we have:

Corollary 3.1 *Let R be a reduced ring and X a nonempty set of commutative indeterminates. Then the following are equivalent:*
(1) *R is a right ACS-ring;*
(2) *$R[X]$ is a right ACS-ring.*

Now we consider the Ore extension of ACS-ring.

Recall that for a ring R with a ring endomorphism $\alpha : R \longrightarrow R$ and an α-derivation $\delta : R \longrightarrow R$, the Ore extension $R[x; \alpha, \delta]$ of R is the ring obtained by giving the polynomial ring over R with new multiplication

$$xr = \alpha(r)x + \delta(r)$$

for all $r \in R$. If $\delta = 0$, then we write $R[x; \alpha]$ for $R[x; \alpha, 0]$ and call it an *Ore extension of endomorphism type* (also called a *skew polynomial ring*).

Let α be an endomorphism of R. α is called a *rigid endomorphism* if $r\alpha(r) = 0$ implies $r = 0$ for all $r \in R$. A ring R is called α-rigid if there is a rigid endomorphism α of R. Any rigid endomorphism is a monomorphism and any

α-rigid ring is a reduced ring. But there is an endomorphism of a reduced ring which is not a rigid endomorphism.

Lemma 3.3. *Let R be an α-rigid ring and $R[x;\alpha,\delta]$ the Ore extension of R. Then we have the following:*
(1) *If $ab=0,\ a,\ b\in R$, then $a\alpha^n(b)=\alpha^n(a)b=0$ for any positive integer n;*
(2) *If $ab=0$, then $a\delta^m(b)=\delta^m(a)b=0$ for any positive integer m;*
(3) *If $a\alpha^k(b)=\alpha^k(a)b=0$ for some positive integer k, then $ab=0$;*
(4) *Let $p=\sum_{i=0}^m a_ix^i$ and $q=\sum_{j=0}^n b_jx^j$ in $R[x;\alpha,\delta]$. Then $pq=0$ if and only if $a_ib_j=0$ for all $0\le i\le m,\ 0\le j\le n$;*
(5) *If $e(x)^2=e(x)\in R[x;\alpha,\delta]$ and $e(x)=e_0+e_1x+\cdots+e_nx^n$, then $e=e_0\in R$.*

Proof. See Lemma 4, Proposition 6 and Corollary 7 of [3]. ∎

Using the lemma above we can show:

Theorem 3.2. *Let R be an α-rigid ring. Then R is a right ACS-ring if and only if the Ore extension $R[x;\alpha]$ is a right ACS-ring.*

Proof. Suppose that $S=R[x;\alpha]$ is a right ACS-ring and let $a\in R$. Then there is an idempotent $e(x)=e_nx^n+e_{n-1}x^{n-1}+\cdots+e_1x+e_0\in R[x;\alpha]$ such that $r_S(a)\le_e e(x)S$. Since R is α-rigid, then $e(x)=e_0\in R$. We now show that $r_R(a)\le_e e_0R$. It is easy to see that $r_R(a)\le e_0R$.

For any $0\ne e_0r_0\in e_0R$, then there is $0\ne h(x)=b_tx^t+b_{t-1}x^{t-1}+\cdots+b_1x+b_0\in S,\ (b_t\ne 0)$ such that $0\ne e_0r_0h(x)\in r_S(a)$. Thus there is $k\in\{0,1,\ldots,t\}$ such that $0\ne e_0r_0b_k\in r_R(a)$. So $r_R(a)\le_e e_0R$ and R is a right ACS-ring.

Conversely, suppose that R is a right ACS-ring. Let

$$g(x)=b_mx^m+b_{m-1}x^{m-1}+\cdots+b_1x+b_0\in S.$$

Then there are $e_i^2=e_i\in R$, such that $r_R(b_i)\le_e e_iR$ for all $i\in\{0,1,\ldots,m\}$. Set $e=e_0e_1\cdots e_m$. Since R is α-rigid, then R is reduced and $e^2=e\in R$. Furthermore, $\cap_{i=0}^m r_R(b_i)\le_e \cap_{i=0}^m e_iR=eR$. We now show that $r_S(g(x))\le_e eS$.

For any $f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0\in r_S(g(x))$, then $g(x)f(x)=0$ and $b_ia_j=0$ for all $0\le i\le m,\ 0\le j\le n$. Thus $a_j\in r_R(b_i)$ for all $0\le i\le m,\ 0\le j\le n$. So $a_j\in eR$ and $f(x)\in eS$. Hence $r_S(g(x))\le eS$.

Let $0\ne eh(x)=ec_tx^t+ec_{t-1}x^{t-1}+\cdots+ec_1x+ec_0\in eS$ with $0\ne ec_t$. We can find $r\in R$ such that $0\ne eh(x)r$ and $ec_j\alpha^j(r)\in\cap_i^n r_R(b_i)$ for all $j\in\{0,1,\ldots,t\}$. By the lemma above, since $b_i\alpha^i(ec_j\alpha^j(r))=0$ for all $0\le i\le m,\ 0\le j\le t$, we have $g(x)eh(x)r=0$. Thus $r_S(g(x))\le_e eS$ and S is a right ACS-ring. ∎

4. Formal Triangular Matrix Rings of ACS-Rings

It is shown in [6] that the class of quasi-Baer rings is closed under $n\times n$ matrix rings and under $n\times n$ upper (or lower) triangular matrix rings. It is natural to ask:

Is the class of ACS-rings closed under $n\times n$ upper (or lower) triangular matrix rings?

Proposition 4.1. *Let $T_n(R)$ be the $n \times n$ upper triangular matrix ring over R. If $T_n(R)$ is a right ACS-ring, so is R.*

Proof. We only show the case $n = 2$. The cases $n \geq 3$ are similar. Let $a \in R$, then $r_{T_2(R)}\left(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\right) \leq_e \begin{pmatrix} e & m \\ 0 & f \end{pmatrix} T_2(R)$ for some idempotent $\begin{pmatrix} e & m \\ 0 & f \end{pmatrix}$ of $T_2(R)$. Obviously $e^2 = e \in R$ and it is easy to show that $r_R(a) \leq eR$.

Let $0 \neq er \in eR$, then $\begin{pmatrix} er & 0 \\ 0 & 0 \end{pmatrix} \in \begin{pmatrix} e & m \\ 0 & f \end{pmatrix} T_2(R)$ and there is nonzero element $\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$ of $T_2(R)$ such that

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} er & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} = \begin{pmatrix} erx & ery \\ 0 & 0 \end{pmatrix}.$$

Thus either $0 \neq erx$ or $ery \neq 0$, we have $erx \in r_R(a)$ or $ery \in r_R(a)$ and hence $r_R(a) \leq_e eR$. So R is a right ACS-ring. ∎

The converse of the proposition above is not true, in general. See:

Example 4.1. Let $\mathbb{Z}$ be the ring of integers, then $\mathbb{Z}$ is an ACS-ring. But the upper matrix ring $T_2(\mathbb{Z})$ is not a right ACS-ring.

Proof. Let $T = T_2(\mathbb{Z})$. It is easy to see that all idempotents of T are:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & b \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & b \\ 0 & 1 \end{pmatrix}$$

where $0 \neq b \in \mathbb{Z}$.

Let $t = \begin{pmatrix} 2 & 3 \\ 0 & 0 \end{pmatrix} \in T$, then $r_T(t) = \left\{ \begin{pmatrix} 0 & y \\ 0 & z \end{pmatrix} \in T \mid 2y + 3z = 0 \right\}$. If T is a right ACS-ring, a calculation shows that $r_T(t)$ must be essential, as a right ideal, in T. Let $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in T$, then there is $\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \in T$ such that $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} = \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \in r_T(t)$. But this is impossible. ∎

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