Vietnam Journal of Mathematics **35**:1 (2007) 1–10

Vietnam Journal
of
MATHEMATICS

Sharp Weighted Inequalities for Multilinear Commutator of Marcinkiewicz Operator

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Received August 25, 2005
Revised September 29, 2006

Abstract. In this paper, we prove the sharp inequality for the multilinear commutator related to the Marcinkiewicz operator. By using the sharp inequality, we obtain the weighted L^p-norm inequality for the multilinear commutator.

2000 Mathematics Subject Classification: 42B20, 42B25.
Keywords: Multilinear commutator; Marcinkiewicz operator; BMO; Sharp inequality.

1. Introduction

Let T be the Calderón–Zygmund singular integral operator, we know that the commutator $[b,T](f) = T(bf) - bT(f)$ (where $b \in BMO(R^n)$) is bounded on $L^p(R^n)$ for $1 < p < \infty$ (see [2]). In [8], the sharp estimates for some multilinear commutators of the Calderón–Zygmund singular integral operators are obtained. The main purpose of this paper is to prove a sharp inequality for some multilinear commutator related to the Marcinkiewicz operator. By using the sharp inequality, we obtain the weighted L^p-norm inequality for the multilinear commutator.

2. Notations and Results

First let us introduce some notations(see [3, 8, 9]). In this paper, Q will denote a cube of R^n with sides parallel to the axes, and for a cube Q let $f_Q = |Q|^{-1}\int_Q f(z)dz$ and the sharp function of f is defined by

$$f^{\#}(x)=\sup_{Q\ni x}\frac{1}{|Q|}\int_Q|f(y)-f_Q|dy.$$

It is well-known that(see [3])

$$f^{\#}(x)=\sup_{Q\ni x}\inf_{c\in C}\frac{1}{|Q|}\int_Q|f(y)-c|dy.$$

We say that b belongs to $BMO(R^n)$ if $b^{\#}$ belongs to $L^\infty(R^n)$ and define $||b||_{BMO}=||b^{\#}||_{L^\infty}$. It has been known that(see [8])

$$||b-b_{2^kQ}||_{BMO}\leqslant Ck||b||_{BMO}.$$

Let M be the Hardy–Littlewood maximal operator, that is

$$M(f)(x)=\sup_{x\in Q}|Q|^{-1}\int_Q|f(y)|dy;$$

we write that $M_p(f)=(M(|f|^p))^{1/p}$ for $0<p<\infty$. For $b_j\in BMO$ $(j=1,\ldots,m)$, set

$$||\vec{b}||_{BMO}=\prod_{j=1}^m||b_j||_{BMO}.$$

Given a positive integer m and $1\leqslant j\leqslant m$, we denote by C_j^m the family of all finite subsets $\sigma=\{\sigma(1),\ldots,\sigma(j)\}$ of $\{1,\ldots,m\}$ of j different elements. For $\sigma\in C_j^m$, set $\sigma^c=\{1,\ldots,m\}\setminus\sigma$. For $\vec{b}=(b_1,\ldots,b_m)$ and $\sigma=\{\sigma(1),\ldots,\sigma(j)\}\in C_j^m$, set $\vec{b}_\sigma=(b_{\sigma(1)},\ldots,b_{\sigma(j)})$, $b_\sigma=b_{\sigma(1)}\cdots b_{\sigma(j)}$ and $\|\vec{b}_\sigma\|_{BMO}=\|b_{\sigma(1)}\|_{BMO}\cdots\|b_{\sigma(j)}\|_{BMO}$.

We denote the Muckenhoupt weights by A_1(see [3]), that is

$$A_1=\{w:M(w)(x)\leqslant Cw(x),\ a.e.\ x\in R^n\}.$$

In this paper, we will study some multilinear commutators as follows.

Definition. *Let $0<\gamma\leqslant 1$ and Ω be homogeneous of degree zero on R^n such that $\int_{S^{n-1}}\Omega(x')d\sigma(x')=0$. Assume that $\Omega\in Lip_\gamma(S^{n-1})$, that is there exists a constant $M>0$ such that for any $x,y\in S^{n-1}$, $|\Omega(x)-\Omega(y)|\leqslant M|x-y|^\gamma$. The Marcinkiewicz multilinear commutator is defined by*

$$\mu_\Omega^{\vec{b}}(f)(x)=\left(\int_0^\infty|F_t^{\vec{b}}(f)(x)|^2\frac{dt}{t^3}\right)^{1/2},$$

where

$$F_t^{\vec{b}}(f)(x)=\int_{|x-y|\leqslant t}\frac{\Omega(x-y)}{|x-y|^{n-1}}\left[\prod_{j=1}^m(b_j(x)-b_j(y))\right]f(y)dy.$$

Set

$$F_t(f)(x)=\int_{|x-y|\leqslant t}\frac{\Omega(x-y)}{|x-y|^{n-1}}f(y)dy,$$

we also define that

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

which is the Marcinkiewicz operator (*see* [10]).

Let H be the space $H = \left\{ h : ||h|| = \left(\int_0^\infty |h(t)|^2 dt/t^3 \right)^{1/2} < \infty \right\}$. Then, it is clear that

$$\mu_\Omega(f)(x) = \|F_t(f)(x)\| \text{ and } \mu_\Omega^{\tilde{b}}(f)(x) = \|F_t^{\tilde{b}}(f)(x)\|.$$

Note that when $b_1 = \cdots = b_m$, $T_{\tilde{b}}$ is just the m order commutator. It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [1, 4 - 8, 10]). Our main purpose is to establish the sharp inequality for the multilinear commutator.

Now we state our main results as follows.

Theorem 1. *Let $b_j \in BMO$ for $j = 1, \ldots, m$. Then for any $1 < r < \infty$, there exists a constant $C > 0$ such that for any $f \in C_0^\infty(R^n)$ and any $x \in R^n$,*

$$(\mu_\Omega^{\vec{b}}(f))^\#(x) \leqslant C \left(||\vec{b}||_{BMO} M_r(f)(x) + \sum_{j=1}^m \sum_{\sigma \in C_j^m} ||\vec{b}_\sigma||_{BMO} M_r(\mu_\Omega^{\vec{b}_{\sigma^c}}(f))(x) \right).$$

Theorem 2. *Let $b_j \in BMO$ for $j = 1, \ldots, m$. Then $\mu_\Omega^{\vec{b}}$ is bounded on $L^p(w)$ for $w \in A_1$ and $1 < p < \infty$.*

3. Proofs of Theorems

To prove the theorem, we need the following lemmas.

Lemma 1.(see [10]) *Let $w \in A_1$ and $1 < p < \infty$. Then μ_Ω is bounded on $L^p(w)$.*

Lemma 2. *Let $1 < r < \infty$, $b_j \in BMO$ for $j = 1, \ldots, k$ and $k \in N$. Then, we have*

$$\frac{1}{|Q|} \int_Q \prod_{j=1}^k |b_j(y) - (b_j)_Q| dy \leqslant C \prod_{j=1}^k ||b_j||_{BMO}$$

and

$$\left(\frac{1}{|Q|} \int_Q \prod_{j=1}^k |b_j(y) - (b_j)_Q|^r dy \right)^{1/r} \leqslant C \prod_{j=1}^k ||b_j||_{BMO}.$$

Proof. For $\sigma \in C_k^m$, where $k \leqslant m$ and $m \in N$, we have

$$\frac{1}{|Q|} \int_Q |(b(y) - (b_j)_Q)_\sigma| dy \leqslant C ||b_\sigma||_{BMO}$$

and

$$\left(\frac{1}{|Q|}\int_Q |(b(y)-(b_j)_Q)_\sigma|^r dy\right)^{1/r} \leqslant C||b_\sigma||_{BMO}.$$

We just need to choose $p_j > 1$ and $q_j > 1$, where $1 \leqslant j \leqslant k$, such that $1/p_1+\cdots+1/p_k = 1$ and $1/q_1+\cdots+1/q_k = 1/r$. After that,using the Hölder's inequality with exponent $1/p_1+\cdots+1/p_k = 1$ and $1/q_1+\cdots+1/q_k = 1/r$ respectively, we may get the conclusions.

Proof of Theorem 1. It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\left(\frac{1}{|Q|}\int_Q |\mu_\Omega^{\vec{b}}(f)(x)-C_0|dx\right) \leqslant C\left(||\vec{b}||_{BMO}M_r(f)(x)+\sum_{j=1}^m\sum_{\sigma\in C_j^m} M_r(\mu_\Omega^{\vec{b}}(f)(x)\right).$$

Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. We first consider the case $m = 1$. We write, for $f_1 = f\chi_{2Q}$ and $f_2 = f\chi_{R^n\setminus 2Q}$,

$$F_t^{b_1}(f)(x) = (b_1(x)-(b_1)_{2Q})F_t(f)(x)-F_t((b_1-(b_1)_{2Q})f_1)(x)-F_t((b_1-(b_1)_{2Q})f_2)(x),$$

then

$$\begin{aligned}
&|\mu_\Omega^{b_1}(f)(x)-\mu_\Omega(((b_1)_{2Q}-b_1)f_2)(x_0)|\\
&=\Big|\|F_t^{b_1}(f)(x)||-||F_t(((b_1)_{2Q}-b_1)f_2)(x_0)\|\Big|\\
&\leqslant \|F_t^{b_1}(f)(x)-F_t(((b_1)_{2Q}-b_1)f_2)(x_0)\|\\
&\leqslant \|(b_1(x)-(b_1)_{2Q})F_t(f)(x)||+||F_t((b_1-(b_1)_{2Q})f_1)(x)\|\\
&\ +\|F_t((b_1-(b_1)_{2Q})f_2)(x)-F_t((b_1-(b_1)_{2Q})f_2)(x_0)\|\\
&=A(x)+B(x)+C(x).
\end{aligned}$$

For $A(x)$, by Hölder's inequality with exponent $1/r+1/r' = 1$, we get

$$\begin{aligned}
&\Big(\frac{1}{|Q|}\int_Q A(x)dx\Big) = \frac{1}{|Q|}\int_Q |b_1(x)-(b_1)_{2Q}||\mu_\Omega(f)(x)|dx\\
&\leqslant \Big(\frac{C}{|2Q|}\int_{2Q}|b_1(x)-(b_1)_{2Q}|^{r'}dx\Big)^{1/r'}\Big(\frac{1}{|Q|}\int_Q|\mu_\Omega(f)(x)|^r dx\Big)^{1/r}\\
&\leqslant C||b_1||_{BMO}M_r(\mu_\Omega(f))(\tilde{x}).
\end{aligned}$$

For $B(x)$, choose p such that $1 < p < r$, by the boundedness of μ_Ω on $L^p(R^n)$(see Lemma 1) and Hölder's inequality with exponent $1/(r/(r-p))+1/(r/p) = 1$, we have

$$\Big(\frac{1}{|Q|}\int_Q B(x)dx\Big) = \frac{1}{|Q|}\int_Q[\mu_\Omega((b_1-(b_1)_{2Q})f_1)(x)]dx$$

$$\leqslant \Big(\frac{1}{|Q|}\int_{R^n}[\mu_\Omega((b_1-(b_1)_{2Q})f\chi_{2Q})(x)]^p dx\Big)^{1/p}$$
$$\leqslant C\Big(\frac{1}{|Q|}\int_{R^n}|b_1(x)-(b_1)_{2Q}|^p|f\chi_{2Q}(x)|^p dx\Big)^{1/p}$$
$$\leqslant C\Big(\frac{1}{|2Q|}\int_{2Q}|b_1-(b_1)_{2Q}|^{rp/(r-p)}dx\Big)^{(r-p)/rp}\Big(\frac{1}{|2Q|}\int_{2Q}|f(x)|^r dx\Big)^{1/r}$$
$$\leqslant C||b_1||_{BMO}M_r(f)(\tilde{x}).$$

For $C(x)$, note that $|x_0-y|\approx|x-y|$ for $y\in Q^c$, we have

$$\begin{aligned}
C(x) &= \big\|F_t((b_1-(b_1)_{2Q})f_2)(x)-F_t((b_1-(b_1)_{2Q})f_2)(x_0)\big\| \\
&= \Big(\int_0^\infty \Big|\int_{|x-y|\leqslant t}\frac{\Omega(x-y)f_2(y)}{|x-y|^{n-1}}(b_1(y)-(b_1)_{2Q}\Big)dy \\
&\quad -\int_{|x_0-y|\leqslant t}\frac{\Omega(x_0-y)f_2(y)}{|x_0-y|^{n-1}}(b_1(y)-(b_1)_{2Q})dy\Big|^2\frac{dt}{t^3}\Big)^{1/2} \\
&\leqslant \Big(\int_0^\infty\Big[\int_{|x_0-y|\leqslant t,|x_0-y|>t}\frac{|\Omega(x-y)||f_2(y)|}{|x-y|^{n-1}}|(b_1(y)-(b_1)_{2Q})|dy\Big]^2\frac{dt}{t^3}\Big)^{1/2} \\
&\quad +\Big(\int_0^\infty\Big[\int_{|x-y|>t,|x_0-y|\leqslant t}\frac{|\Omega(x_0-y)||f_2(y)|}{|x_0-y|^{n-1}}\big|(b_1(y)-(b_1)_{2Q})\big|dy\Big]^2\frac{dt}{t^3}\Big)^{1/2} \\
&\quad +\Big(\int_0^\infty\Big[\int_{|x-y|\leqslant t,|x_0-y|\leqslant t}\Big|\frac{|\Omega(x-y)|}{|x-y|^{n-1}}-\frac{|\Omega(x_0-y)|}{|x_0-y|^{n-1}}\Big||(b_1(y) \\
&\quad -(b_1)_{2Q})||f_2(y)|dy\Big]^2\frac{dt}{t^3}\Big)^{1/2} \\
&\equiv S_1+S_2+S_3,
\end{aligned}$$

thus, by Minkowski's inequality and Hölder's inequality with exponent $1/r'+1/r=1$,

$$\begin{aligned}
S_1 &\leqslant C\int_{(2Q)^c}|(b_1(y)-(b_1)_{2Q})|\frac{|f(y)|}{|x-y|^{n-1}}\Big(\int_{|x-y|\leqslant t<|x_0-y|}\frac{dt}{t^3}\Big)^{1/2}dy \\
&\leqslant C\int_{(2Q)^c}|(b_1(y)-(b_1)_{2Q})|\frac{|f(y)|}{|x-y|^{n-1}}\Big|\frac{1}{|x-y|^2}-\frac{1}{|x_0-y|^2}\Big|^{1/2}dy \\
&\leqslant C\int_{(2Q)^c}|(b_1(y)-(b_1)_{2Q})|\frac{|f(y)|}{|x-y|^{n-1}}\frac{|x_0-x|^{1/2}}{|x-y|^{3/2}}dy \\
&\leqslant C\sum_{k=1}^\infty\int_{2^{k+1}Q\setminus 2^kQ}|(b_1(y)-(b_1)_{2Q})|\frac{|Q|^{1/(2n)}|f(y)|}{|x_0-y|^{n+1/2}}dy \\
&\leqslant C\sum_{k=1}^\infty 2^{-k/2}(|2^{k+1}Q|^{-1}\int_{2^{k+1}Q}|(b_1(y)-(b_1)_{2Q})||f(y)|dy)
\end{aligned}$$

$$\leqslant C\sum_{k=1}^{\infty}2^{-k/2}(|2^{k+1}Q|^{-1}\int_{2^{k+1}Q}|(b_1(y)-(b_1)_{2Q})|^{r'}dy)^{1/r'}(|2^{k+1}Q|^{-1}$$
$$\times\int_{2^{k+1}Q}|f(y)|^r dy)^{1/r}$$
$$\leqslant C\sum_{k=1}^{\infty}2^{-k/2}||b_1||_{BMO}M_r(f)(\tilde{x})$$
$$\leqslant C\|b_1\|_{BMO}M_r(f)(\tilde{x});$$

similarly, we have $S_2\leqslant C||b_1||_{BMO}M_r(f)(\tilde{x})$.

We now estimate S_3. By the following inequality (see [10]):

$$|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x_0-y)}{|x_0-y|^{n-1}}|\leqslant\Big(\frac{|x-x_0|}{|x_0-y|^n}+\frac{|x-x_0|^\gamma}{|x_0-y|^{n-1+\gamma}}\Big),$$

we gain

$$S_3\leqslant C\int_{(2Q)^c}|b_1(y)-(b_1)_{2Q}|\frac{|f(y)||x-x_0|}{|x_0-y|^n}\Big(\int_{|x_0-y|\leqslant t,|x-y|\leqslant t}\frac{dt}{t^3}\Big)^{1/2}dy$$
$$+C\int_{(2Q)^c}|b_1(y)-(b_1)_{2Q}|\frac{|f(y)||x-x_0|^\gamma}{|x_0-y|^{n-1+\gamma}}\Big(\int_{|x_0-y|\leqslant t,|x-y|\leqslant t}\frac{dt}{t^3}\Big)^{1/2}dy$$
$$\leqslant C\sum_{k=1}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}|b_1(y)-(b_1)_{2Q}|\Big(\frac{|Q|^{1/n}}{|x_0-y|^{n+1}}+\frac{|Q|^{\gamma/n}}{|x_0-y|^{n+\gamma}}\Big)|f(y)|dy$$
$$\leqslant C\sum_{k=1}^{\infty}(2^{-k}+2^{-k\gamma})|2^{k+1}Q|^{-1}\int_{2^{k+1}Q}|b_1(y)-(b_1)_{2Q}||f(y)|dy$$
$$\leqslant C\sum_{k=1}^{\infty}(2^{-k}+2^{-k\gamma})||b_1||_{BMO}M_r(f)(\tilde{x})$$
$$\leqslant C||b_1||_{BMO}M_r(f)(\tilde{x}).$$

This completes the proof of the case $m=1$.

Now, we consider the case $m\geq 2$. We write, for $\tilde{b}=(b_1,...,b_m)$,

$$F_t^{\tilde{b}}(f)(x)=\int_{|x-y|\leqslant t}\Big[\prod_{j=1}^m(b_j(x)-b_j(y))\Big]f(y)\Omega(x-y)|x-y|^{1-n}dy$$
$$=\int_{|x-y|\leqslant t}\Big[\prod_{j=1}^m((b_j(x)-(b_j)_{2Q})-(b_j(y)-(b_j)_{2Q}))\Big]f(y)\Omega(x-y)|x-y|^{1-n}dy$$
$$=\sum_{j=0}^m\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x)-(b)_{2Q})_\sigma$$
$$\times\int_{|x-y|\leqslant t}(b(y)-(b)_{2Q})_{\sigma^c}f(y)\Omega(x-y)|x-y|^{1-n}dy$$

$$= (b_1(x)-(b_1)_{2Q})\cdots(b_m(x)-(b_m)_{2Q})F_t(f)(x)$$
$$+ (-1)^m F_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f)(x)$$
$$+ \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x)-(b)_{2Q})_\sigma F_t^{\tilde{b}_{\sigma^c}}(f)(x),$$

thus

$$|\mu_\Omega^{\tilde{b}}(f)(x)-\mu_\Omega((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f_2)(x_0)|$$
$$\leqslant ||F_t^{\tilde{b}}(f)(x)-(-1)^m F_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f_2)(x_0)||$$
$$\leqslant ||(b_1(x)-(b_1)_{2Q})\cdots(b_m(x)-(b_m)_{2Q})F_t(f)(x)||$$
$$+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||(b(x)-(b)_{2Q})_\sigma F_t^{\tilde{b}_{\sigma^c}}(f)(x)||$$
$$+||F_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f_1)(x)||$$
$$+||F_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f_2)(x)$$
$$-F_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f_2)(x_0)||$$
$$= I_1(x)+I_2(x)+I_3(x)+I_4(x).$$

For $I_1(x)$, by Hölder's inequality with exponent $1/r'+1/r=1$ and Lemma 2, we get

$$\frac{1}{|Q|}\int_Q I_1(x)dx \leqslant C\frac{1}{|Q|}\int_Q |\prod_{j=1}^m(b_j(x)-(b_j)_{2Q})||\mu_\Omega(f)(x)|dx$$
$$\leqslant C\Big(\frac{1}{|2Q|}\int_{2Q}|\prod_{j=1}^m(b_j(x)-(b_j)_{2Q})|^{r'}dx\Big)^{1/r'}\Big(\frac{1}{|Q|}\int_Q|\mu_\Omega(f)(x)|^r dx\Big)^{1/r}$$
$$\leqslant C||\vec{b}||_{BMO}M_r(\mu_\Omega(f))(\tilde{x}).$$

For $I_2(x)$, by Hölder's inequality with exponent $1/r'+1/r=1$, we get

$$\frac{1}{|Q|}\int_Q I_2(x)dx = \frac{1}{|Q|}\int_Q\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||(b(x)-(b)_{2Q})_\sigma F_t^{\vec{b}_{\sigma^c}}(f)(x)||dx$$
$$\leqslant \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|Q|}\int_Q|(b(x)-(b)_{2Q})_\sigma||\mu_\Omega^{\vec{b}_{\sigma^c}}(f)(x)|dx$$
$$\leqslant C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\left(\frac{1}{|2Q|}\int_{2Q}|(b(x)-(b)_{2Q})_\sigma|^{r'}dx\right)^{1/r'}\left(\frac{1}{|Q|}\int_Q|\mu_\Omega^{\vec{b}_{\sigma^c}}(f)(x)|^r dx\right)^{1/r}$$
$$\leqslant C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||\vec{b}_\sigma||_{BMO}M_r(\mu_\Omega^{\vec{b}_{\sigma^c}}(f))(\tilde{x}).$$

For $I_3(x)$, we choose $1 < p < r$, by the boundedness of μ_Ω on $L^p(R^n)$ and Hölder's inequality, we get

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q I_3(x)dx \\
&= \frac{1}{|Q|}\int_Q ||F_t(\prod_{j=1}^m (b_j(y)-(b_j)_{2Q})f_1)(x)||dx \\
&\leqslant \Big(\frac{1}{|Q|}\int_{R^n} |\mu_\Omega(\prod_{j=1}^m (b_j(y)-(b_j)_{2Q})f\chi_{2Q})(x)|^p dx\Big)^{1/p} \\
&\leqslant C\Big(\frac{1}{|Q|}\int_{R^n} \Big|\prod_{j=1}^m (b_j(y)-(b_j)_{2Q})\Big|^p |f\chi_{2Q}|^p dx\Big)^{1/p} \\
&\leqslant C\Big(\frac{1}{|2Q|}\int_{2Q} |\prod_{j=1}^m (b_j(y)-(b_j)_{2Q})|^{rp/(r-p)} dx\Big)^{(r-p)/rp}\Big(\frac{1}{|2Q|}\int_{2Q} |f(x)|^r dx\Big)^{1/r} \\
&\leqslant C\|\vec{b}\|_{BMO} M_r(f)(\tilde{x}).
\end{aligned}$$

For $I_4(x)$, similar to the proof of $C(x)$ in the case $m = 1$, we get

$$\begin{aligned}
I_4(x) &= \Big\|F_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f_2)(x) \\
&\qquad - F_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f_2)(x_0)\Big\| \\
&= \Big(\int_0^\infty |\int_{|x-y|\leqslant t} \frac{\Omega(x-y)f_2(y)}{|x-y|^{n-1}}\Big[\prod_{j=1}^m (b_j(y)-(b_j)_{2Q})\Big]dy \\
&\quad - \int_{|x_0-y|\leqslant t} \frac{\Omega(x_0-y)f_2(y)}{|x_0-y|^{n-1}}\Big[\prod_{j=1}^m (b_j(y)-(b_j)_{2Q})\Big]dy|^2 \frac{dt}{t^3})^{1/2} \\
&\leqslant \Big(\int_0^\infty \Big[\int_{|x-y|\leqslant t, |x_0-y|>t} \frac{|\Omega(x-y)||f_2(y)|}{|x-y|^{n-1}}\Big|\prod_{j=1}^m (b_j(y)-(b_j)_{2Q})\Big|dy\Big]^2 \frac{dt}{t^3}\Big)^{1/2} \\
&\quad + \Big(\int_0^\infty \Big[\int_{|x-y|>t, |x_0-y|\leqslant t} \frac{|\Omega(x_0-y)||f_2(y)|}{|x_0-y|^{n-1}}\Big|\prod_{j=1}^m (b_j(y)-(b_j)_{2Q})\Big|dy\Big]^2 \frac{dt}{t^3}\Big)^{1/2} \\
&\quad + \Big(\int_0^\infty \Big[\int_{|x-y|\leqslant t, |x_0-y|\leqslant t} \Big|\frac{|\Omega(x-y)|}{|x-y|^{n-1}} - \frac{|\Omega(x_0-y)|}{|x_0-y|^{n-1}}\Big|\Big|\prod_{j=1}^m (b_j(y) \\
&\quad - (b_j)_{2Q})\Big||f_2(y)|dy\Big]^2 \frac{dt}{t^3}\Big)^{1/2} \\
&\equiv J_1 + J_2 + J_3,
\end{aligned}$$

thus

$$J_1 \leqslant C\int_{(2Q)^c} \Big|\prod_{j=1}^m (b_j(y)-(b_j)_{2Q})\Big|\frac{|f(y)|}{|x-y|^{n-1}}\Big(\int_{|x-y|\leqslant t<|x_0-y|} \frac{dt}{t^3}\Big)^{1/2} dy$$

$$
\begin{aligned}
&\leqslant C\int_{(2Q)^c}\Big|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\Big|\frac{|f(y)|}{|x-y|^{n-1}}\Big|\frac{1}{|x-y|^2}-\frac{1}{|x_0-y|^2}\Big|^{1/2}dy\\
&\leqslant C\int_{(2Q)^c}\Big|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\Big|\frac{|f(y)|}{|x-y|^{n-1}}\frac{|x_0-x|^{1/2}}{|x-y|^{3/2}}dy\\
&\leqslant C\sum_{k=1}^\infty\int_{2^{k+1}Q\setminus 2^kQ}\Big|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\Big|\frac{|Q|^{1/(2n)}|f(y)|}{|x_0-y|^{n+1/2}}dy\\
&\leqslant C\sum_{k=1}^\infty 2^{-k/2}|2^{k+1}Q|^{-1}\int_{2^{k+1}Q}\Big|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\Big||f(y)|dy\\
&\leqslant C\sum_{k=1}^\infty 2^{-k/2}(|2^{k+1}Q|^{-1}\int_{2^{k+1}Q}\Big|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\Big|^{r'}dy)^{1/r'}\\
&\quad\times\Big(|2^{k+1}Q|^{-1}\int_{2^{k+1}Q}|f(y)|^r dy\Big)^{1/r}\\
&\leqslant C\sum_{k=1}^\infty 2^{-k/2}\prod_{j=1}^m||b_j||_{BMO}M_r(f)(\tilde{x})\\
&\leqslant C\|\vec{b}\|_{BMO}M_r(f)(\tilde{x});
\end{aligned}
$$

similarly, we have $J_2\leqslant C\|\vec{b}\|_{BMO}M_r(f)(\tilde{x})$.

We now estimate J_3. By the following inequality that we use in the case $m=1$:

$$
\Big|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x_0-y)}{|x_0-y|^{n-1}}\Big|\leqslant\Big(\frac{|x-x_0|}{|x_0-y|^n}+\frac{|x-x_0|^\gamma}{|x_0-y|^{n-1+\gamma}}\Big),
$$

we gain

$$
\begin{aligned}
J_3&\leqslant C\int_{(2Q)^c}\Big|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\Big|\frac{|f(y)||x-x_0|}{|x_0-y|^n}\Big(\int_{|x_0-y|\leqslant t,|x-y|\leqslant t}\frac{dt}{t^3}\Big)^{1/2}dy\\
&\quad+C\int_{(2Q)^c}\Big|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\Big|\frac{|f(y)||x-x_0|^\gamma}{|x_0-y|^{n-1+\gamma}}\Big(\int_{|x_0-y|\leqslant t,|x-y|\leqslant t}\frac{dt}{t^3}\Big)^{1/2}dy\\
&\leqslant C\sum_{k=1}^\infty\int_{2^{k+1}Q\setminus 2^kQ}\Big|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\Big|\Big(\frac{|Q|^{1/n}}{|x_0-y|^{n+1}}+\frac{|Q|^{\gamma/n}}{|x_0-y|^{n+\gamma}}\Big)|f(y)|dy\\
&\leqslant C\sum_{k=1}^\infty(2^{-k}+2^{-k\gamma})|2^{k+1}Q|^{-1}\int_{2^{k+1}Q}\Big|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\Big||f(y)|dy\\
&\leqslant C\sum_{k=1}^\infty(2^{-k}+2^{-k\gamma})\prod_{j=1}^m\|b_j\|_{BMO}M_r(f)(\tilde{x})\\
&\leqslant C\|\vec{b}\|_{BMO}M_r(f)(\tilde{x}).
\end{aligned}
$$

This completes the proof of Theorem 1. ■

Proof of Theorem 2. We choose $1 < r < p$ in Theorem 1 and by using Lemma 1, we may get the conclusion of Theorem 2. This finishes the proof. ■

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