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H-Cofinitely Supplemented Modules

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Abstract. Let M be a right R-module. We call M H*-cofinitely supplemented* if for every cofinite submodule A of M (i.e. the factor module M/A is finitely generated) there exists a direct summand D of M such that $M = A + X$ holds if and only if $M = D + X$. It is shown that in this paper: (1) Let M be an H-cofinitely supplemented Duo module. Then every direct summand of M is an H-cofinitely supplemented module. (2) Let $M = M_1 \oplus M_2$ be a Duo module. If M_1 and M_2 are H-cofinitely supplemented modules, then M is H-cofinitely supplemented. (3) Assume $\text{Rad}\,(M) << M$. Then M is H-cofinitely supplemented if and only if every cofinite submodule of $M/\text{Rad}\,(M)$ is a direct summand and each cofinite direct summand of $M/\text{Rad}\,(M)$ lifts to a direct summand of M. In addition, let M be a Duo right R-module. M is H-cofinitely supplemented if and only if every maximal submodule of M has an H-supplement in M.

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I. Introduction

Throughout this paper R denotes an associative ring with unity and all R-modules are unitial right R-modules. We review some basic definitions. A submodule N of a module M is called *small*, written $N \ll M$, if $M \neq N + L$ for every proper submodule L of M. For submodules A and B of M with $M = A + B$, B is called a *supplement* of A if it is minimal with respect to this property, equivalently if $A \cap B$ is small in B. An R-module M is *supplemented* if every submodule of M has a supplement in M. If for any submodule A of M, there exists a direct summand D with say $M = D \oplus D'$ for some submodule D' of M, such that $M = A + X$ holds if and only if $M = D + X$ then M is called *H-supplemented* (in this case D' is a supplement of A). The module M is called $\oplus$*-supplemented* if every submodule of M has a supplement that is a direct summand of M. These are generalizations of lifting (i.e. dual extending) modules. An account of modules to these concepts can be found in the texts by Mohammed and Muller and Wisbauer, referenced in the paper as [11] and [15], respectively. For more properties of, $\oplus$-supplemented modules and H-supplemented modules, we refer to [6, 7, 10]. In [6], they called a module M completely $\oplus$-supplemented if every direct summand of M is $\oplus$-supplemented.

A module M is called *local* if the sum of all proper submodules is also a proper submodule of M and is called *hollow* if every proper submodule of M is small. Clearly, every local module is hollow and the supplement of a maximal submodule of M is local.

A submodule N of M is called *cofinite* (in M) if the factor module M/N is finitely generated. The module M is called *cofinitely supplemented* if every cofinite submodule of M has a supplement in M (see [1] and [13]), M is called $\oplus$-*cofinitely supplemented* if every cofinite submodule of M has a supplement that is a direct summand of M (see [8] and [3]) and M is called *completely* $\oplus$*-cofinitely supplemented* if every direct summand of M is $\oplus$-cofinitely supplemented (see [8]). By definitions, $((\oplus)$ -)supplemented modules are $((\oplus)$ -)cofinitely supplemented modules and also the converse is true if M is finitely generated. In this note, H-supplemented modules are generalized by requiring the same condition as above for cofinite submodules A of M, that is if for any cofinite submodule A of M, there exists a direct summand D of M such that $M = A + X$ holds if and only if $M = D + X$. Such modules are called *H-cofinitely supplemented modules*. Clearly every H-supplemented module is H-cofinitely supplemented and every H-cofinitely supplemented module is $\oplus$-cofinitely supplemented. On the other hand, every finitely generated H-cofinitely supplemented module is H-supplemented.

Recall that A module M has the *Summand Intersection Property*, if the intersection of any two direct summands of M is again a direct summand (see [5,14]) and M has the *Summand Sum Property*, if the sum of any two direct summands of M is again a direct summand (see [4]).

Let M be a module. A submodule X of M is called *fully invariant* if for every $h \in End_R(M)$, $h(X) \subseteq X$. The module M is called *Duo module*, if every submodule of M is fully invariant.

Among the main problems investigated in this paper is when property of being H-cofinitely supplemented is inherited by direct sums or summands: The later case holds for an H-cofinitely supplemented module M if (i) the sum of any two direct summands of M is also a direct summand, or (ii) M is distributive in which case all factors of M are H-cofinitely supplemented (Theorem 2.1). Let $M = M_1 \oplus M_2$ be a Duo module. Then M is H-cofinitely supplemented module if and only if so is each M_i, $i = 1, 2$ (Corollary 2.3 and Theorem 2.5). Also,

H-cofinitely supplemented modules M with $Rad(M) \ll M$, and H-cofinitely supplemented Duo modules are characterized (Theorem 2.7 and Theorem 2.12, respectively).

For the other definitions in this note we refer to [2, 11, 15].

2. H-cofinitely Supplemented Modules

By definitions, we have the following hierarchy;

$$\begin{array}{ccc} H-cofinitely\ supplemented & \Rightarrow & \oplus-cofinitely\ supplemented \\ & & \Downarrow \\ & & cofinitely\ supplemented. \end{array}$$

Also note that if M is finitely generated module then we have [11, Proposition A.2]. In [1, Lemma 2.1], they shown that any homomorphic image of a cofinitely supplemented module is also cofinitely supplemented module. Our aim in this section is to investigate conditions which ensure that a factor submodule of an H-cofinitely supplemented module will be an H-cofinitely supplemented module.

A module M is called *distributive* if its lattice of submodules is a distributive lattice, equivalently for submodules K, L, N of M, $N + (K \cap L) = (N + K) \cap (N + L)$ or $N \cap (K + L) = (N \cap K) + (N \cap L)$.

Theorem 2.1.

(1) *Let M be an H-cofinitely supplemented module and X a submodule of M. If for every direct summand K of M, $(X + K)/X$ is a direct summand of M/X then M/X is H-cofinitely supplemented.*

(2) *Let M be an H-cofinitely supplemented module with the SSP. Then every direct summand of M is an H-cofinitely supplemented module.*

(3) *Let M be an H-cofinitely supplemented distributive module. Then M/N is H-cofinitely supplemented for every submodule N of M.*

Proof.
1. Any cofinite submodule of M/N has the form T/N where T is a cofinite submodule of M and $N \subseteq T$. Since M is H-cofinitely supplemented, there exists a direct summand D of M such that $M = T + Y$ if and only if $M = D + Y$. By hypothesis, $(D + N)/N$ is a direct summand of M/N. Therefore $M/N = T/N + L/N$ if and only if $M/N = (D + N)/N + L/X$ for every $L/N \leq M/N$.

2. Let N be a direct summand of M. Let $M = N \oplus N'$ for some $N' \leq M$. We want to show that M/N' is H-cofinitely supplemented. Assume that L is a direct summand of M. Since M has the SSP, $L + N$ is a direct summand of M. Let $M = (L+N') \oplus K$ for some $K \leq M$. Then $M/N' = (L+N')/N' \oplus (K+N')/N'$. Therefore M/N' is an H-cofinitely supplemented module by (1).

3. Let D be a direct summand of M. Let $M = D \oplus D'$ for some $D' \leq M$. Now $M/N = (D + N)/N + (D' + N)/N$ for every submodule N of M. Note that $N = N + (D \cap D') = (N + D) \cap (N + D')$ by distributive of M. Now

$M/N = (D+N)/N \oplus (D'+N)/N$. By (1), it is an H-cofinitely supplemented module. ∎

Theorem 2.2. ([9, Corollary 18]) *Let M be a Duo module. Then M has the SIP and the SSP.*

As a result of Theorem 2.1 and Theorem 2.2, we can obtain the following corollary.

Corollary 2.3 *Let M be an H-cofinitely supplemented Duo module. Then every direct summand of M is an H-cofinitely supplemented module.*

By hierarchy, every H-cofinitely supplemented module is a $\oplus$-cofinitely supplemented module. Now we give an equivalent condition.

Proposition 2.4. *Assume that M is $\oplus$-cofinitely supplemented such that whenever $M = M_1 \oplus M_2$ then M_1 and M_2 are relatively projective. Then M is an H-cofinitely supplemented module.*

Proof. Let N be a cofinite submodule of M. Since M is a $\oplus$-cofinitely supplemented module, there exists a decomposition $M = M_1 \oplus M_2$ such that $M = N + M_2$ and $N \cap M_2 << M_2$ for some submodules M_1 and M_2. By hypothesis, M_1 is M_2-projective. By [11, Lemma 4.47], we obtain $M = A \oplus M_2$ for some submodule A of M such that $A \leq N$. Then $N = A \oplus (M_2 \cap N)$. Let $X \leq M$ with $M = N + X$. Then $M = A + (M_2 \cap N) + X$. Since $M_2 \cap N$ is small in M_2 and so is small in M, $M = A + X$. Hence $M = N + X$ if and only if $M = A + X$. Thus M is an H-cofinitely supplemented module. ∎

Theorem 2.5. *Let $M = M_1 \oplus M_2$ be a Duo module. If M_1 and M_2 are H-cofinitely supplemented modules, then M is H-cofinitely supplemented.*

Proof. Assume M_1 and M_2 are H-cofinitely supplemented modules. Take any cofinite submodule L of M. By [9], $L = (L \cap M_1) \oplus (L \cap M_2)$. Clearly, $L \cap M_1$ and $L \cap M_2$ are cofinite submodules of M_1 and M_2, respectively. For each i, there exist some direct summands D_i of M_i such that $M_i = D_i + Y_i$ if and on ly if $M_i = A_i + Y_i$ for any $Y_i \leq M_i$. Put $D = D_1 \oplus D_2$, and let $X \leq M$. Then $X = X_1 \oplus X_2$, where $X_i = X \cap M_i$ by Duo assumption. Hence $M = A + X$ if and only if $M_i = A_i + X_i$ $(i = 1, 2)$ if and only if $M_i = D_i + X_i$ $(i = 1, 2)$ if and only if $M = D + X$. ∎

Lemma 2.6. *Let R be any ring and let M be a $\oplus$-cofinitely supplemented $R-$module. Then every cofinite submodule of the module $M/Rad(M)$ is a direct summand.*

Proof. Let $N/Rad(M)$ be any cofinite submodule of $M/Rad(M)$. Then N is a cofinite submodule of M and by hypothesis there exists a submodule K of M such that $M = N + K = K \oplus K'$ and $N \cap K$ is small in K. Since $N \cap K$ is also small in M, $N \cap K \leq Rad(M)$. Thus $M/Rad(M) = (N/Rad(M)) \oplus ((K +$

$Rad(M))/Rad(M))$, as required. ■

Theorem 2.7. *Let $Rad(M) \ll M$. Then M is H-cofinitely supplemented if and only if every cofinite submodule of $M/Rad(M)$ is a direct summand and each cofinite direct summand of $M/Rad(M)$ lifts to a direct summand of M.*

Proof.
$(\Rightarrow)$: By Lemma 2.6, we prove only the last statement and let $N/Rad(M) = \bar{N} \leq \bar{M} = M/Rad(M)$ be a cofinite submodule. Then M/N is finitely generated and so $M = N + K$ with $N \cap K \leq Rad(M)$ for some $K \leq M$. By assumption, there exists a direct summand L of M such that $M = L \oplus L'$, for some submodule L' of M, and $M = N + X$ if and only if $M = L + X$. Hence $M = N + L'$ and $N \cap L'$ is small in L'. It follows that $\bar{M} = \bar{N} \oplus \bar{L}'$. Now we show $\overline{N} = \overline{L}$. Since N is cofinite, $N + L$ is cofinite and so $\overline{N + L}$ is cofinite in $\overline{M}$. By hypothesis $\overline{M} = \overline{N + L} \oplus \overline{U}$ for some $\overline{U} \leq \overline{M}$. It implies $M = N + L + U$ with $(N + L) \cap U = RadM$. Then $M = N + U = L + U$. By modularity $N + L = L + ((N + L) \cap U) = N + ((N + L) \cap U)$. It follows that $\overline{N} = \overline{L}$ since $(N + L) \cap U = RadM$.

$(\Leftarrow)$: Let N be a cofinite submodule of M. Then $(N + Rad(M))/Rad(M) = \bar{N}$ is a cofinite submodule of $\bar{M}$. There exists a submodule $\bar{K}$ of $\bar{M}$ such that $\bar{M} = \bar{N} \oplus \bar{K}$ and $\bar{N} = \bar{L}$ for some submodule L of M with $M = L \oplus L'$. Since $Rad(M)$ is small in M, it follows that $M = N + X$ if and only if $M = L + X$.

■

While the property supplemented is inherited by direct summands, it is unknow that the same is true for H-supplemented and H-cofinitely supplemented modules. In this vein we call a module M *completely H-cofinitely supplemented* if every direct summand of M is H-cofinitely supplemented. In Theorem 2.1, we proved that if M is an H-cofinitely supplemented module with the SSP, then every direct summand of M is an H-cofinitely supplemented module.

Theorem 2.8. *Let M be an H-cofinitely supplemented Duo module. Then M is a completely H-cofinitely supplemented module.*

Proof. This is a repetition of Corollary 2.3. ■

Example 2.9. Let F be a field and R the upper triangular matrix ring $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$. For submodules $A = \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}$ and $B = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$, let $M = A \oplus (R/B)$. Then M is an H-supplemented and completely H-supplemented module by [10, Example 2.14]. Also M has the SSP property. Therefore M is H-cofinitely supplemented and so it is a completely H-cofinitely supplemented module by Theorem 2.1.

For any submodule M, we shall denote the socle of M by $Soc(M)$.

Lemma 2.10. (see [1, Lemma 2.7]) *Let R be a ring. The following statements* *are equivalent for an $R-$ module M.*

1. *Every cofinite submodule of M is a direct summand of M.*
1. *Every maximal submodule of M is a direct summand of M.*
2. *$M/Soc(M)$ does not contain a maximal submodule.*

A module M is called *local* if the sum of all proper submodules of M is a proper submodule of M. Note that 0 is a local submodule and also a cofinitely supplemented submodule of M. For any module M, $Cof(M)$ will denote the sum of all cofinitely supplemented submodules of M, $Loc(M)$ will denote the sum of all local submodules of M (see [1]) and, in case M does not contain a local submodule, $Loc(M)$ is the zero submodule. By [15, 41.16], $Loc(M)$ is the sum of all cofinitely supplemented submodules of M. Thus;

$$\text{Loc}\ (M) \leq Cof(M) \cdots \tag{1}$$

Theorem 2.11. (see [1, Theorem 2.8]) *The following statements are equivalent for an $R-$module M.*

(1) *M is cofinitely supplemented.*
(2) *Every maximal submodule of M has a supplement in M.*
(3) *The module $M/Loc(M)$ does not contain a maximal submodule.*
(4) *The module $M/Cof(M)$ does not contain a maximal submodule.*

$\oplus - Cof(M)$, $\oplus - Cof_1(M)$ and $Loc_1(M)$ will denote the sum of all $\oplus$-cofinitely supplemented submodules of M, the sum of all $\oplus-$cofinitely supplemented submodules which are direct summand of M and the sum of all local submodules which are direct summand of M, respectively (see [8]). By [8], we have

$$Loc(M) \leq \oplus - Cof(M) \leq Cof(M) \cdots \tag{2}$$

and

$$Loc_1(M) \leq \oplus - Cof_1(M) \cdots \tag{3}$$

We consider the case R is the ring $\mathbb{Z}$ of rational integers. Then
(i). $Loc_1(M) = Loc(M) = 0$ for every torsionfree R-module M because a local and torsionfree $\mathbb{Z}$-module is zero.
(ii). $Cof(M) = \oplus - Cof(M) = \oplus - Cof_1(M)$ for every injective $R-$module M by [1] and [15, 41.23].
(iii). Let M denote the Prüfer $p-$group $\mathbb{Z}(p^{\infty})$ for some prime integer p. Then $Loc_1(M) = 0$, $\oplus - Cof_1(M) = M$.

Now $H - Cof(M)$ and $H - Cof_1(M)$ will denote the sum of all H-cofinitely supplemented submodules of M and the sum of all H-cofinitely supplemented submodules which are direct summand of M, respectively. Since every local module is hollow and hollow modules are H-supplemented by [11, Proposition A.2], so local modules are H-supplemented. Therefore, by (2), we have

$$Loc(M) \leq H - Cof(M) \leq \oplus - Cof(M) \leq Cof(M) \cdots \tag{2'}$$

and

$$Loc_1(M) \leq H - Cof_1(M) \leq \oplus - Cof_1(M) \cdots \tag{3'}$$

by (3) and (2′).

Question.
(1) When $Loc(M) = H - Cof(M) = \oplus - Cof(M) = Cof(M)$?
(2) When $Loc_1(M) \leq H - Cof_1(M) \leq \oplus - Cof_1(M)$?

Our aim in this section is to prove an analog of Theorem 2.11 for H-cofinitely supplemented modules.

Lemma 2.12. *Let M be an $R-$module. If every maximal submodule A of M there exists a direct summand D of M such that $M = A + X$ holds if and only if $M = D + X$ then $M/Loc_1(M)$ does not contain a maximal submodule of M.*

Proof. Suppose that $M/Loc_1(M)$ contains a maximal submodule $Q/Loc_1(M)$ of $M/Loc_1(M)$. Then Q is a maximal submodule of M. By hypothesis, there exists a direct summand P of M such that $M = Q+X$ if and only if $M = P+X$. Let $M = P \oplus P'$ for some submodule P' of M. Hence $M = Q + P'$. Clearly, $Q \cap P'$ is small in P'. This shows that P' is a local summand of M by [15, 41.1.(3)]. Therefore $P' \leq Loc_1(M)$. Thus $Q/Loc_1(M) = M/Loc_1(M)$. It is a contradiction. ■

Theorem 2.13. *Let R be a ring and M be a Duo right R-module. Then the following statements are equivalent:*
(1) *M is H-cofinitely supplemented.*
(2) *Every maximal submodule of M has an H-supplement in M.*
(3) *The module $M/Loc_1(M)$ does not contain a maximal submodule.*
(4) *The module $M/H - Cof_1(M)$ does not contain a maximal submodule.*

Proof.
(1) ⇒ (2) Clear.
(2) ⇒ (3) By Lemma 2.12.
(3) ⇒ (4) It follows from equation (3′).
(4) ⇒ (1) Let N be a cofinite submodule of M. Let $S = H - Cof_1(M)$. Then $M/(N+S)$ is finitely generated. By (4), $M = N+S$. Since $M/N = (N+S)/N \cong S/(N \cap S)$ is finitely generated, there exist H-cofinitely supplemented modules N_i $(i = 1, 2, ..., t)$, such that $M = N + N_1 + ... + N_t$. The rest is similar to [8, Theorem 1.12]. ■

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