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Instability of Impulsive Hybrid State Dependent Delay Differential Systems

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Abstract. In this paper, we present the sufficient conditions ensuring the instability of the zero solution of an impulsive state dependent delay differential system for different conditions on the delay function. We assume the instability of the associated linear impulsive system and apply the idea of dichotomies together with Schauder–Tychonoff Theorem to establish the sufficient conditions for the unstable solution of the problem at hand.

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1. Introduction

The linear nonautonomous delay systems and the state dependent delay equations are intensively investigated because of their theoretical importance to the theory of functional differential equations and their applications [5 - 8]. While the stability of the solution of the delay systems has received preeminent attention, the problem of giving sufficient conditions in case the solutions are unstable has not been studied enough [9, 12 - 13]. The notion of dichotomies is found to be quite elegant in describing the unstable properties of the nonlinear differential equations, see for example, [1 - 2, 15 - 20]. In relation to numerous applications in science and technology such as biological phenomena involving thresholds, bursting rhythm models in medicine and biology, optimal control models in economics and frequency modulated systems, the theory of impulsive differential equations has been developed intensively [4, 10, 11, 14].

In this paper, we study the instability of the nonautonomous impulsive differential system with state dependent delay by applying the idea of dichotomies and Schauder–Tychonoff Theorem [3]. State dependent time delay often occurs in engineering systems, medicine, hydro dynamics, laser physics, chemistry etc. Since the existence of state dependent time delay usually causes the instability to the systems, the study of state dependent time delay systems has received considerable attentions in the recent years, especially in Internet congestion, congestive heart failure in humans, traffic congestion in automated highway, mobile networking, population growth, etc. In Sec. 2, we set the notations and terminology relative to the impulsive differential system with state dependent delay. Section 3 addresses the instability of the zero solution of the impulsive state dependent delay system subject to non-delay dependent conditions. In Sec. 4, we enhance the scope of the instability results developed in Sec. 3 by allowing the delay function to behave like a time lag function. In fact, we study the effect of delay on the instability of the impulsive differential system.

2. Preliminary Notes

Let $t_0 < t_1 < t_2 < ... < t_i < ...$, $\lim_{i\to\infty} t_i = \infty$ be a given sequence of real numbers. We define $J = [t_0, \infty)$, $J_\tau = [t_0 - \tau, \infty)$ for a constant $\tau > 0$ and $PC([-\tau, 0], R^n) = \{\phi : [-\tau, 0] \to R^n, \phi(t)$ is continuous everywhere except at finite number of points $\hat{t}$ at which $\phi(\hat{t} + 0)$ and $\phi(\hat{t} - 0)$ exist, and $\phi(\hat{t} + 0) = \phi(\hat{t})\}$. We equip the linear space $PC([-\tau, 0], R^n)$ with the norm defined by $|\phi|_\tau = \max_{[-\tau,0]} |\phi|$ for $\phi \in PC([-\tau, 0], R^n)$ and $u_t \in PC([-\tau, 0], R^n)$ denotes the function $u_t(s) = u(t + s),\ t \in J,\ s \in [-\tau, 0]$.

Consider the impulsive differential system with state dependent delay

$$\begin{cases} u'(t) = A(t)u(t) + B(t)u(t - r(t, u(t))), & t \neq t_i, \\ u_{t_0} = \phi, & \\ u(t_i + 0) = C_i u(t_i), & i = 1, 2, ..., \end{cases} \tag{1}$$

where the matrices $A, B : [t_0 - \tau, \infty) \to R^{n\times n}$ are piecewise continuous with points of discontinuity of the first kind at $t = t_i$, the impulse matrices C_i are constant and nonsingular, and $r(t, u(t))$ is a nonnegative bounded delay function. The underlying vector space V is R^n and $|.|$ denotes a fixed norm in V. For a matrix $A \in R^{n\times n}$, $|A|$ will denote the corresponding matrix norm. By a solution of (1), we mean a piecewise uniformly continuous function $u(t, t_0, \phi)$ on $[t_0, \infty)$ which is left continuous on each interval $J_i = (t_i, t_{i+1}]$ and is defined by

$$u(t, t_0, \phi) = \begin{cases} \phi, & t_0 - \tau \leqslant t \leqslant t_0, \\ u_0(t, t_0, \phi), & t_0 \leqslant t \leqslant t_1, \\ u_1(t, t_1, \phi_1), & t_1 < t \leqslant t_2, \\ \quad \cdots & \\ u_i(t, t_i, \phi_i), & t_i < t \leqslant t_{i+1}, \\ \quad \cdots & \end{cases} \tag{2}$$

where $u_i(t, t_i, \phi_i)$ is the solution of the following delay differential equation

$$u'(t) = A(t)u(t) + B(t)u(t - r(t, u(t))),$$

$$u(t_i + 0) = \phi_i, \ \ i = 1, 2, ...,$$

with ϕ_i as the initial function on $(t_i, t_{i+1}]$. Let h, k denote the positive continuous functions having bounded growth with $h^{-1}(t) = 1/h(t)$. For a bounded function f, let $|f|^\infty = \sup\{|f(t)| : t \in J_\tau\}$, $C_h(J_\tau) = \{f : J_\tau \to V : h^{-1}f$ is uniformly continuous on all intervals $J_i\}$; for $f \in C_h(J_\tau)$, let $|f|_h = |h^{-1}f|^\infty$ and there exists $\sigma > 0$ such that $B_h[0, \sigma] = \{f \in C_h(J_\tau) : |f|_h \leqslant \sigma\}$. Moreover, $L^1(J)$ will denote the space of absolutely integrable functions defined on J, and $|f|^1 = \int_{t_0}^{\infty} |f(s)|ds$. For the forthcoming analysis, for $t \in [t_i + 0, t_{i+1}]$, the fundamental matrix $\Phi(t)$ of

$$\begin{cases} u'(t) = A(t)u(t), \ \ t \neq t_i, \\ u(t_i + 0) = C_i u(t_i), \ \ i = 1, 2, ..., \end{cases} \tag{3}$$

admits the representation

$$\Phi(t) = \Psi(t)\Psi^{-1}(t_i + 0)C_i\Psi(t_i)\Psi^{-1}(t_{i-1} + 0)C_{i-1}...C_1\Psi(t_1)\Psi^{-1}(t_0),$$

where $\Psi(t)$ is the fundamental matrix of the equation $u'(t) = A(t)u(t)$. The matrix $\Phi(t)$ is continuously differentiable for $t \neq t_i$ with points of discontinuity of the first kind at $t = t_i$, that is, $\Phi(t_i + 0) = C_i\Phi(t_i)$. The matrix $\Phi(t)$ is invertible if and only if the impulse matrices $C_i, \ i = 1, 2, ...$ are nonsingular.

We will use the notion of (h, k)-dichotomies [1, 2, 16] to discuss the unstable properties of (3).

Definition 1. *We say that* (3) *has an* (h, k)*-dichotomy on the interval* J_τ *if and only if there exists a constant* L *and a projection matrix* P $(P^2 = P)$ *such that*

$$|\Phi(t)P\Phi^{-1}(s)| \leqslant Lh(t)h^{-1}(s), \ t_0 - \tau \leqslant s \leqslant t,$$

$$|\Phi(t)(I - P)\Phi^{-1}(s)| \leqslant Lk(t)k^{-1}(s), \ \ t_0 - \tau \leqslant t \leqslant s, \tag{4}$$

where I *is an identity matrix. Moreover, there exists a positive constant* M *such that*

$$h(t)h^{-1}(s) \leqslant Mk(t)k^{-1}(s), \ \ t \geq s, \ s, t \in J. \tag{5}$$

Remark 1. In case $h = k$, we will say that (3) possesses an h-dichotomy. If (3) has an (h, k)-dichotomy, then (5) implies that (3) has both an h-dichotomy and a k-dichotomy, each one with the same projection matrix P and constant ML.

Definition 2. *We say that the function* $h : J_\tau \to (0, \infty)$ *is of class* $G_{\tau,N}$ *if and only if* $h(s)h^{-1}(t) \leqslant N, \ s \in [t - \tau, t + \tau], \ t \geq t_0$ *for a positive constant* N.

In passing, it is worth remarking that the constants M, N are greater than or equal to 1.

Definition 3. *The zero solution of* (1) *is said to be h-stable on the interval J if and only if for every $\epsilon > 0$, there exists a $\delta > 0$ ($\delta = \delta(t_0, \epsilon)$) such that if $\phi \in C([-\tau, 0])$ and $|\phi|_\tau < \delta$, then the solution $u(t, t_0, \phi)$ exists on all J and $h^{-1}(t)|u(t, t_0, \phi)| < \epsilon$ for all $t \geq t_0$. Moreover, if $|\phi|_\tau < \delta$ implies*

$$\lim_{t\to\infty} h^{-1}(t)u(t, t_0, \phi) = 0, \tag{6}$$

then the zero solution of (1) is called h-asymptotically stable.

Definition 4. *The zero solution of* (1) *is said to be h-unstable on J if and only if there exists an $\epsilon > 0$ such that for every $\delta > 0$ there exists an initial value function $\phi_\delta \in C([-\tau, 0])$, $|\phi_\delta|_\tau < \delta$ and a $\tau_\delta \geq t_0$ such that $|u(\tau_\delta, t_0, \phi_\delta)| > \epsilon$.*

3. Non-delay Dependent Conditions

We assume that (3) possesses the dichotomy (4)–(5). For $t \neq t_i$, we define the dichotomic operator U associated with (1) as

$$\begin{aligned} U[u](t) = &\int_{t_0}^{t} \Phi(t)P\Phi^{-1}(s)B(s)u(s - r(s, u(s)))ds \\ &- \int_{t}^{\infty} \Phi(t)(I - P)\Phi^{-1}(s)B(s)u(s - r(s, u(s)))ds, \end{aligned}$$

and for $t = t_i$,

$$\begin{aligned} U[u](t_i + 0) &= \int_{t_0}^{\infty} \chi(t_i + 0, s)B(s)u(s - r(s, u(s)))ds \\ &= \int_{t_0}^{\infty} C_i\chi(t_i, s)B(s)u(s - r(s, u(s)))ds = C_iU[u](t_i), \end{aligned}$$

where

$$\chi(t, s) = \begin{cases} \Phi(t)P\Phi^{-1}(s), & t_0 - \tau \leqslant s \leqslant t, \\ \Phi(t)(I - P)\Phi^{-1}(s), & t_0 - \tau \leqslant t \leqslant s. \end{cases}$$

Since the function $U[u]$ is not defined on $[t_0 - \tau, t_0]$, we complete its definition on this interval as

$$\Omega[u](t) = \begin{cases} U[u](t_0), & t_0 - \tau \leqslant t < t_0, \\ U[u](t), & t \geq t_0. \end{cases}$$

Let the delay function $r(t, u)$ be defined on the set

$$C_h(\sigma_0) = \{(t, h(t)u) : t \in [t_0, \infty),\, |u| \leqslant \sigma_0,\, \sigma_0 > 0\},$$

where $0 < \sigma \leqslant \sigma_0$ and $r(t, u)$ is is assumed to be bounded, that is, there exists a constant τ such that

$$0 \leqslant r(t, h(t)u) \leqslant \tau, \text{ for all } (t, h(t)u) \in C_h(\sigma_0). \tag{7}$$

Lemma 1. *For $h \in G_{\tau,N}$, let* (3) *have the dichotomy* (4)–(5), *the condition* (7) *is satisfied and*

$$LMN^2|B|^1 < 1. \tag{8}$$

Then, for every σ $(0 < \sigma \leqslant \sigma_0)$, we have $\Omega : B_h[0,\sigma] \to B_h[0,\sigma]$.

Proof. For $t \neq t_i$ with $t \geq t_0$, we have the estimate

$$\begin{aligned} |h^{-1}(t)\Omega[u](t)| &\leqslant L\int_{t_0}^{t} h^{-1}(s)|B(s)||u(s-r(s,h(s)h^{-1}(s)u(s)))|ds \\ &\quad + L\int_{t}^{\infty} h^{-1}(t)k(t)k^{-1}(s)|B(s)||u(s-r(s,h(s)h^{-1}(s)u(s)))|ds \\ &\leqslant LMN\int_{t_0}^{\infty}|B(s)|ds|u|_h. \end{aligned}$$

On the same pattern, for $t = t_i$, we have

$$|h^{-1}(t_i)\Omega[u](t_i)| \leqslant LMN|C_i|\int_{t_0}^{\infty}|B(s)|ds|u|_h, \ \forall i.$$

For $t \in [t_0 - \tau, t_0]$, we may have

$$|h^{-1}(t)\Omega[u](t)| = |h^{-1}(t)h(t_0)h^{-1}(t_0)\Omega[u](t_0)| \leqslant LMN^2\int_{t_0}^{\infty}|B(s)|ds|u|_h.$$

The conclusion of the lemma follows from these estimates. ∎

Let us define the subspaces of initial conditions for (3) as

$$V_h = \{\xi \in V : \Phi(t)\xi \in C_h\}, \ \ V_{k,0} = \{\xi \in V_k : \lim_{t\to\infty} k(t)^{-1}(t)\Phi(t)\xi = 0\}.$$

Now, we state a known result which is needed to prove that the zero solution of (1) is $h-$unstable. For the proof of this theorem, see [19].

Theorem 1. *If* (3) *possesses an (h,k)-dichotomy with projection P, then* (3) *allows an h-dichotomy with projection Q if and only if $V_{k,0} \subset Q[V] \subset V_h$.*

Theorem 2. *Assume that the hypotheses of Lemma 1 hold and $V_h \neq V$. Then the zero solution of* (1) *is h-unstable.*

Proof. For the sake of contradiction, we assume that the zero solution of (1) is h-stable. Then for $0 < \epsilon < \min(\sigma, \sigma_0)$, there exists a $\delta > 0$ such that $|h^{-1}(t^*)u(t^*, t_0, \phi)| < \sigma$ for $t^* > t_0$ ($t_0 \in R_+$ is given) with $t_i < t^* \leqslant t_{i+1}$ and $|h^{-1}(t)u(t,t_0,\phi)| \geq \sigma$ for $t_0 \leqslant t \leqslant t_i$ for some i if $|\phi|_\tau < \delta$. Also, $|h^{-1}(t_i+0)u(t_i+0,t_0,\phi)| = |h^{-1}(t_i)C_iu(t_i,t_0,\phi)| \geq \epsilon$. Hence, we can find a t^0 satisfying $t_i < t^0 \leqslant t^*$ such that $\epsilon \leqslant |h^{-1}(t^0)u(t^0,t_0,\phi)| < \sigma$. Define

$$z(t^0) = u(t^0,t_0,\phi) - \Omega[u](.;t_0,\phi)(t^0), \ \ t_0 - \tau \leqslant t^0,$$

where the initial value function ϕ satisfies

$$0 \neq |\phi|_\tau < \delta, \ \ |h_{t_0}^{-1}\phi|_\tau = |h^{-1}(t_0)\phi(0)|, \ \ P\phi(0) = 0. \tag{9}$$

Notice that $V_h \neq V$ implies that $P \neq I$, by Theorem 1. From Lemma 1, it follows that the function $z \in C_h(J_\tau)$ and it can readily be verified that z is a solution of (3) for $t_0 \leqslant t^0$. Hence $z(t_0) \in V_h$ and by Theorem 1, we may assume that $z(t_0) \in P[V]$. Further

$$z(t_0) = \phi(0) + (I - P)\int_{t_0}^{\infty} \Phi^{-1}(s)B(s)u(s - r(s, u(s)))ds,$$

which, in view of $P\phi(0) = 0$, implies that $z(t_0) = 0$ and hence $z(t^0) = 0$ for all $t_0 \leqslant t^0$. However, if $u(.; t_0, \phi)$ satisfies the integral equation

$$u(t^0, t_0, \phi) = \Omega[u](.; t_0, \phi)(t^0), \ \ t_0 \leqslant t^0,$$

then, by virtue of the first estimate obtained in Lemma 1,we obtain

$$\sup_{t^0 \in [t_0, \infty)} |h^{-1}(t^0)u(t^0, t_0, \phi)| \leqslant LMN^2|B|^1|u(.; t_0, \phi)|_h, \ N \geq 1.$$

But the choice of ϕ (condition (9)) implies that

$$|u(.; t_0, \phi)|_h \leqslant LMN^2|B|^1|u(.; t_0, \phi)|_h.$$

Since $LMN^2|B|^1 < 1$, therefore we get

$$u(t^0, t_0, \phi) = 0, \ \ t_0 - \tau \leqslant t^0,$$

which contradicts that $u(t^0, t_0, \phi) = \phi(0) \neq 0$. This completes the proof of the theorem. ∎

Remark 2. If (3) has the (h, k)-dichotomy (4)–(5), $k \in G_{\tau,N}$, the conditions $LMN^2|B|^1 < 1$ and

$$0 \leqslant r(t, k(t)u) \leqslant \tau, \ \ \text{for all } (t, k(t)u) \in C_k(\sigma_0), \tag{10}$$

are satisfied, then the zero solution of (1) is $k-$unstable if $V_k \neq V$. This result follows from Theorem 2.

In order to cope with the situation when $V_h \neq V$ in Remark 2 ($V_k \neq V$ in Theorem 3) is not satisfied, we present the following theorem to deal with the problem of instability.

Theorem 3. *If* (3) *has the* (h, k)*-dichotomy* (4)–(5), $h \in G_{\tau,N}$, $LMN^2|B|^1 < 1$ *and* (7) *holds, then the zero solution of* (1) *is not asymptotically h-stable if* $V_{h,0} \neq V_h$ *(if* $k \in G_{\tau,N}$, $V_{k,0} \neq V_k$ *and* (10) *is valid, then the zero solution of* (1) *is not asymptotically k-stable).*

Proof. In view of Remark 1, it follows by Theorem 1 that it is reasonable to assume that the projection P defining the h-dichotomy satisfies

$$\lim_{t\to\infty} h^{-1}(t)\Phi(t)P = 0. \tag{11}$$

Let us suppose that the zero solution of (1) is asymptotically h-stable. Then for $0 < \epsilon < \min(\sigma, \sigma_0)$, there exists a $\delta > 0$ such that $|\phi|_\tau < \delta$ implies that $|h^{-1}(t^*)u(t^*, t_0, \phi)| < \sigma$ for $t^* > t_0$ with $t_i < t^* \leqslant t_{i+1}$ and $|h^{-1}(t)u(t, t_0, \phi)| \geq \sigma$ for $t_0 \leqslant t \leqslant t_i$ for some i if $|\phi|_\tau < \delta$. Also, $|h^{-1}(t_i + 0)u(t_i + 0, t_0, \phi)| = |h^{-1}(t_i)C_i u(t_i, t_0, \phi)| \geq \epsilon$. Hence, we can find a t^0 satisfying $t_i < t^0 \leqslant t^*$ such that $\epsilon \leqslant |h^{-1}(t^0)u(t^0, t_0, \phi)| < \sigma$. Let $\sigma_1 \leqslant \min\{1, \sigma\}$ be a positive number such that

$$\sigma_1 |h_{t_0}|_\tau < \delta, \tag{12}$$

and there exists a positive number β such that

$$\beta + LMN^2 \int_{t_0}^{\infty} |B(s)| ds \sigma_1 \leqslant \sigma_1. \tag{13}$$

We fix a vector $z_0 \in V_h \setminus V_{h,0}$ satisfying $|\Phi_{z_0}|_h \leqslant \beta$ and introduce the operator Λ given by

$$\Lambda[u](t^0) = \Phi(t^0)z_0 + \Omega[u](t^0),\ t^0 \geq t_0 - \tau. \tag{14}$$

From the choice of β and by Lemma 1, it follows that $\Lambda : B_h[0, \sigma_1] \to B_h[0, \sigma_1]$.

Let $\{u_n\}$ be a sequence of functions in $B_h[0, \sigma_1]$ and uniformly converging to a function u on any compact interval $[t_0 - \tau, t_1^0]$, $t_1^0 \geq t_0$. For a chosen value of $\epsilon > 0$, we can have a sufficiently large number t_2^0 ($\geq t_1^0$) such that

$$LMN^2 \int_{t^0}^{\infty} |B(s)| ds < \epsilon,\ \forall t^0 \geq t_2^0.$$

Now, we define a function $\alpha(\epsilon) : [t_0 - \tau, \infty) \to V$ by

$$\alpha(\epsilon)(t^0) = \begin{cases} 0, & t_0 - \tau \leqslant t^0 \leqslant t_2^0, \\ -\int_{t^0}^{\infty} \Phi(t^0)(I-P)\Phi^{-1}(s)B(s)u_n(s - r(s, u_n(s)))ds, & t^0 > t_2^0. \end{cases}$$

In view of the last estimate, we have $|h(t^0)\alpha(\epsilon)(t^0)| \leqslant \epsilon$. Thus

$$\begin{aligned} \Lambda[u_n](t^0) = \Phi(t^0)z_0 &+ h(t^0)\alpha(\epsilon) \\ &+ \int_{t_0}^{t^0} \Phi(t^0)P\Phi^{-1}(s)B(s)u_n(s - r(s, u_n(s)))ds \\ &- \int_{t^0}^{t_2^0} \Phi(t^0)(I-P)\Phi^{-1}(s)B(s)u_n(s - r(s, u_n(s)))ds. \end{aligned}$$

From here, it follows that $\Lambda[u_n](t^0)$ converges uniformly to $\Lambda[u](t^0)$ on $[t_0, t_1^0]$. Hence the operator Λ is continuous in the sense required in Schauder–Tychonoff

Theorem [3]. Now, for a piecewise uniformly continuous function u on $[t_0 - \tau, \infty)$, we notice that

$$\Lambda[u]'(t^0) = A(t^0)\Lambda[u](t^0) + B(t^0)u(t^0 - r(t, u(t^0)))ds,\ \ t^0 \geq t_0.$$

Therefore, the family of functions $\Lambda[B_h[0, \sigma_1]]$ is equicontinuous at any $t^0 \geq t_0$. Moreover, this family of functions is also equicontinuous on $[t_0 - \tau, t_0]$ since $\Lambda[u](t^0) = \Phi(t^0)z_0 + \Omega[u](t_0),\ \ t_0 - \tau \leqslant t^0 \leqslant t_0$. So, by Schauder–Tychonoff Theorem [3], this operator has a fixed point in $B_h[0, \sigma_1]$. Let u be one of these fixed points which in fact is a solution of (1) on $[t_0, \infty]$. Hence from (11) and the dominated convergence theorem, it follows that

$$u(t^0) = \Phi(t^0)z_0 + o(h)(t^0), \tag{15}$$

where $o(h)$ satisfies $\lim_{t^0 \to \infty} h^{-1}(t^0)o(h)(t^0) = 0$. On the other hand, as $u = \Lambda[u]$; for $t^0 \in [t_0 - \tau, t_0]$, we have

$$|u(t^0)| \leqslant (|\Phi_{z_0}|_h + |\Omega[u]|_h)h(t^0) \leqslant (\beta + LMN^2 \int_{t_0}^{\infty} |B(s)|ds\sigma_1)h(t^0),$$

which together with (12) and (13) yields

$$|u(t^0)| \leqslant \sigma_1 |h_{t_0}|_\tau < \delta,\ \ t^0 \in [t_0 - \tau, t_0].$$

Therefore, $|u_{t_0}|_\tau < \delta$ which implies $u(t^0) = o(h)(t^0)$. But $\lim_{t^0 \to \infty} h^{-1}(t^0)\Phi(t^0)z_0 \neq 0$, which contradicts (15). This completes the proof. ∎

4. Effect of Delay on the Instability

The results obtained in Sec. 3 are limited in the sense that the condition (8) does not involve the the time lag function $r(t, u)$. For example, Theorems 2 and 3 cannot be applied to study the instability of equation $u''(t) = u(t - e^{-t}|u(t)|^\beta),\ \ 0 \leqslant \beta < 1$. In this section, we will study delay dependent instability. We do not assume that $B \in L^1$ and rewrite (1) as

$$\begin{cases} u'(t) = (A(t) + B(t))u(t) + B(t)(u(t - r(t, u(t))) - u(t)),\ \ t \neq t_i, \\ u_{t_0} = \phi, \\ u(t_i + 0) = C_i u(t_i),\ \ i = 1, 2, ... \end{cases}$$

Moreover, the notation Φ now represents the fundamental matrix of the system

$$\begin{cases} z'(t) = (A(t) + B(t))z(t),\ \ t \neq t_i, \\ z(t_i + 0) = C_i z(t_i),\ \ i = 1, 2, ..., \end{cases} \tag{16}$$

and the subspaces V_h, V_k now refer to (16) and the dichotomic operator is defined by

$$\begin{aligned} W[u](t) = &\int_{t_0}^{t} \Phi(t)P\Phi^{-1}(s)B(s)(u(s - r(s, u(s))) - u(s))ds \\ &- \int_{t}^{\infty} \Phi(t)(I - P)\Phi^{-1}(s)B(s)(u(s - r(s, u(s))) - u(s))ds,\ \ t \neq t_i, \end{aligned}$$

and for $t = t_i$,

$$W[u](t_i+0) = \int_{t_0}^{\infty} \chi(t_i+0,s)B(s)(u(s-r(s,u(s))) - u(s))ds$$
$$= \int_{t_0}^{\infty} C_i\chi(t_i,s)B(s)(u(s-r(s,u(s))) - u(s))ds = C_iW[u](t_i).$$

We set

$$\Upsilon[u](t) = \begin{cases} W[u](t_0), & t_0-\tau \leqslant t < t_0, \\ W[u](t), & t \geq t_0. \end{cases}$$

The delay function $r(t,u)$ is assumed to be satisfying

$$r(t,h(t)(u)) \leqslant r_1(t) + r_2(t)\psi(|u|) \leqslant \tau, \ \ \forall (t,h(t)(u)) \in C_h(\sigma_0), \tag{17}$$

where r_1, r_2 are assumed to be continuous functions and ψ is monotone increasing with $\psi(0) = 0$. Notice that the condition (17) implies

$$r(t,u(t)) = r(t,h(t)h^{-1}(t)u(t)) \leqslant r_1(t) + r_2\psi(\sigma), \ \ \forall u \in B_h[0,\sigma_0].$$

Motivated by [5, 13], we introduce the set $B_h^*[0,\sigma]$ which consists of those functions of $B_h[0,\sigma]$ which satisfy

$$h^{-1}(t)|u(t) - u(t')| \leqslant N\alpha(t)(t-t')\sigma, \tag{18}$$

where $t_0 - \tau \leqslant t' \leqslant t, \ t' \geq t_0, \ h \in G_{\tau,N}$ and

$$\alpha(t) = \max\{1, |(A+B)_t|_\tau\}, \ t \geq t_0.$$

Further, it follows from the standard arguments [5,13] that $B_h^*[0,\sigma]$ is a closed set in $B_h[0,\sigma]$.

Lemma 2. *Assume that* (16) *has the dichotomy* (4)–(5) *and satisfies the condition* (17). *Further, we require that*

$$2LMN^2\{|\alpha r_1 B|^1 + |\alpha r_1 B|^\infty\} < 1, \tag{19}$$

and the function $\alpha r_2 B$ is bounded and integrable, then for every σ_1 $(0 < \sigma_1 \leqslant \sigma)$, we have $\Upsilon : B_h^*[0,\sigma_1] \to B_h^*[0,\sigma_1]$.

Proof. For every $\sigma > 0, \ t \geq t_0, \ t \neq t_i, \ u \in B_h^*[0,\sigma]$, using the definition of the operator Υ, we have

$$|h^{-1}(t)\Upsilon[u](t)| \leqslant LM\int_{t_0}^{\infty} h^{-1}(s)|B(s)||u(s-r(s,u(s))) - u(s)|ds$$
$$\leqslant LMN\int_{t_0}^{\infty} \alpha(s)r(s,u(s))|B(s)|ds \ \sigma.$$

In a similar fashion, one can obtain

$$\begin{aligned}|h^{-1}(t_i)\Upsilon[u](t_i)| &\leqslant LM\int_{t_0}^{\infty} h^{-1}(s)|B(s)||u(s-r(s,u(s)))-u(s)|ds\\ &\leqslant LMN|C_i|\int_{t_0}^{\infty}\alpha(s)r(s,u(s))|B(s)|ds\ \sigma.\end{aligned}$$

If $t\in[t_0-\tau,t_0]$, then

$$\begin{aligned}|h^{-1}(t)\Upsilon[u](t)| &= |h^{-1}(t)h(t_0)h^{-1}(t_0)\Upsilon[u](t_0)|\\ &\leqslant LMN^2\int_{t_0}^{\infty}\alpha(s)r(s,u(s))|B(s)|ds\ \sigma.\end{aligned}$$

From these estimates together with (17), we get

$$|h^{-1}(t)\Upsilon[u](t)|\leqslant LMN^2\{|\alpha r_1 B|^1+|\alpha r_2 B|^1\psi(\sigma)\}\sigma.$$

Since $\psi(\sigma)\to 0$ as $\sigma\to 0$, so from (19), we obtain $|\Upsilon[u]|_h\leqslant\sigma$ provided σ is sufficiently small. In order to prove (18), we proceed as follows:

$$\Upsilon[u](t)-\Upsilon[u](t')=\widehat{S}_1(t)+\widehat{S}_2(t)+\widehat{S}_3(t)-\widehat{S}_4(t),\ t_0-\tau\leqslant t'\leqslant t,\ t'\geq t_0,\ t\neq t_i,$$

where

$$\begin{aligned}\widehat{S}_1(t) &= \int_{t'}^{t}\Phi(t)P\Phi^{-1}(s)B(s)(u(s-r(s,u(s)))-u(s))ds,\\ \widehat{S}_2(t) &= \int_{t_0}^{t'}[\Phi(t)-\Phi(t')]P\Phi^{-1}(s)B(s)(u(s-r(s,u(s)))-u(s))ds,\\ \widehat{S}_3(t) &= \int_{t'}^{t}\Phi(t')(I-P)\Phi^{-1}(s)B(s)(u(s-r(s,u(s)))-u(s))ds,\\ \widehat{S}_4(t) &= \int_{t}^{\infty}[\Phi(t)-\Phi(t')](I-P)\Phi^{-1}(s)B(s)(u(s-r(s,u(s)))-u(s))ds.\end{aligned}$$

From (17), for each $u\in B_h^*[0,\sigma]$, we have

$$\begin{aligned}|h^{-1}(t)\widehat{S}_1(t)| &\leqslant LN\int_{t'}^{t}\alpha(s)|B(s)|r(s,u(s))ds\ \sigma\\ &\leqslant LN\alpha(t)\int_{t'}^{t}\alpha(s)|B(s)|r(s,u(s))ds\ \sigma\\ &\leqslant LN\alpha(t)\int_{t'}^{t}\alpha(s)|B(s)|(r_1(t)+r_2\psi(\sigma))ds\ \sigma\\ &\leqslant LN\alpha(t)\{|\alpha r_1 B|^{\infty}+|\alpha r_2 B|^{\infty}\psi(\sigma)\}(t-t')\,\sigma. \qquad (20)\end{aligned}$$

In view of the following estimate

$$\begin{aligned}|(\Phi(t)-\Phi(t'))P\Phi^{-1}(s)| &= |\int_{t'}^{t}(A+B)(\nu)\Phi(\nu)d\nu P\Phi^{-1}(s)|\\ &\leqslant L\alpha(t)\int_{t'}^{t}h(\nu)h^{-1}(s)d\nu\\ &\leqslant LN\alpha(t)h(t)h^{-1}(s)(t-t'),\end{aligned} \tag{21}$$

$(t' \leqslant s \leqslant t,\ t-t' \leqslant \tau,\ t \neq t_i)$, we obtain

$$\begin{aligned}|h^{-1}(t)\widehat{S}_2(t)| &\leqslant LN^2\alpha(t)\int_{t_0}^{\infty}\alpha(s)|B(s)|r(s,u(s))ds(t-t')\sigma\\ &\leqslant LN^2\alpha(t)\{|\alpha r_1B|^1+|\alpha r_2B|^1\psi(\sigma)\}(t-t')\sigma.\end{aligned} \tag{22}$$

Furthermore, we have

$$\begin{aligned}|h^{-1}(t)\widehat{S}_3(t)| &\leqslant LMN[\int_{t'}^{t}\frac{h(t')}{h(t)}\alpha(s)|B(s)|r(s,u(s))ds]\,\sigma\\ &\leqslant LMN^2\alpha(t)\{|\alpha r_1B|^\infty+|\alpha r_2B|^\infty\psi(\sigma)\}(t-t')\,\sigma.\end{aligned} \tag{23}$$

Following the procedure used in obtaining (21), we have

$$\begin{aligned}|h^{-1}(t)\widehat{S}_4(t)| &\leqslant LMN^2\alpha(t)\int_{t}^{\infty}\alpha(s)|B(s)|r(s,u(s))ds(t-t')\sigma\\ &\leqslant LMN^2\alpha(t)\{|\alpha r_1B|^1+|\alpha r_2B|^1\psi(\sigma)\}(t-t')\sigma.\end{aligned} \tag{24}$$

From (20) and (22)–(24), it follows that

$$\begin{aligned}h^{-1}(t)|\Upsilon[u](t)-\Upsilon[u](t')| &\leqslant 2LMN^2\alpha(t)\{|\alpha r_1B|^1+|\alpha r_1B|^\infty\\ &\quad+(|\alpha r_2B|^1+|\alpha r_2B|^\infty)\psi(\sigma)\}(t-t')\sigma.\end{aligned} \tag{25a}$$

Repeating the above procedure for $t=t_i$, we have

$$\begin{aligned}h^{-1}(t_i)|\Upsilon[u](t_i)-\Upsilon[u](t')| &\leqslant 2LMN^2|C_i|\alpha(t_i)\{|\alpha r_1B|^1+|\alpha r_1B|^\infty\\ &\quad+(|\alpha r_2B|^1+|\alpha r_2B|^\infty)\psi(\sigma)\}(t_i-t')\sigma.\end{aligned} \tag{25b}$$

Taking into account $\lim_{\sigma\to 0}\psi(\sigma)=0$, the proof of the lemma follows from (19) and (25) for small values of σ. ■

Lemma 3. *If $h \in G_{\tau,N}$ and $z_0 \in V_h$, then $\Phi z_0 \in B_h^*[0,\sigma]$.*

Proof. If $t_0 - \tau \leqslant t' \leqslant t, t \geq t_0$ with $t \neq t_i$, then

$$\begin{aligned} h^{-1}(t)|\Phi(t)z_0 - \Phi(t')z_0| &= |h^{-1}(t)\int_{t'}^{t}(A+B)(\xi)\Phi(\xi)z_0 d\xi| \\ &\leqslant h^{-1}(t)\int_{t'}^{t}|(A+B)(\xi)|h(\xi)|h^{-1}(\xi)\Phi(\xi)z_0|d\xi \\ &\leqslant N\alpha(t)(t-t')\sigma. \end{aligned}$$

Also, we have $h^{-1}(t_i)|\Phi(t_i)z_0 - \Phi(t')z_0| \leqslant N|C_i|\alpha(t_i)(t_i - t')\sigma$. This completes the proof of the lemma. ■

Theorem 4. *Assume that* (16) *possesses the dichotomy* (4)–(5). *Let* $h, k \in G_{\tau,N}$ *and* (19) *is satisfied. Moreover, we require that*

$$r(t, h(t)(u)) \leqslant r_1(t) + r_2(t)\psi(|u|) \leqslant \tau, \;\; \forall (t, h(t)(u)) \in C_h(\sigma_0), \tag{26}$$

$$r(t, k(t)(u)) \leqslant r_1(t) + r_2(t)\psi(|u|) \leqslant \tau, \;\; \forall (t, k(t)(u)) \in C_k(\sigma_0). \tag{27}$$

If the function $\alpha r_2 B$ *is bounded and integrable and* $V_h \neq V_k$, *then the zero solution of* (1) *is not* h*-stable.*

Proof. Let us assume that the zero solution of (1) is h-stable. Then for $\sigma > 0$ and $0 < \epsilon < \min(\sigma, \sigma_0)$, there exists a $\delta > 0$ such that $|u(t^*, t_0, \phi)| < \sigma$ for $t^* > t_0$ with $t_i < t^* \leqslant t_{i+1}$ and $|u(t, t_0, \phi)| \geq \sigma$ for $t_0 \leqslant t \leqslant t_i$ for some i provided $|\phi|_\tau < \delta$. Also, $|u(t_i + 0, t_0, \phi)| = |C_i u(t_i, t_0, \phi)| \geq \epsilon$. Hence, we can find a t^0 satisfying $t_i < t^0 \leqslant t^*$ such that $\epsilon \leqslant |u(t^0, t_0, \phi)| < \sigma$. For a sufficiently small number σ_1 ($\sigma_1 \leqslant \sigma$), we have

$$2LMN^2\{|\alpha r_1 B|^1 + |\alpha r_2 B|^1 \psi(\sigma_1)\}\sigma_1 \leqslant \sigma_1 \leqslant \sigma, \;\; \sigma_1 |k_{t_0}|_\tau < \delta.$$

For a small $\beta > 0$, we fix an initial condition $z_0 \in V_k \backslash V_h$ such that $|z(t^0, t_0, z_0)|_k \leqslant \beta$, where β satisfies

$$\beta + 2LMN^2\{|\alpha r_1 B|^1 + |\alpha r_2 B|^1 \psi(\sigma_1)\}\sigma_1 \leqslant \sigma_1, \tag{28}$$

and $z(t^0, t_0, z_0)$ represents the solution of (16) satisfying $z(t_0, t_0, z_0) = z_0$. Now we consider the integral equation $u = \Theta[u]$, where Θ is defined by

$$\Theta[u](t^0) = \Phi(t^0)z_0 + \Upsilon[u](t^0), \;\; t^0 \geq t_0 - \tau.$$

Using k instead of h in Lemma 2 and Lemma 3 together with the choice of β, it follows that $\Theta : B_k^*[0, \sigma_1] \to B_k^*[0, \sigma_1]$.

Employing the procedure used in the proof of Theorem 3, it can be shown that the operator Θ satisfies Schauder–Tychonoff Theorem. Letting u to be a fixed point of the operator Θ, it is straightforward to show that u is a solution of (1). Moreover, $t_0 - \tau \leqslant t^0 \leqslant t_0$ and (28) imply that

$$|u(t^0)| \leqslant (|\Phi z_0|_k + |\Upsilon[u]|_k)k(t^0) \leqslant \sigma_1 |k_{t_0}|_\tau < \delta,$$

which evidently shows that u is an $h-$bounded function. From the hypothesis of the theorem, it follows that $\Upsilon : B_h^*[0,\sigma] \to B_h^*[0,\sigma]$. Thus, $\Upsilon[u]$ is h-bounded. Since $u(t^0) = z(t^0, t_0, z_0) + \Upsilon[u](t^0)$, therefore, $z(t^0, t_0, z_0)$ is $h-$bounded. This contradicts the choice of z_0. ■

Example. As an application of Theorem 4, we study the instability of the zero solution of the equation

$$u''(t) = u(t - e^{-t}|u(t)|^{\beta}), \ \ t \neq t_i, \ \ 0 \leqslant \beta < 1,$$

whose corresponding first order system has an associated equation (16), which has an $(1, e^t)$-dichotomy with $V_1 \neq V_{e^t}$. In this case, h^{-1} and k^{-1} are bounded. The conditions (26) and (27) are satisfied with $\sigma_0 = 1$, $\tau = 1$, $r_1(t) = 0$, $r_2(t) = e^{(\beta-1)t}$, $\psi(\sigma) = \sigma^{\beta}$. Moreover, we emphasize that $r(t, h(t)(u))$ and $r(t, k(t)(u))$ are respectively bounded on the sets $C_h(1)$ and $C_k(1)$ by the constant $\tau = 1$. Condition (19) is satisfied in view of $r_1(t) = 0$ and using the notations of Theorem 4, $\alpha r_2 B = e^{(\beta-1)t}$ is bounded and integrable. Thus, by Theorem 4, it follows that the zero solution of this equation will be Liapunov unstable.

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