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On the Characterizations and Their Stabilities of the Composed Random Variables by Constant-Regression

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Abstract. Let us consider the composed random variable (r.v) $\eta = \sum_{k=1}^{\nu} \xi_k$ where $\xi_1, \xi_2, ...$ are i.i.d r.vs and ν is positive $r.v$, independent of all ξ_k.

In [1, 2], we gave some characterizations of the distribution function of η. In this paper, we give another characteristic function of η satisfying some differential equations and prove the stabilities of those theorems.

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1. Introduction

Let us consider the random variables $(r.v)$:

$$\eta = \sum_{k=1}^{\nu} \xi_k$$

where $\xi_k, \xi_2, ..., \xi_k, ...$ are independent identically distributed random variables with the distribution function $F(x)$ and the characteristic function $\varphi(t)$; ν is a positive value r.v independent of all ξ_k and ν has the distribution function $A(x)$ with the generating function $a(z)$.

In [1] and [2], η is called the composed r.v of ν and ξ_k and has the characteristic function $\psi(t)$

$$\psi(t) = a[\varphi(t)]. \tag{1}$$

In those papers, we have already considered the case ξ_k has the exponential law with the characteristic function

$$\psi(t) = \frac{1}{1 - i\theta t},$$

and ν has the geometric law with the generating function

$$a(z) = \frac{\alpha z}{1 - \beta z} \quad (\alpha + \beta = 1).$$

In this case

$$\psi(t) = a[\varphi(t)] = \frac{\alpha}{\alpha - i\theta t} \tag{2}$$

and we showed that the composed r.v η has the stability in the following sense: If the condition "ξ_k has the exponential law" is changed by the condition: For sufficiently small ε, ξ_k has ε-exponential law then the distribution function $G(x)$ of η will have $\delta(\varepsilon)$-exponential law too (see [1, 2]).

In this paper, we also consider the above case and another case: ν has also the geometric law with the generating function $a(z)$ in (2) but ξ_k has also the geometric law with the characteristic function

$$\psi(t) = \frac{p}{1 - qe^{it}} \quad (p + q = 1). \tag{3}$$

We shall give some characterizations of the distribution of η by the zero-regression of T and λ_1 which are too stabilities on the space of values of η.

After that, we shall show the stability of those characterizations when the condition zero- regression of T and λ_1 was changed by the condition ε- zero regression of T and λ_1.

2. Characteristic Theorems

At first, we consider the case: ν has the geometric law and ξ_k has the exponential law. As we know, in this case, η has also the exponential law with the characteristic function which has the form (2).

Theorem 2.1. *The characteristic function $\psi(t)$ of the composed r.v η has also the form (2) if and only if it satisfies the following differential equation:*

$$[3\psi^{''}(t)]^2 - 2\psi^{'}(t)\psi^{'''}(t) = 0, \tag{4}$$

with the initial conditions:

$$\psi(0) = 1; \psi'(0) = i\frac{\theta}{\alpha}; \psi''(0) = -2(\frac{\theta}{\alpha})^2. \tag{5}$$

At second, in the case ν still has geometric law and ξ_k has also the geometric distribution function with the characteristic function in (3), the characteristic function of η will have the following form:

$$\psi(t) = a[\varphi(t)] = \frac{\alpha p}{(1-\beta p) - qe^{it}} \tag{6}$$

and we have the following characteristic theorem:

Theorem 2.2. *The characteristic function $\psi(t)$ of the composed r.v η has the form (6) if and only if it satisfies the differential equation:*

$$\left\{\left[\psi^{''}(t)\right]^2 - \psi^{'}(t)\psi^{'''}(t)\right\}\psi^2(t) + 2[\psi^{'}(t)]^2\psi^{''}(t)\psi(t) - 2[\psi^{'}(t)]^4 = 0, \tag{7}$$

with the initial conditions

$$\psi(0) = 1; \quad \psi^{'}(0) = \frac{iq}{p\alpha}; \quad \psi^{''}(0) = -\Big(\frac{q(1-\beta p + q^2)}{\alpha^2 p^2}\Big). \tag{8}$$

Since the proofs of those characteristic theorems are similar, we shall give the proof of Theorem 2.2.

Proof. Suppose that $\psi(t)$ has the form (6), putting $u = (1-\beta p) - qe^{it}$ then

$$\begin{aligned}
\psi^{'}(t) &= \frac{i\alpha pqe^{it}}{u^2}\\
\psi^{''}(t) &= \frac{-\alpha pqe^{it}(1-\beta p) - \alpha pq^2e^{2it}}{u^3}\\
\psi^{'''}(t) &= \frac{-i(\alpha-\beta p)^2\alpha pqe^{it} - 4i\alpha pq^2(1-\beta p)e^{2it} - i\alpha pq^3e^{3it}}{u^4}.
\end{aligned} \tag{9}$$

and

$$\begin{aligned}
[\psi^{''}(t)]^2 &= \frac{(1-\beta p)^2\alpha^2p^2q^2e^{2it} + \alpha^2p^2q^4e^{4it} + 2(1-\beta p)\alpha^2p^2q^3e^{3it}}{u^6}\\
\psi^{'}(t)\psi^{'''}(t) &= \frac{(1-\beta p)^2\alpha^2p^2q^2e^{2it} + \alpha^2p^2q^4e^{4it} + 4(1-\beta p)\alpha^2p^2q^3e^{3it}}{u^6}.
\end{aligned}$$

Therefore

$$\left\{[\psi^{''}(t)]^2 - \psi^{'}(t)\psi^{'''}(t)\right\}\psi^2(t) = \frac{-2(1-\beta p)\alpha^4p^4q^3e^{3it}}{u^8}.$$

On the other hand

$$\begin{aligned}
[\psi^{'}(t)]^2\psi^{''}(t)\psi(t) &= \frac{(1-\beta p)\alpha^4p^4q^3e^{3it} + \alpha^4p^4q^4e^{4it}}{u^8}\\
[\psi^{'}(t)]^4 &= \frac{\alpha^4p^4q^4e^{4it}}{u^8}.
\end{aligned}$$

then we have

$$\{[\psi^{''}(t)]^2 - \psi^{'}(t)\psi^{'''}(t)\}\psi^2(t) + 2[\psi^{'}(t)]^2\psi^{''}(t)\psi(t) - 2[\psi^{'}(t)]^4 = 0.$$

The proof by contradiction based on the existence theorem of a unique solution in a neighborhood of $t = 0$ of differntial equations (7), (8), see [5] and the analysis of the complex function $\psi(t)$.

Let us consider now $(X_1, X_2, ..., X_n)$, a sample in the space of values of η and $\lambda_k = \sum_{i=1}^{n} X_i^k$ is a statistic in this space.

Definition 2.1. *The r.v Y is called to be constant regression with the r.v X if $E(Y/X) = E(Y)$.*

Proposition 2.1. (Lukacs Lemma, see [3]) *The r.v Y is constant - regression with the r.v X if and only if*

$$E(Ye^{itX}) = EY.Ee^{itX}. \tag{10}$$

Remark 2.1. With the above statistics λ_k, we always have

$$\begin{aligned}
&Ee^{it\lambda_1} = [\psi(t)]^n.\\
&iE\lambda_1 e^{it\lambda_1} = n\psi^{n-1}(t)\psi^{'}(t).\\
&i^4 E\lambda_4 e^{it\lambda_1} = n\psi^{n-1}(t)\psi^{(4)}(t).\\
&i^4 E\lambda_3\lambda_1 e^{it\lambda_1} = n\psi^{n-1}(t)\psi^{(4)}(t) + n(n-1)\psi^{'''}(t)\psi^{'}(t)\psi^{n-2}(t).\\
&i^4 E\lambda_2^2 e^{it\lambda_1} = n\psi^{n-1}(t)\psi^{(4)}(t) + n(n-1)[\psi^{''}(t)]^2\psi^{n-2}(t).\\
&i^4 E\lambda_2\lambda_1^2 e^{it\lambda_1} = n\psi^{n-1}(t)\psi^{(4)}(t) + 2n(n-1)\psi^{'''}(t)\psi^{'}(t)\psi^{n-2}(t)\\
&+ n(n-1)[\psi^{''}(t)]^2\psi^{n-2}(t) + n(n-1)(n-2)\psi^{''}(t)[\psi^{'}(t)]^2\psi^{n-3}(t)\\
&i^4 E\lambda_1^4 e^{it\lambda_1} = n\psi^{n-1}(t)\psi^{(4)}(t) + 4n(n-1)\psi^{'''}(t)\psi^{'}(t)\psi^{n-2}(t)\\
&+ 3n(n-1)[\psi^{''}(t)]^2\psi^{n-2}(t) + 6n(n-1)(n-2)\psi^{''}(t)[\psi^{'}(t)]^2\psi^{n-3}(t)\\
&+ n(n-1)(n-2)(n-3)[\psi^{'}(t)]^4\psi^{n-4}(t).
\end{aligned} \tag{11}$$

Theorem 2.3. *The characteristic function of η has the form (2) if and only if the statistic $T_1 = 3\lambda_2^2 - 2\lambda_1\lambda_3 - \lambda_4$ is zero - regression with λ_1.*

Proof. According to Remark 2.1, we can see

$$\begin{aligned}
E(T_1 e^{it\lambda_1}) &= i^4\{3n\psi^{(4)}(t)\psi^{n-1}(t) + 3n(n-1)[\psi^{''}(t)]^2\psi^{n-2}(t) - 2n\psi^{(4)}(t)\psi^{n-1}(t)\\
&\quad - 2n(n-1)\psi^{'''}(t)\psi^{'}(t)\psi^{n-2}(t) - n\psi^{(4)}(t)\psi^{n-1}(t)\}\\
&= i^4 n(n-1)\psi^{n-2}(t)\{3[\psi^{''}(t)]^2 - 2\psi^{'''}(t)\psi^{'}(t)\} = 0.
\end{aligned} \tag{12}$$

Applying Theorem 2.1 and Proposition 2.1 we shall have the conclusion. ∎

Theorem 2.4. *The characteristic function of composed r.v* η *has the form* (6) *if and only if the statistic* T_2*:*

$$T_2 = A\lambda_3\lambda_1 + B\lambda_2^2 + C\lambda_1^2\lambda_2 + D\lambda_1^4 + H\lambda_4$$

is zero - regression with the statistic λ_1

$$(\textit{where } A = n^2 - n + 10, B = -n^2 + 7n - 6, C = -2(n+3), D = 2, H = -4n). \tag{13}$$

Proof. According to Remark 2.1, we have

$$\begin{aligned}
i^{-4}E(T_2e^{it\lambda_1}) &= A.n\psi^{(4)}(t)\psi^{n-1}(t) + A(n-1)n(\psi^{'''}(t)\psi^{'}(t)\psi^{n-2}(t)) \\
&+ Bn\psi^{(4)}(t)\psi^{n-1}(t) + Bn(n-1)[\psi^{''}(t)]^2\psi^{n-2}(t) + Cn\psi^{(4)}(t)\psi^{n-1}(t) \\
&+ 2Cn(n-1)\psi^{'''}(t)\psi^{'}(t)\psi^{n-2}(t) + Cn(n-1)[\psi^{''}(t)]^2\psi^{n-2}(t) \\
&+ Cn(n-1)(n-2)\psi^{''}(t)[\psi^{'}(t)]^2\psi^{n-3}(t) + Dn\psi^{(4)}(t)\psi^{n-1}(t) \\
&+ 4Dn(n-1)\psi^{'''}(t)\psi^{'}(t)\psi^{n-2}(t) + 3Dn(n-1)[\psi^{''}(t)]^2\psi^{n-2}(t) \\
&+ 6Dn(n-1)(n-2)\psi^{''}(t)[\psi^{'}(t)]^2\psi^{n-3}(t) + \\
&+ Dn(n-1)(n-2)(n-3)[\psi^{'}(t)]^4\psi^{n-4}(t) + Hn\psi^{(4)}(t)\psi^{n-1}(t) \\
&= n\psi^{(4)}(t)\psi^{n-1}(t)(A + B + C + D + H) \\
&+ n(n-1)[\psi^{''}(t)]^2\psi^{n-2}(t)(B + C + 3D) \\
&+ n(n-1)\psi^{'}(t)\psi^{'''}(t)\psi^{n-2}(t)(A + 2C + 4D) \\
&+ n(n-1)(n-2)\psi^{''}(t)[\psi^{'}(t)]^2\psi^{n-3}(t)(C + 6D) \\
&+ n(n-1)(n-2)(n-3)[\psi^{'}(t)]^4\psi^{n-4}(t)D.
\end{aligned}$$

Thus

$$\begin{aligned}
i^{-4}E(T_2e^{it\lambda_1}) = &-n(n-1)(n-2)(n-3)\psi^{n-4}(t)\{[\psi^{''}(t)]^2\psi^2(t) \\
&- \psi^{'}(t)\psi^{'''}(t)\psi^2(t) + 2\psi^{''}(t)[\psi^{'}(t)]^2\psi(t) - 2[\psi^{'}(t)]^4\} = 0.
\end{aligned}$$

This completes the proof. ■

3. Stability Theorems

Now we shall consider the stability of those characteristic theorems by the zero-regression.

Definition 3.1. *Assume that* X *and* Y *are two r.v with* $EY < +\infty$. Y *is called* ε*- zero regression with respect to* X *(with small enough* ϵ*) if*

$$\big|E(Y/X)\big| \leq \varepsilon. \tag{14}$$

Theorem 3.1. *If $G(x)$ and $\psi(t)$ are the distribution function and characteristic function of composed r.v η, respectively with $\mu_1 = E|X_j| < +\infty$ (for all j) and the statistic*

$$T_1 = 3\lambda_2^2 - 2\lambda_1\lambda_3 - \lambda_4$$

is ε- zero regression with respect to the statistic λ_1 for some sufficiently small ϵ $(0 < \epsilon < 1)$, then

$$\rho(G; G_0) = \sup_{x\in R^1} |G(x) - G_0(x)| \leqslant C_1\varepsilon^{\frac{1-\delta}{2}}, \tag{15}$$

where $G_0(x)$ is a distribution function with the characteristic function (2) *respectively and C_1 is a constant independent of ϵ; $0 < \delta < 1$.*

Theorem 3.2. *If $G(x)$ and $\psi(t)$ are the distribution function and characteristic function respectively of composed r.v η and $\mu_1 = E|X_j| < +\infty$ (for all j) and the statistic*

$$T_2 = A\lambda_3\lambda_1 + B\lambda_2^2 + C\lambda_1^2\lambda_2 + D\lambda_1^4 + H\lambda_4,$$

(where A, B, C, D, H in (13)*) is ε-zero regression with respect to the statistic λ_1 for some sufficiently small $\varepsilon(0 < \varepsilon < 1)$, then*

$$\rho(G; G_1) = \sup_{x\in R^1} |G(x) - G_1(x)| \leq C_2\epsilon\frac{1-\delta}{8}, \tag{16}$$

where C_2 is a constant independent of ϵ; $0 < \delta < 1$ and $G_1(x)$ is the distribution with the characteristic function $\psi(t)$ in (6) *and has $\sup\limits_{x\in R} |G_1'(x)| < +\infty$.*

Because the proof of Theorem 3.1 and that of Theorem 3.2 are similar, we shall prove only Theorem 3.2.

Lemma 3.1. (see [3]) *If Y is ϵ- zero regression with respect to$X$ and $EY < +\infty$, then*

$$E(Ye^{itX}) = r(t), \tag{17}$$

where

$$|r(t)| \leqslant \epsilon,\ \ \overline{r(t)} = r(-t);\ \ r(0) = 0.$$

($\overline{r(t)}$ is the conjugate value of the complex number $r(t)$).

Lemma 3.2. *Suppose that statistic T_2 is ϵ-zero regression with respect to the statistic λ_1, for some sufficiently small ϵ $(0 < \epsilon < 1)$, then the characteristic function of η will satisfy the following differential equation:*

$$\begin{aligned}\psi^{n-4}(t)\{[\psi^{''}(t)]^2\psi^2(t) - \psi^{'}(t)\psi^{'''}(t)\psi^2(t) + 2\psi^{''}(t)[\psi^{'}(t)]^2\psi(t) - 2[\psi^{'}(t)]^4\}\\ = \frac{r(t)}{-i^4 n(n-1)(n-2)(n-3)},\end{aligned} \tag{18}$$

with the initial conditions

$$\psi(0) = 1;\ \ \psi^{'}(0) = \frac{iq}{p\alpha};\ \ \psi^{''}(0) = -\Big(\frac{q(1-\beta p+q^2)}{\alpha^2 p^2}\Big), \tag{19}$$

where

$$|r(t)| < \epsilon,\ \ \overline{r(t)} = r(-t);\ \ r(0) = 0.$$

Proof. Applying Lemma 3.1 for T_2 and λ_1 and according to Theorem 2.4, we have $E(T_2 e^{it\lambda_1}) = -i^4 n(n-1)(n-2)(n-3)\psi^{n-4}(t)\{[\psi^{''}(t)]^2\psi^2(t) - \psi^{'}(t)\psi^{'''}(t)\psi^2(t) + 2\psi^{''}(t)[\psi^{'}(t)]^2\psi(t) - 2[\psi^{'}(t)]^4\} = r(t)$.

Then we shall derive (18) and (19).

Proof of Theorem 3.2. Let $\psi_1(t)$ be the characteristic function having the from (6) corresponding to the distribution function $G_1(x)$, we shall estimate $(G(x) - G_1(x))$ for all $x \in R^1$.

Put $\frac{\psi^2(t)}{\psi^{'}(t)} = -e^{u(t)}$, we have $u^{'}(t) = 2\frac{\psi^{'}(t)}{\psi(t)} - \frac{\psi^{''}(t)}{\psi^{'}(t)}$ and

$$\begin{aligned} u^{''}(t) &= 2\frac{\psi^{''}(t)\psi(t) - [\psi^{'}(t)]^2}{\psi^2(t)} - \frac{\psi^{'''}(t)\psi^{'}(t) - [\psi^{''}(t)]^2}{[\psi^{'}(t)]^2} \\ &= \frac{2[\psi^{'}(t)]^2(\psi^{''}(t)\psi(t) - [\psi^{'}(t)]^2) + \psi^2(t)([\psi^{''}(t)]^2 - \psi^{'}(t)\psi^{'''}(t))}{\psi^2(t)[\psi^{'}(t)]^2}. \end{aligned} \tag{20}$$

Therefore

$$u^{''}(t) = \frac{r(t)}{-i^4 n(n-1)(n-2)(n-3)\psi^{n-2}(t)[\psi'(t)]^2} = Q(t). \tag{21}$$

Since $\psi(t)$ and $\psi^{'}(t)$ are continuous functions, for sufficiently small ε, we can determine

$$T_1(\epsilon) = \sup\{c;\ \ |\psi(t)| \geq \epsilon^{\frac{\delta_1}{n-2}}; \forall t, |t| \leqslant c\}, \tag{22}$$

$$T_2(\epsilon) = \sup\{c;\ \ |\psi^{'}(t)| \geq \epsilon^{\frac{\delta_2}{2}}; \forall t, |t| \leqslant c\}, \tag{23}$$

where δ_1, δ_2 are positive numbers satisfing the conditions

$$1 - \delta_2 - \frac{\delta_1(n-1)}{n-2} > 0, \tag{24}$$

$$\frac{\delta_2}{2} - \frac{2\delta_1}{n-2} \geq 0. \tag{25}$$

Notice that for sufficiently small ε, the following inequality always holds

$$\frac{q}{p\alpha} \geq \varepsilon^{\frac{\delta_2}{2}}. \tag{26}$$

Put $T(\varepsilon) = \min\{T_1(\epsilon); T_2(\epsilon)\}$, we have

$$|Q(t)| \leqslant \frac{\varepsilon^{1-\delta_1-\delta_2}}{n(n-1)(n-2)(n-3)} = \frac{\varepsilon^{1-\delta_1-\delta_2}}{n_1} = C(\varepsilon);$$

($C(\varepsilon) \to 0$ when $\varepsilon \to 0$).

At first, we consider the case $0 \leqslant t \leqslant T(\varepsilon)$ (Proving theorem in the case $-T(\varepsilon) \leqslant t \leqslant 0$ is similar). Therefore,

$$u(s) = \int_0^s \int_0^u Q(v)dvdu - is + \ln \frac{-\alpha p}{iq} \tag{27}$$

From (19) we can see

$$u_1(s) = -is + \ln \frac{-\alpha p}{iq}. \tag{28}$$

and $u_1(t)$ can be defined by $\psi_1(t)$ which was the solution of the equations (7) and (8). Since we put $\frac{\psi^2(t)}{\psi'(t)} = -e^{u(t)}$, then $\frac{\psi'(t)}{\psi^2(t)} = -e^{-u(t)}$.

Therefore

$$\left(\psi^{-1}(s)\right)' = e^{-u(s)}$$

From the initial conditions $\psi(0) = 1$, so $\psi^{-1}(t) = 1 + \int_0^t e^{-u(s)}ds$ that means

$$\psi(t) = \frac{1}{1 + \int_0^t e^{-u(s)}ds}, \quad 0 \leqslant s \leqslant t.$$

On the other hand, since $c(\varepsilon) \geq 0$ we have the estimation

$$\left|e^{-u(s)}\right| \leqslant \frac{q}{\alpha p} e^{\frac{C(\varepsilon)t^2}{2} + t}. \tag{29}$$

It is well known that $e^x \geq x \quad (\forall x > 0)$, so

$$\begin{aligned} \left|1 + \int_0^t e^{-u(s)}ds\right| &\leqslant 1 + \int_0^t |e^{-u(s)}|ds \leqslant 1 + \frac{q}{p\alpha} t e^{\frac{C(\varepsilon)t^2}{2} + t} \\ &\leqslant 1 + \frac{q}{p\alpha} e^{C(\varepsilon)t^2 + 2t} \\ &\leqslant \left(1 + \frac{q}{p\alpha}\right) e^{C(\varepsilon)t^2 + 2t} \end{aligned} \tag{30}$$

and it follows that

$$|\psi(t)| \geq \frac{\alpha p}{(\alpha p + q)e^{C(\varepsilon)t^2} + 2t}. \tag{31}$$

We consider the inequality

$$\frac{\alpha p}{(\alpha p + q)e^{C(\varepsilon)t^2 + 2t}} \geq \varepsilon^{\frac{\delta_1}{n-2}}.$$

Thus

$$e^{C(\epsilon)t^2 + 2t} \leqslant \frac{\alpha p}{\alpha p + q} \epsilon^{-\frac{\delta_1}{n-2}}. \tag{32}$$

Therefore

$$C(\varepsilon)t^2 + 2t + \frac{\delta_1}{n-2} \ln \varepsilon - \ln \frac{\alpha p}{\alpha p + q} \leqslant 0. \tag{33}$$

We have

$$\Delta' = 1 - C(\varepsilon)\Big[\frac{\delta_1}{n-2}\ln\varepsilon - \ln\frac{\alpha p}{\alpha p + q}\Big].$$

$C(\varepsilon) \geq 0$, and when $\varepsilon \to 0$ then $\ln\varepsilon \to -\infty$ so $\Delta' > 0$.
For $t \geq 0$, the solution of (33)

$$0 \leqslant t \leqslant \frac{\sqrt{\Delta'} - 1}{C(\varepsilon)}.$$

From the definition of $T_1(\varepsilon)$, we have

$$T_1(\varepsilon) \geq \frac{\sqrt{\Delta'} - 1}{C(\varepsilon)}. \tag{34}$$

Now, we define $T_2(\varepsilon)$. We have

$$\frac{\psi^2(t)}{\psi'(t)} = -e^{u(t)} \Rightarrow |\psi^{'}| = \frac{|\psi|^2}{|e^{u(t)}|}$$

so

$$|e^{u(t)}| \leqslant \frac{\alpha p}{q} e^{\frac{C(\varepsilon)t^2}{2} + t}.$$

$$|\psi^{'}| \geq \frac{q|\psi(t)|^2}{\alpha p} e^{-\frac{C(\varepsilon)t^2}{2} - t}$$

with defined $T_1(\varepsilon)$, we have: $|\psi(t)|^2 \geq \epsilon^{\frac{2\delta_1}{n-2}} \quad \forall t,\ 0 \leqslant t \leqslant T_1(\epsilon)$; Therefore

$$|\psi^{'}| \geq \frac{q}{\alpha p}\varepsilon^{\frac{2\delta_1}{n-2}} e^{\frac{-C(\varepsilon)t^2}{2} - t}.$$

We consider the inequality

$$\frac{q}{\alpha p}\epsilon^{\frac{2\delta_1}{n-2}} e^{-\frac{C(\epsilon)}{2} - t} \geq \epsilon^{\frac{\delta_2}{2}}$$

or

$$C(\epsilon)t^2 + 2t + 2\Big(\frac{\delta_2}{2} - \frac{2\delta_1}{n-2}\Big)\ln\epsilon - 2\ln\frac{q}{\alpha p} \leqslant 0. \tag{35}$$

We have

$$\Delta_1^{'} = 1 - 2C(\epsilon)\Big[\Big(\frac{\delta_2}{2} - \frac{2\delta_1}{n-2}\Big)\ln\epsilon - \ln\frac{q}{\alpha p}\Big].$$

Because $C(\varepsilon) \geq 0, \dfrac{\delta_2}{2} - \dfrac{2\delta_1}{n-2} \geq 0$ and $\varepsilon \to 0$ then $\ln\epsilon \to -\infty$ that means $\Delta_1^{'} > 0$. The solution of (35) is

$$0 \leqslant t \leqslant \frac{\sqrt{\Delta_1^{'}} - 1}{C(\epsilon)}.$$

From the definition of $T_2(\varepsilon)$, we have

$$T_2(\epsilon) \geq \frac{\sqrt{\Delta_1^{'}} - 1}{C(\epsilon)}. \tag{36}$$

By using Essen's inequality (see [4]), we get

$$
\begin{aligned}
\sup_{x\in R}|G_1(x)-G_0(x)| &\leqslant \int_{-T}^{T}\Big|\frac{\psi(t)-\psi_1(t)}{t}\Big|dt+\frac{m_1}{T}, \quad (m_1=\sup_{x\in R}G_1'(x))\\
&\leqslant \int_{-T}^{T}\frac{1}{|t|}\Big|\frac{1}{1+\int_0^t e^{-u(s)}ds}-\frac{1}{1+\int_0^t e^{-u_1(s)}ds}\Big|dt+\frac{m_1}{T}\\
&\leqslant \int_{-T}^{T}\frac{1}{|t|}\Big|\int_0^t(e^{-u(s)}-e^{-u_1(s)})ds\Big|dt+\frac{m_1}{T} \qquad (37)\\
&\leqslant \int_{-T}^{T}\frac{1}{|t|}\Big|\int_0^t\big|e^{-u_*(s)}\big|\,\big|u(s)-u_1(s)\big|ds\Big|dt+\frac{m_1}{T}\\
&\leqslant \int_{-T}^{T}\frac{1}{|t|}\Big|\int_0^t\big|e^{-u_*(s)}\big|\frac{C(\epsilon)s^2}{2}ds\Big|dt+\frac{m_1}{T}
\end{aligned}
$$

where $u_*(s)$ is a function which satisfies the estimation

$$\min\big\{|u_1(s)|,\ \ |u(s)|\big\}\leqslant|u_*(s)|\leqslant\max\big\{|u(s)|,\ \ |(s)|\big\}$$

for all $s\in R$.
From (29) and (32) we always have

$$|e^{-u(s)}|\leqslant\frac{q}{\alpha p}e^{\frac{C(\varepsilon)t^2}{2}+t}\leq\frac{q}{\alpha p}\sqrt{\frac{\alpha p}{\alpha p+q}}\,\varepsilon^{\frac{-\delta_1}{2(n-2)}}.$$

Futhermore, $\big|e^{u_*(s)}\big|\leqslant\big|e^{-u(s)}\big|$ for all $s\in R$.

So, for all t, $|t|\leqslant T(\varepsilon)\leqslant\min\big\{T_1(\varepsilon),T_2(\varepsilon)\big\}$ we always have

$$
\begin{aligned}
\sup_{x\in R}|G(x)-G_1(x)| &\leqslant \frac{q}{\alpha p}\sqrt{\frac{\alpha p}{\alpha p+q}}\epsilon^{\frac{-\delta_1}{2(n-2)}}\frac{C(\epsilon)}{\delta}\int_{-T}^{T}t^2dt+\frac{m_1}{T}\\
&= MT^3C(\epsilon)\epsilon^{\frac{-\delta_1}{2(n-2)}}+\frac{m_1}{T};\ \ \Big(with\ \ M=\frac{q}{9\alpha p}\sqrt{\frac{q}{\alpha p+q}}\Big) \qquad (38)\\
&= MT^3\epsilon^{1-\delta_1-\delta_2-\frac{\delta_1}{2(n-2)}}+\frac{m_1}{T}.
\end{aligned}
$$

If we choose $T=T(\varepsilon)=\varepsilon^{\frac{-1+\delta_2+\frac{\delta_1(2n-3)}{2(n-2)}}{4}}$, with δ_1,δ_2 satisfying the conditions (24), (25) and (26) we have: $-1+\delta_2+\dfrac{\delta_1(2n-3)}{2(n-2)}<0$ and for some sufficiently small ε we have

$$T(\varepsilon)\leqslant\min\{T_1(\varepsilon),T_2(\varepsilon)\}.$$

We can conclude that

$$
\begin{aligned}
\rho(G;G_1) &= \sup_{x\in R}\big|G(x)-G_1(x)\big|\\
&\leqslant (M+m_1)\varepsilon^{\frac{1-\delta_2-\frac{\delta_1(2n-3)}{2(n-2)}}{4}}\leqslant C_2\varepsilon^{\frac{1-\delta}{8}},
\end{aligned}
$$

where $C_2 = M + m_1$; $\delta = \delta_1 + \delta_2$. Thus the proof is complet.

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