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Short Communication

Using Quasi-Interpolant Wavelet Representations for Non-Linear Sampling Recovery

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1. We consider the problem of non-linear sampling recovery of functions defined on the interval $\mathbb{I} := [0, 1]$. We begin with recalling some well-known settings.

Suppose that $\xi = \{\xi^k\}_{k=1}^n$ is a fixed sequence of n points in $\mathbb{I}$, and we want to approximately recover a function f on $\mathbb{I}$ from the sampled values $f(\xi^1), f(\xi^2), ...,$ $f(\xi^n)$. Using this information we can approximately recover a continuous function f on $\mathbb{I}$, by the linear sampling recovery method L defined by

$$L(f) \;=\; L(\Phi, \xi, f) := \sum_{k=1}^{n} f(\xi^k)\varphi_k, \tag{1}$$

where $\Phi = \{\varphi_k\}_{k=1}^n$ is a fixed sequence of n functions $\mathbb{I}$. Denote by L_q the normed space of functions on $\mathbb{I}$ with the usual qth integral norm $\|\cdot\|_q$ for $1 \leqslant q < \infty$, and the normed space $\mathbf{C}(\mathbb{I})$ of continuous functions on $\mathbb{I}$ with the max-norm $\|\cdot\|_\infty$ for $q = \infty$. We will measure the error of the approximate recovery (1) by $\|f - L(\Phi, \xi, f)\|_q$. For a subset $W \subset L_q$, the worst case error of the recovery of $f \in W$ by $L(f)$ can be represented by $\sup_{f\in W} \|f - L(\Phi, \xi, f)\|_q$. To study optimal sampling linear methods of the form (1) for recovering $f \in W$, we can use the quantity

$$\lambda_n(W)_q := \inf_{\Phi,\xi} \sup_{f\in W} \|f - L(\Phi, \xi, f)\|_q, \tag{2}$$

where the infimum is taken over all pairs (Φ, ξ) with $\xi = \{\xi^k\}_{k=1}^n$ and $\Phi = \{\varphi_k\}_{k=1}^n$.

We can consider some more general sampling recovery methods. One of them is defined by

$$G(\Phi, \xi, a, f) := \sum_{k=1}^{n} a_k(f(\xi^1), ..., f(\xi^n))\varphi_k, \tag{3}$$

where $a = \{a_k\}_{k=1}^n$ is a given sequence of n functions on $\mathbb{R}^n$. Similarly to (2), to study optimal linear methods of the form (3) for recovering $f \in W$, we can use the quantity

$$\gamma_n(W)_q \ := \ \inf_{\Phi,\xi,a} \ \sup_{f\in W} \ \|f - G(\Phi, \xi, a, f)\|_q,$$

where the infimum is taken over all triples (Φ, ξ, a) with $\xi = \{\xi^k\}_{k=1}^n$, $a = \{a_k\}_{k=1}^n$ and $\Phi = \{\varphi_k\}_{k=1}^n$. Another is the sampling method R given by

$$R(H, \xi, f) := H(f(\xi^1), ..., f(\xi^n)) \tag{4}$$

where H is a mapping from $\mathbb{R}^n$ into L_q. To study optimal sampling methods of recovery for $f \in W$ from n their values, we can use the quantity

$$\varrho_n(W)_q \ := \ \inf_{H,\xi} \ \sup_{f\in W} \ \|f - R(H, \xi, f)\|_q,$$

where the infimum is taken over all sequences $\xi = \{\xi^k\}_{k=1}^n$ and all mappings H from $\mathbb{R}^n$ into L_q.

We use the notations: $x_+ := \max\{0, x\}$ for $x \in \mathbb{R}$; $A_n \ll B_n$ if $A_n \leqslant CB_n$ with C an absolute constant not depending on n, $A_n \asymp B_n$ if $A_n \ll B_n$ and $B_n \ll A_n$.

For $1 \leqslant p \leqslant \infty, \ 0 < \theta \leqslant \infty$, denote by $U^\alpha_{p,\theta}$ the unit ball of the Besov space $B^\alpha_{p,\theta}$ of functions on $\mathbb{I}$. The following results are known (see [4, 6 - 8]).

Theorem 1. *Let $1 \leqslant p, q \leqslant \infty, \ 0 < \theta \leqslant \infty$ and $\alpha > 1/p$. Then there are the relations*

$$\varrho_n(U^\alpha_{p,\theta})_q \ \asymp \ \lambda_n(U^\alpha_{p,\theta})_q \ \asymp \ \gamma_n(U^\alpha_{p,\theta})_q \ \asymp \ n^{-\alpha+(1/p-1/q)_+}.$$

Moreover, we can explicitly construct an asymptotically optimal linear sampling recovery method L^* of the form (1)*, that is,*

$$\sup_{f\in U^\alpha_{p,\theta}} \|f - L^*(f)\|_q \ \asymp \ n^{-\alpha+(1/p-1/q)_+}.$$

2. In a sampling recovery method of the forms (1), (3) and (4) the points $\xi = \{\xi^k\}_{k=1}^n$ at which the sampled values are taken, and the mappings L, G, R which can be linear or non-linear are the same for all functions. This is not flexible even for functions from a class of common properties, say, of a common smoothness. To have a good approximate recovery the choice of points $\{\xi^k\}_{k=1}^n$ and a recovery approximant constructed from the sampled values at these points should depend on a concrete function. We will introduce a setting of a problem of non-linear sampling recovery of a functions having this flexibility.

Let $\Phi = \{\varphi_k\}_{k \in K}$ be a family of functions in L_q. Let us have the freedom to choose n terms φ_k from Φ and n sampled values for constructing an approximate recovery. More precisely, given a function $f \in W$, we choose a sequence $\xi = \{\xi^k\}_{k=1}^n$ of n points in $\mathbb{I}$, a sequence $a = \{a_k\}_{k=1}^n$ of n functions defined on $\mathbb{R}^n$ and a sequence $\Phi_n = \{\varphi_{k_s}\}_{s=1}^n$ of n functions from Φ. This choice defines an sampling recovery method given by

$$S(f) = S(\Phi_n, a, \xi, f) := \sum_{s=1}^n a_s(f(\xi^1), ..., f(\xi^n))\varphi_{k_s}.$$

Then we consider the approximate recovery of f from its values $f(\xi^s), s = 1, 2, ..., n$, by $S(f)$. Clearly, an efficient choice essentially depends on f, and this dependence is non-linear. Unlike sampling recovery methods of the forms (1), (3) and (4), for each function f we will first search an optimal sampling recovery method with regard to Φ

$$\nu_n(f, \Phi)_q := \inf_{\Phi_n, a, \xi} \|f - S(\Phi_n, a, \xi, f)\|_q,$$

where the infimum is taken over all sequences $\xi = \{\xi^k\}_{k=1}^n$ of n points in $\mathbb{I}$, $a = \{a_k\}_{k=1}^n$ of n functions defined on $\mathbb{R}^n$, and $\Phi_n = \{\varphi_{k_s}\}_{s=1}^n$ of n functions from Φ. We want to know the worst case of non-linear sampling recovery with regard to Φ for $f \in W$ by considering the quantity

$$\nu_n(W, \Phi)_q := \sup_{f \in W} \nu_n(f, \Phi)_q.$$

The idea of non-linear sampling recovery in term of the quantity $\nu_n(W, \Phi)_q$ naturally comes from the non-linear n-term approximation. The reader can find in [3] a survey on various aspects of this approximation and its applications.

For a given even natural number $r = 2\rho$, let N_r be the B-spline of order r with knots at the points $0, 1, ..., r$, and

$$M_r := N_r(\cdot + \rho)$$

be the centered B-spline. Denote by $\mathbf{M}$ the set of all such B-spline wavelets

$$M_{k,s}(x) := M_r(2^k x - s),$$

which do not vanish identically on $\mathbb{I}$.

The main result of the present paper is as follows.

Theorem 2. *Let $1 \leqslant p, q \leqslant \infty$, $0 < \theta \leqslant \infty$, and $1 < \alpha < r$. Then for the unit ball $U^\alpha_{p,\theta}$ of the Besov space, there is the following asymptotic order*

$$\nu_n(U^\alpha_{p,\theta}, \mathbf{M})_q \asymp n^{-\alpha}. \tag{6}$$

For $1 \leqslant p < q \leqslant \infty$, an optimal non-linear sampling recovery method for $\nu_n(U^\alpha_{p,\theta}, \mathbf{M})_q$ is better than any linear sampling recovery method of the form (1) or more generally, any sampling recovery method of the form (4). Namely, the asymptotic orders of λ_n, γ_n and ϱ_n are $n^{-\alpha+1/p-1/q}$, while the asymptotic order of ν_n is $n^{-\alpha}$.

3. To construct an asymptotically optimal non-linear sampling recovery method S^* for $\nu_n(U^\alpha_{p,\theta}, \mathbf{M})_q$ which gives the upper bound of (6) we used a quasi-interpolant wavelet representation of functions in the Besov space in terms of the B-splines $M_{k,s}$. Let $\Lambda = \{\lambda_j\}_{|j| \leqslant J}$ be a finite even sequence, i.e., $\lambda_{-j} = \lambda_j$. We define the operator Q by

$$Q(f,x) := \sum_{s=-\infty}^{\infty} \Lambda(f,s) M_r(x-s) \tag{7}$$

for a function f defined on $\mathbb{R}$, where

$$\Lambda(f,s) := \sum_{|j| \leqslant J} \lambda_j f(s-j). \tag{8}$$

It is a bounded linear operator in $C(\mathbb{R})$. Moreover, Q is local in the following sense. There exists a positive number $\delta > 0$ such that for any $f \in C(\mathbb{R})$, and $x \in \mathbb{R}$, $Q(f,x)$ depends only on the value $f(y)$ at a finite number of points y with $|y - x| \leqslant \delta$. In the present paper, we will require it to reproduce the space $\mathcal{P}_{r-1}$ of polynomials of order at most $r-1$, that is,

$$Q(p) \ = \ p, \ p \in \mathcal{P}_{r-1}.$$

Then, such an operator Q will be a quasi-interpolant in the normed space $C(\mathbb{R})$(see [2, p. 63]). A method of construction of such a quasi-interpolant via Neumann series was suggested in [1].

Let a quasi-interpolant Q of the form (7)–(8) be given. For $h > 0$ and a function f on $\mathbb{R}$, we define the operator Q^h by

$$Q^h(f) = \ \sigma_h \circ Q \circ \sigma_{1/h}(f),$$

where

$$\sigma_h(f,x) = \ f(x/h).$$

If a function f on $\mathbb{R}$ possesses a smoothness α in a neighborhood of $\mathbb{I}$, then the approximation by means of Q^h has the asymptotic order [2, p. 63–65]

$$\|f - Q^h f\|_q \ = \ O(h^\alpha).$$

We consider, however, the sampling recovery only for functions which are defined in $\mathbb{I}$. The quasi-interpolant Q^h is not defined for a function f on $\mathbb{I}$, and therefore, not appropriate for an approximate sampling recovery of f from its sampled values at points in $\mathbb{I}$. An approach to construct a quasi-interpolant for a function on $\mathbb{I}$ is to extend it by interpolation Lagrange polynomials.

For a non-negative integer k, we put $x_j = j2^{-k}, j \in \mathbb{Z}$. If f is a function on $\mathbb{I}$, let

$$U_k(f,x) := \ f(x_0) \ + \ \sum_{s=1}^{r-1} \frac{2^{sk}\Delta^s_{2^{-k}} f(x_0)}{s!} \prod_{j=0}^{s-1} (x - x_j),$$

$$V_k(f,x) := \ f(x_{2^k-r+2}) \ + \ \sum_{s=1}^{r-1} \ \frac{2^{sk}\Delta^s_{2^{-k}} f(x_{2^k-r+2})}{s!} \ \prod_{j=0}^{s-1} (x - x_{2^j-r+2+j}).$$

be the $(r-1)$th Lagrange polynomials interpolating f at the left end points $x_0, x_1, ..., x_{r-2}$, and right end points $x_{2^k-r+2}, x_{2^k-r+3}, ..., x_{2^k}$, of the interval $\mathbb{I}$, respectively. We define the function $\bar{f}$ as an extension of f on $\mathbb{R}$ by the formula

$$\bar{f}(x) := \begin{cases} U_k(f,x), & x < 0 \\ f(x), & 0 \leqslant x \leqslant 1 \\ V_k(f,x), & x > 1. \end{cases}$$

We introduce the operator Q_k by

$$Q_k(f,x) := Q^{2^{-k}}(\bar{f},x) = \sum_{s \in J(k)} a_{k,s}(f) M_{k,s}(x), \quad \forall x \in \mathbb{I},$$

where $J(k) := \{s \in \mathbb{Z} : -\rho < s < 2^k + \rho\}$ is the set of s for which $M_{k,s}$ do not vanish identically on $\mathbb{I}$, and

$$a_{k,s}(f) := \Lambda^{2^{-k}}(\bar{f}, s) = \sum_{|j| \leqslant J} \lambda_k \bar{f}(2^{-k}(s-j)).$$

Notice that the number of the terms in $Q_k(f)$ is of the size $\approx 2^k$.

An important property of Q_k is that the function $Q_k(f)$ is completely determined from the values of f at the points $x_0, x_1, ..., x_{2^k}$ which are in $\mathbb{I}$. For each pair k, s the coefficient $a_{k,s}(f)$ is a linear combination of the values $f(2^{-k}(s-j))$, $|j| \leqslant J$, and maybe, $f(2^{-k}j)$ for $j = 0, 1, ..., r-1$ or $j = 2^k - r + 2, 2^k - r + 3, ..., 2^k$, if the point $2^{-k}s$ is near to the ends 0 or 1 of the interval $\mathbb{I}$, respectively. Thus, the number of these values does not exceed the $2J + r$ and not depend on neither functions f and nor k, s. The operator Q_k also has properties similar to the properties of the quasi-interpolants Q and Q^h. Namely, it is a local bounded linear mapping in $C(\mathbb{I})$ and reproducing $\mathcal{P}_{r-1}$, more precisely,

$$Q_k(p^*) \;=\; p, \; p \in \mathcal{P}_{r-1},$$

where p^* is the restriction of p on $\mathbb{I}$. We will call Q_k a *quasi-interpolant for* $C(\mathbb{I})$.

Put

$$q_k(f) := Q_k(f) - Q_{k-1}(f) \quad \text{with} \quad Q_{-1}(f) := 0.$$

and for $s \in J(k)$,

$$c_{k,s}(f) := \begin{cases} a_{k,s}(f), & \text{if } 2^{k-1} + \rho \leqslant s < 2^k + \rho, \\ a_{k,s}(f) - a'_{k,s}(f), & \text{if } -\rho < s < 2^{k-1} + \rho. \end{cases}$$

where

$$a'_{k,s}(f) := 2^{-r+1} \sum_{j=s+\rho}^{s+3\rho} \binom{r}{j-s-\rho} a_{k-1,j}(f).$$

Theorem 3. *Under the assumptions of Theorem 2 a function f on $\mathbb{I}$ belongs to the Besov space $B^\alpha_{p,\theta}$ if and only if f has a quasi-interpolant wavelet representation*

$$f = \sum_{k=0}^{\infty} q_k(f) = \sum_{k=0}^{\infty} \sum_{s \in J(k)} c_{k,s}(f) M_{k,s}$$

with the convergence in the space $B^\alpha_{p,\theta}$*, and in addition the quasi-norm of the Besov space* $\|f\|_{B^\alpha_{p,\theta}}$ *is equivalent to the discrete quasi-norm*

$$\Big(\sum_{k=0}^{\infty}\big(2^{(\alpha-1/p)k}\|\{c_{k,s}(f)\}\|_p\big)^\theta\Big)^{1/\theta}$$

where

$$\|\{c_{k,s}(f)\}\|_p := \Big(\sum_{s\in J(k)}|c_{k,s}(f)|^p\Big)^{1/p}.$$

4. From Theorem 3 we obtain the following

Corollary 1. *Under the assumptions of Theorem 2 there is the inequality*

$$\sup_{f\in U^\alpha_{p,\theta}}\|f - Q_k(f)\|_q \ \leqslant\ C2^{-(\alpha-(1/p-1/q)_+)k},$$

and the number of sampled values of a function in $Q_k(f)$ *does not exceed* $\lambda 2^k$ *with some absolute constants* C *and* λ*. Moreover, the linear sampling method* $Q_k(f)$ *with* $n \asymp \lambda 2^k \leqslant n$*, is asymptotically optimal for* $\gamma_n(U^\alpha_{p,\theta})_q$ *and* $\varrho_n(U^\alpha_{p,\theta})_q$*.*

5. We give a sketch of the proof of Theorem 2.

The lower bound of (6) in Theorem 2 can be obtained from the lower bound for the quantity of n-term approximation $\sigma_n(U^\alpha_{p,\theta},\mathbf{M})_q$ established in [5], and the inequality

$$\nu_n(U^\alpha_{p,\theta},\mathbf{M})_q \geq \sigma_n(U^\alpha_{p,\theta},\mathbf{M})_q.$$

The upper bound for the case where $1\leqslant q\leqslant p\leqslant\infty$ follows from Corollary 1. The most difficult and interesting is the case where $1\leqslant p<q\leqslant\infty$. For this case a linear sampling recovery method does not work and therefore, we should construct a non-linear one. It will be constructed on the basic of the following representation of functions from $U^\alpha_{p,\theta}$.

Theorem 3 says that for arbitrary positive integer m, a function $f\in U^\alpha_{p,\theta}$ can be represented by a series

$$f \ = \ Q_m(f) + \sum_{k>m} q_k(f) \tag{9}$$

where

$$q_k(f) := \sum_{s\in J(k)} c_{k,s}(f)M_{k,s}(x). \tag{10}$$

Moreover, q_k satisfy the condition

$$\|q_k(f)\|_p \ \ll\ 2^{-k/p}\|\{c_{k,s}(f)\}\|_p, \ \ll\ 2^{-\alpha k}, \quad k=m+1,m+2,... \tag{11}$$

Our strategy of using the representation (9)–(11) for construction of a recovery approximant $S_n(f)$ is as follows. We will choose two appropriate integers $\bar{k}$ and k^*. Then we take the quasi-interpolant $Q_{\bar{k}}(f)$ as the main linear part of

$S_n(f)$. The non-linear part is constructed as a sum of greedy algorithms G_k with regard to the representations (10) for non-linear approximation of each component function $q_k(f),\ k = 0, 1, ...$ for $\bar{k} < k \leqslant k^*$.

Let $m_k := |J(k)| = 2^k + 2\rho - 1$. We define an integer $\bar{k}$ from the condition $(2J + r)m_{\bar{k}+2} \leqslant n < (2J + r)m_{\bar{k}+3}$. Next, we will select an integer k^* and a sequence of non-negative integers $\{n_k\}_{k=\bar{k}+1}^{k^*}$ such that

$$(2J + r)m_{\bar{k}} + (2J + r) \sum_{k=\bar{k}+1}^{k^*} n_k \leqslant n. \tag{12}$$

To do this we fix a number ε satisfying the inequalities $0 < \varepsilon < (\alpha - 1/p + 1/q)/(1/p - 1/q)$. Then an appropriate selection of k^* and $\{n_k\}_{k=\bar{k}+1}^{k^*}$ is $k^* := [\varepsilon^{-1} \log(\lambda n)] + \bar{k} + 1$ and $n_k = [\lambda n 2^{-\varepsilon(k-\bar{k})}], \quad k = \bar{k} + 1, \bar{k} + 2, ..., k^*$, with a positive constant λ chosen such that there holds the inequalities (12) and $n_k < m_k$. Here $[t]$ denotes the integer part of $t \in \mathbb{R}$.

Thus, the integers $\bar{k}$ and k^* as well the sequence $\{n_k\}_{k=\bar{k}+1}^{k^*}$ have been selected. We are now in position to construct a non-linear sampling recovery method which will give the upper bound of (6) for the case where $1 \leqslant p < q \leqslant \infty$.

For a non-linear approximation of $q_k(f)$ we define the greedy algorithms G_k with regard to the decomposition (10) as follows. We reorder the indices $s \in J(k)$ as $\{s_j\}_{j=1}^{m_k}$ so that

$$|c_{k,s_1}(f)| \geq |c_{k,s_2}(f)| \geq \cdots |c_{k,s_n}(f)| \geq \cdots |c_{k,m_k}(f)|,$$

and then take the first largest n term for a non-linear approximation of $q_k(f)$ by forming the linear combination

$$G_k(q_k(f)) := \sum_{j=1}^{n_k} c_{k,s_j}(f) M_{k,s_j}.$$

We define the non-linear operator S_n^* by

$$S_n^*(f, x) := Q_{\bar{k}}(f, x) + G_n^*(f, x)$$

where

$$G_n^*(f, x) := \sum_{k=\bar{k}+1}^{k^*} G_k(q_k(f), x).$$

We have

$$S_n^*(f, x) = \sum_{s \in J(\bar{k})} a_{k,s}(f) M_{k,s}(x) + \sum_{k=\bar{k}+1}^{k^*} \sum_{j=1}^{n_k} c_{k,s_j}(f) M_{k,s_j}(x). \tag{13}$$

Theorem 4. *The non-linear sampling recovery method* S_n^* *given in* (13) *is of the form* (5) *for the family* $\Phi = \mathbf{M}$. *Moreover, it gives the upper bound of* (6) *in Theorem* 2 *for the case where* $1 \leqslant p < q \leqslant \infty$. *Namely, there are the following upper estimates*

$$\nu_n(U^\alpha_{p,\theta}, \mathbf{M})_q \ \leqslant \sup_{f \in U^\alpha_{p,\theta}} \|f - S^*_n(f)\|_q \ \ll n^{-\alpha}.$$

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