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# Hopfian and Co-Hopfian Modules Over Commutative Rings*

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**Abstract.** The structures of Hopfian and co-Hopfian modules over commutative rings are studied. The notation of semi Hopfian (resp. semi co-Hopfian) modules as a generalization of that of Hopfian (resp. co-Hopfian) modules was introduced in [2]. A characterization of semi Hopfian modules by using certain sets of prime ideals is given. Also, it is shown the analogue of Hilbert's Basis Theorem is valid for semi Hopficity, to the effect that an $R$-module $M$ is semi Hopfian if and only if $M[X]$ is a semi Hopfian $R[X]$-module. Moreover, we shall prove the dual of these results for semi co-Hopfian modules.

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## 1. Introduction

Throughout this article, $R$ denotes a commutative ring with identity. In [3],

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Hiremath introduced the notion of Hopfian modules. The dual notion is defined by Varadarajan in [6]. An $R$-module $M$ is said to be Hopfian if any surjective endomorphism of $M$ is automatically an isomorphism. Also, an $R$-module $M$ is said to be co-Hopfian if any injective endomorphism of $M$ is an isomorphism. We refer the reader to [6] for reviewing the most important properties of Hopfian and co-Hopfian modules.

The aim of the present paper is to investigate the Hopfian and co-Hopfian modules over commutative rings. Also, we will examine semi Hopfian and semi co-Hopfian modules more extensively. These notions were introduced in [2]. An $R$-module $M$ is called semi Hopfian (resp. semi co-Hopfian) if for any $x \in R$, the endomorphism of $M$ induced by multiplication by $x$ is an isomorphism, provided it is surjective (resp. injective). In [2], by using these concepts we give a new characterization of Artinian rings. Clearly, any Hopfian module is semi-Hopfian and any co-Hopfian module is semi co-Hopfian. Also, it is obvious that as an $R$-module the ring $R$ is Hopfian (resp. co-Hopfian) if and only if it is semi Hopfian (resp. semi co-Hopfian). In Sec. 2, we prove the following characterization of Artinian semi Hopfian and also of Noetherian semi co-Hopfian modules.

**Theorem 1.1.**

i) *A Noetherian $R$-module $M$ is semi co-Hopfian if and only if* $\mathrm{Coass}_R(M) \subseteq \mathrm{Ass}_R(M)$.

ii) *An Artinian $R$-module $M$ is semi Hopfian if and only if* $\mathrm{Ass}_R(M) \subseteq \mathrm{Coass}_R(M)$.

Here $\mathrm{Ass}\,_R M$ denotes the set of associated prime ideals of $M$. Also, the set of coassociated prime ideals of $M$ is denoted by $\mathrm{Coass}_R M$. By using this characterization of Noetherian semi co-Hopfian modules, we study behavior of co-Hopficity under ring extensions. Namely, we show that:

**Theorem 1.2.**

i) *If $(R, \mathfrak{m})$ is a Noetherian local ring, then $\hat{R}$ is co-Hopfian as an $\hat{R}$-module if and only if $R$ is co-Hopfian as an $R$-module.*

ii) *Let $f : (R, \mathfrak{m}) \longrightarrow (T, \mathfrak{n})$ be an injective homomorphism of Noetherian local rings. If $f$ is integral and $R$ is co-Hopfian as an $R$-module, then $T$ is co-Hopfian as a $T$-module.*

In Sec. 3, functorial nature of semi Hopfian and of semi co-Hopfian modules is studied. Varadarajan [6, Theorem 2.1] shows that the analogue of Hilbert's Basis Theorem is valid for Hopficity. Then, in [9] Xue extends Varadarajan's result. He also proved the dual result for co-Hopfian modules. Among other things here, we extend [9, Theorems 2 and 5] to semi Hopfian and semi co-Hopfian modules respectively. To be precise, we prove the following.

**Theorem 1.3.** *Let $X_1, \ldots, X_n$ be $n$ commutating indeterminates over $R$. For any $R$-module $M$, the following are equivalent:*

i) *$M$ is semi Hopfian (resp. semi co-Hopfian) $R$-module.*

ii) *$M[X_1, \ldots, X_n]$ (resp. $M[X_1^{-1}, \ldots, X_n^{-1}]$) is a semi Hopfian (resp. semi co-Hopfian) $R[X_1, \ldots, X_n]$-module.*
iii) *$M[[X_1, \ldots, X_n]]$ (resp. $M[X_1^{-1}, \ldots, X_n^{-1}]$) is semi Hopfian (resp. semi co-Hopfian) $R[[X_1, \ldots, X_n]]$-module.*

## 2. Semi Hopfian and Semi Co-Hopfian Modules

Let $M$ be an $R$-module. An element $x$ of $R$ is called *$M$-regular* if the map $M \xrightarrow{x} M$ is injective. Also, an element $x$ of $R$ is said to be *$M$-coregular* if the map $M \xrightarrow{x} M$ is surjective. The set of all elements of $R$ which are not $M$-regular (resp. $M$-coregular) is denoted by $Z_R(M)$ (resp. $W_R(M)$).

**Definition 2.1.**
i) *An $R$-module $M$ is called semi Hopfian if every $M$-coregular element of $R$ is also $M$-regular.*
ii) *An $R$-module $M$ is called semi co-Hopfian if every $M$-regular element of $R$ is also $M$-coregular.*

It is clear that any Hopfian module is semi Hopfian and that any co-Hopfian module is semi co-Hopfian.

Recall that an $R$-module $M$ is called *good* if its zero submodule possesses a primary decomposition. A non-zero $R$-module $S$ is said to be *secondary* if for any $x \in R$, the endomorphism of $M$ induced by multiplication by $x$ is either surjective or nilpotent. We say that the $R$-module $M$ is *representable* if there are secondary submodules $S_1, S_2, \ldots, S_k$ such that $M = S_1 + S_2 + \cdots + S_k$. The two notions of primary decomposition and of secondary representation are dual concepts. We refer the reader to [4, Appendix to §6], for more details about secondary representation. Although, the following result is proved in [2], we include it here to provide more examples of semi Hopfian and semi co-Hopfian modules.

**Lemma 2.2.**
i) *Every finitely generated $R$-module is Hopfian.*
ii) *Every Artinian $R$-module is co-Hopfian.*
iii) *Every good $R$-module is semi-Hopfian.*
iv) *Every representable $R$-module is semi co-Hopfian.*

*Proof.*
i) See [7, Proposition 1.2].
ii) is well known and one can check it easily.
iii) Let $x$ be an $M$-coregular element of $R$. Let $0 = \bigcap_{i=1}^{n} Q_i$, be a primary decomposition of the zero submodule of $M$. Fix $1 \leq i \leq n$. Since $Q_i$ is a proper submodule of $M$ and $\frac{M}{Q_i} \xrightarrow{x} \frac{M}{Q_i}$ is either injective or nilpotent, it follows that $x$ is $\frac{M}{Q_i}$-regular. Assume that, $xm = 0$ for some element $m$ in $M$. Then $xm \in Q_i$ and so $m \in Q_i$. Hence $m \in \bigcap_{i=1}^{n} Q_i = 0$ and $x$ is $M$-regular, as required.
iv) is similar to (iii). ∎

*Example 2.2.*
i) Let $N$ be a non-zero co-Hopfian $R$-module. Set $M = \oplus_{i\in\mathbb{N}} N$. Then $M$ is semi co-Hopfian, but it is not co-Hopfian. Define the $R$-homomorphism $\psi : M \longrightarrow M$ by $\psi(m_1, m_2, \dots) = (0, m_1, m_2, \dots)$ for all $(m_1, m_2, \dots) \in M$. Then $\psi$ is injective, but it is not surjective.
ii) Let $\mathfrak{p}$ be non-maximal prime ideal of a Noetherian ring $R$. It is easy to see that $E(R/\mathfrak{p})$ is co-Hopfian, although it is not Artinian.
iii) Let $P$ be the set of all prime integers. Set $M = \oplus_{p\in P}\mathbb{Z}_p$. Then $M$ is both Hopfian and co-Hopfian as an $\mathbb{Z}$-module (see [6, p. 300]). However, the Goldei dimension of $M$ is not finite.
iv) Let $M$ be an $R$-module such that $\mathrm{Ann}_R M$ is a maximal ideal of $R$. Then, it is easy to see that $M$ is both semi Hopfian and semi co-Hopfian.

The set of *weakly associated* prime ideals of an $R$-module $M$ is defined as:

$$\mathrm{Assf}_R(M) = \{\mathfrak{p} \in \mathrm{Spec} R : \mathfrak{p} \text{ is minimal over } (0 :_R m) \text{ for some element } m \text{ of } M\}.$$

Recall that $\mathrm{Spec}R$ denotes the set of prime ideals of $R$. Also, the set of maximal ideals of $R$ is denoted by $\mathrm{Max}R$. For an $R$-module $M$, let $E(M)$ denote its injective envelop. An $R$-module $M$ is said to be finitely embedded (f.e.) if

$$E(M) = E(R/\mathfrak{m}_1) \oplus \cdots \oplus E(R/\mathfrak{m}_k),$$

where each $\mathfrak{m}_i$ is a maximal ideal of $R$. This is the dual of the notion of *finitely generated*. The set of *coassociated* (resp. *weakly coassociated*) prime ideals of an $R$-module $M$ is defined as:

$$\mathrm{Coass}_R(M) = \{\mathfrak{p} \in \mathrm{Spec} R : \mathfrak{p} = (0 :_R L) \text{ for some f.e. homomorphic image } L \text{ of } M\}$$

(resp. $\mathrm{Coassf}_R(M) = \{\mathfrak{p} \in \mathrm{Spec}R : \mathfrak{p}$ is minimal over $(0 :_R L)$ for some f.e. homomorphic image $L$ of $M\}$).

We summarize some important properties of weakly associated (resp. weakly coassociated) prime ideals from [10] and [1] in the following lemma and we may use them without further comment.

**Lemma 2.4.** *Let $M$ be an $R$-module.*

i) *If $R$ is Noetherian, then $\mathrm{Assf}_R(M) = \mathrm{Ass}_R(M)$ and $\mathrm{Coassf}_R(M) = \mathrm{Coass}_R(M)$.*
ii) *If $M$ is either Noetherian or Artinian, then $\mathrm{Assf}_R(M) = \mathrm{Ass}_R(M)$ and $\mathrm{Coassf}_R(M) = \mathrm{Coass}_R(M)$.*
iii) $Z_R(M) = \bigcup_{\mathfrak{p}\in \mathrm{Assf}_R(M)} \mathfrak{p}$ *and* $W_R(M) = \bigcup_{\mathfrak{p}\in \mathrm{Coassf}_R(M)} \mathfrak{p}$.
iv) *$M = 0$ if and only if $\mathrm{Assf}_R(M) = \phi$.*
v) *$M = 0$ if and only if $\mathrm{Coassf}_R(M) = \phi$.*
vi) *If $S$ is a multiplicatively closed subset of $R$, then*

$$\mathrm{Assf}_{S^{-1}R}(S^{-1}M) = \{S^{-1}\mathfrak{p} : \mathfrak{p} \in \mathrm{Assf}_R(M) \text{ with } \mathfrak{p} \cap S = \phi\}.$$

vii) *$\mathrm{Coass}_R(R) = \mathrm{Max}R$ and for any finitely generated $R$-module $M$, $\mathrm{Coass}_R(M) \subseteq \mathrm{Max}R$.*

Now, we are ready to present a characterization of semi Hopfian and of semi co-Hopfian modules by using the notions of weakly associated and weakly coassociated primes.

**Theorem 2.5.** *Let $M$ be an $R$-module.*

i) *If $\mathrm{Coassf}_R(M) \subseteq \mathrm{Assf}_R(M)$, then $M$ is semi co-Hopfian. Conversely, if $M$ is semi co-Hopfian, then $\mathrm{Coassf}_R(M) \subseteq \mathrm{Assf}_R(M)$ provided $\mathrm{Assf}_R(M)$ is finite and $\mathrm{Coassf}_R(M) \subseteq \mathrm{Max}R$.*
ii) *If $\mathrm{Assf}_R(M) \subseteq \mathrm{Coassf}_R(M)$, then $M$ is semi Hopfian. Conversely, if $M$ is semi Hopfian, then $\mathrm{Assf}_R(M) \subseteq \mathrm{Coassf}_R(M)$ provided $\mathrm{Coassf}_R(M)$ is finite and $\mathrm{Assf}_R(M) \subseteq \mathrm{Max}R$.*

*Proof.* We only prove (i) and the proof of (ii) is similar. Suppose $\mathrm{Coassf}_R(M) \subseteq \mathrm{Assf}_R(M)$. Then

$$W_R(M) \subseteq Z_R(M),$$

and so $R \setminus Z_R(M) \subseteq R \setminus W_R(M)$. Hence any $M$-regular element of $R$ is also $M$-coregular.

For the second assertion, first note that as $M$ is semi co-Hopfian, we have $R \setminus Z_R(M) \subseteq R \setminus W_R(M)$. Hence $\bigcup_{\mathfrak{p} \in \mathrm{Coassf}_R(M)} \mathfrak{p} \subseteq \bigcup_{\mathfrak{p} \in \mathrm{Assf}_R(M)} \mathfrak{p}$. But $\mathrm{Assf}_R(M)$ is finite and so any element of $\mathrm{Coassf}_R(M)$ is contained in an element of $\mathrm{Assf}_R(M)$, by the Prime Avoidance Theorem. Now, the claim is clear because each element of $\mathrm{Coassf}_R(M)$ is maximal. ∎

By using 2.4(iii), the following is immediate.

**Corollary 2.6.** *Let $(R, \mathfrak{m})$ be a quasi-local ring and $M$ an $R$-module.*

i) *If $\mathfrak{m} \in \mathrm{Assf}_R(M)$, then $M$ is semi co-Hopfian.*
ii) *If $\mathfrak{m} \in \mathrm{Coassf}_R(M)$, then $M$ is semi Hopfian.*

**Corollary 2.7.**

i) *A Noetherian $R$-module $M$ is semi co-Hopfian if and only if $\mathrm{Coass}_R(M) \subseteq \mathrm{Ass}_R(M)$.*
ii) *An Artinian $R$-module $M$ is semi Hopfian if and only if $\mathrm{Ass}_R(M) \subseteq \mathrm{Coass}_R(M)$.*

*Proof.*
i) Since $M$ is Noetherian $\mathrm{Ass}_R(M)$ is finite. Also, by 2.4(ii) and 2.4(vii), we have $\mathrm{Assf}_R(M) = \mathrm{Ass}_R(M)$, $\mathrm{Coassf}_R(M) = \mathrm{Coass}_R(M)$ and $\mathrm{Coass}_R(M)$ is contained in $\mathrm{Max}R$. Thus the claim is clear, by Theorem 2.5.
ii) is similar to (i). ∎

**Theorem 2.8.** *A Noetherian ring $R$ is co-Hopfian (as an $R$-module) if and only if $\mathrm{Max}R \subseteq \mathrm{Ass}_R(R)$.*

*Proof.* First of all note that as an $R$-module $R$ is co-Hopfian if and only if it is semi co-Hopfian. On the other hand by Lemma 2.4 (vii), we have $\mathrm{Coass}_R(R) = \mathrm{Max}R$. Hence the result follows by Corollary 2.7(i). ■

**Corollary 2.9.**

i) *Let $\mathfrak{p} \in \mathit{Assf}_R(R)$. Then $R_{\mathfrak{p}}$ is co-Hopfian as an $R_{\mathfrak{p}}$-module.*

ii) *If $(R, \mathfrak{m})$ is a Noetherian local ring, then $\hat{R}$ is co-Hopfian as an $\hat{R}$-module if and only if $R$ is co-Hopfian as an $R$-module.*

iii) *Let $f : (R, \mathfrak{m}) \longrightarrow (T, \mathfrak{n})$ be an injective homomorphism of Noetherian local rings. If $f$ is integral and $R$ is co-Hopfian as an $R$-module, then $T$ is co-Hopfian as a $T$-module.*

*Proof.* i) By Lemma 2.4(vi), $\mathfrak{p}R_{\mathfrak{p}} \in \mathrm{Assf}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}})$. Hence, by Corollary 2.6(i), it turns out that $R_{\mathfrak{p}}$ is co-Hopfian as an $R_{\mathfrak{p}}$-module.
ii) First, assume that $R$ is co-Hopfian as an $R$-module. By Theorem 2.8, $\mathfrak{m} \in \mathrm{Ass}\ _R(R)$, and so there is $x \neq 0$ in $R$ such that $(0 :_R x) = \mathfrak{m}$. Hence

$$\mathfrak{m}\hat{R} \subseteq (0 :_{\hat{R}} x) \subset \hat{R},$$

and so $(0 :_{\hat{R}} x) = \mathfrak{m}\hat{R}$. That is $\mathrm{Max}(\hat{R}) \subseteq \mathrm{Ass}\ _{\hat{R}}(\hat{R})$. Thus, by Theorem 2.8, we deduce that $\hat{R}$ is co-Hopfian as an $\hat{R}$-module.

Now, assume that $\hat{R}$ is co-Hopfian as an $\hat{R}$-module. Then $\hat{\mathfrak{m}} \in \mathrm{Ass}\ _{\hat{R}}(\hat{R})$. It follows from [4, Theorem 23.2], that $\mathfrak{m} = \hat{\mathfrak{m}} \cap R \in \mathrm{Ass}\ _R R$. Thus, as an $R$-module, $R$ is co-Hopfian, by Theorem 2.8.
iii) By Theorem 2.8, we have $\mathfrak{m} \in \mathrm{Ass}\ _R(R)$. Hence, there exists $0 \neq x$ in $R$ such that $\mathfrak{m} = (0 :_R x)$. Since $f$ is an integral extension of rings, it follows that $f(x) \neq 0$ and that the ideal $\mathfrak{n}$ is the unique ideal of $T$ which contracts to $\mathfrak{m}$. Thus

$$\mathfrak{m}T \subseteq (0 :_T f(x)) \subset T,$$

and so $\mathfrak{n}$ is minimal over $(0 :_T f(x))$. Therefore $\mathfrak{n} \in \mathrm{Assf}_T(T) = \mathrm{Ass}\ _T(T)$, and so by Theorem 2.8, $T$ is co-Hopfian as an $T$-module. ■

Vasconcelos [8, Theorem] proves that over a commutative ring $R$, every finitely generated $R$-module is co-Hopfian if and only if every prime ideal of $R$ is maximal. This result has the following immediate consequence.

**Corollary 2.10.** *Any finitely generated 0-dimensional $R$-module is co-Hopfian.*

*Proof.* Let $M$ be 0-dimensional finitely generated $R$-module. Set $R_1 = R/\mathrm{Ann}_R(M)$. Then every prime ideal of $R_1$ is maximal. The module $M$ possesses the structure of an $R_1$-module in a natural way. This $R_1$-module structure is such that $M$ remains finitely generated. Thus, by the above mentioned result of Vasconcelos, $M$ is co-Hopfian as an $R_1$-module. Therefore $M$ is also co-Hopfian as an $R$-module. ■

## 3. Functorial Results

In this section, we study functorial properties of Hopfian (resp. co-Hopfian) and semi Hopfian (resp. semi co-Hopfian) $R$-modules. In the sequel, $\mathcal{C}_R$ denotes the category of all $R$-modules and all $R$-homomorphisms.

**Proposition 3.1.** *Let $T : \mathcal{C}_R \longrightarrow \mathcal{C}_R$ be a linear faithfully exact functor and $M$ an $R$-module.*

i) *If $T$ is covariant, then $M$ is semi Hopfian (resp. semi co-Hopfian) if and only if $T(M)$ is semi Hopfian (resp. semi co-Hopfian).*
ii) *If $T$ is contravariant, then $M$ is semi Hopfian (resp. semi co-Hopfian) if and only if $T(M)$ is semi co-Hopfian (resp. semi Hopfian).*
iii) *If $T$ is covariant and $T(M)$ is Hopfian (resp. co-Hopfian), then $M$ is Hopfian (resp. co-Hopfian).*
iv) *If $T$ is contravariant and $T(M)$ is Hopfian (resp. co-Hopfian), then $M$ is co-Hopfian (resp. Hopfian).*

*Proof.*
i) Let $x \in R$. Since $T$ is a linear faithfully exact functor, it follows that the map $M \xrightarrow{x} M$ is injective if and only if the map $T(M) \xrightarrow{x} T(M)$ is injective. Similarly, the map $M \xrightarrow{x} M$ is surjective if and only if the map $T(M) \xrightarrow{x} T(M)$ is surjective. Hence $M$ is semi Hopfian (resp. semi co-Hopfian) if and only if $T(M)$ is semi Hopfian (resp. semi co-Hopfian).
ii) is similar to the proof of (i).

Now, we prove (iv) and the proof of (iii) is similar. Let $M \xrightarrow{f} M$ be an injective (resp. surjective) homomorphism. Then the map $T(M) \xrightarrow{T(f)} T(M)$ is surjective (resp. injective), because $T$ is contravariant and linear faithfully exact. Thus the map $T(M) \xrightarrow{T(f)} T(M)$ is injective (resp. surjective). Thus $M \xrightarrow{f} M$ is surjective (resp. injective), by the assumption on $T$. Therefore $M$ is a co-Hopfian (resp. Hopfian) $R$-module. ∎

**Corollary 3.2.** *Suppose $(R, \mathfrak{m})$ is a Noetherian local ring and $M$ a finitely generated R-module. Then $M$ is semi co-Hopfian if and only if $\hat{M}$ is semi co-Hopfian as an R-module.*

**Proposition 3.3.** *Suppose $T : \mathcal{C}_R \longrightarrow \mathcal{C}_R$ is a linear functor and $M$ an R-module.*

i) *If $M$ is semi Hopfian and $Ann_R(M) = W_R(M)$, then $T(M)$ is both semi Hopfian and semi co-Hopfian.*
ii) *If $M$ is semi co-Hopfian and $Ann_R(M) = Z_R(M)$, then $T(M)$ is both semi Hopfian and semi co-Hopfian.*

*Proof.* We only consider the case $M$ is semi Hopfian and the proof of the other case is similar.

If $T(M) = 0$, there is nothing to prove. Hence we may assume that $T(M) \neq 0$. Suppose $x \in R$ is such that the map $T(M) \xrightarrow{x} T(M)$ is surjective or injective. Since $T$ is linear and $T(M) \neq 0$, it follows that $x \notin \mathrm{Ann}_R(M)$. But $\mathrm{Ann}_R M = W_R(M)$ and $M$ is semi Hopfian, by the assumption. So the map $M \xrightarrow{x} M$ is an isomorphism. Therefore the map $T(M) \xrightarrow{x} T(M)$ is also an isomorphism, as required. ■

The following is the analogue of Hilbert's Basis Theorem for semi co- Hopficity.

**Theorem 3.4.** *Let $X_1, \dots, X_n$ be $n$ commutating indeterminates over $R$. For an $R$-module $M$, the following are equivalent:*

i) *$M$ is semi Hopfian $R$-module.*

ii) *$M[X_1, \dots, X_n]$ is semi Hopfian $R[X_1, \dots, X_n]$-module.*

iii) *$M[[X_1, \dots, X_n]]$ is semi Hopfian $R[[X_1, \dots, X_n]]$-module.*

*Proof.* We only prove (i)$\Longleftrightarrow$(ii) and the proof of (i)$\Longleftrightarrow$(iii) is similar. Since $R[X_1, \dots, X_n] = (R[X_1, \dots, X_{n-1}])[X_n]$ and $M[X_1, \dots, X_n] = (M[X_1, \dots, X_{n-1}])[X_n]$, by using induction on $n$, we may only consider the case $n = 1$.

Suppose $M[X]$ is a semi Hopfian $R[X]$-module. We show that $M$ is a semi Hopfian $R$-module. Let $y$ be an $M$-coregular element of $R$. Clearly, $y \in R[X]$ and $y$ is $M[X]$-coregular. Hence $y$ is also $M[X]$-regular, by the assumption on $M[X]$. Thus $y$ is also $M$-regular, because $M \subseteq M[X]$.

Now, suppose that $M$ is a semi Hopfian $R$-module and that $f(X) = \sum_{i=0}^{m} a_i X^i$ is an $M[X]$-coregular element of $R[X]$. Let $\rho(X) = \sum_{j=0}^{n} b_j X^j \in M[X]$ be such that $f(X)\rho(X) = 0$. We have

$$f(X)\rho(X) = \sum_{k=0}^{m+n} c_k X^k,$$

where $c_k = \sum_{i=0}^{k} a_i b_{k-i}$ for $k = 0, \dots, m+n$. We show that $\rho(X) = 0$, by using induction on $n$, the degree of $\rho(X)$. It follows easily that $a_0$ is an $M$-coregular element, and so it is also $M$-regular. Thus $b_0 = 0$, because $0 = c_0 = a_0 b_0$. So $\rho(X) = \rho_1(X)X$, where $\rho_1(X) = \sum_{j=1}^{n} b_j X^{j-1}$. Now $f(X)\rho_1(X) = 0$ and so $\rho_1(X) = 0$, by induction hypothesis. Therefore $\rho(X) = 0$, as required. ■

Let $M$ be an $R$-module. The set of all elements of the form $m_0 + m_1 X^{-1} + \cdots + m_n X^{-n}$, where $n \in \mathbb{N}_0$ and each $m_i \in M$ is denoted by $M[X^{-1}]$. Then $M[X^{-1}]$ possesses the structure of an $R[[x]]$-module in a natural way. The addition in $M[X^{-1}]$ is given componently and the scalar multiplication is an additive extension of

$$(rx^j)(mx^{-i}) = \begin{cases} 0 & , j > i \\ rmx^{j-i} & , j \leq i. \end{cases}$$

where $rx^j \in R[[X]]$ and $mx^{-i} \in M[X^{-1}]$. Also, for each $n \geq 2$, inductively $M[X_1^{-1}, \dots, X_n^{-1}]$ is defined as $(M[X_1^{-1}, \dots, X_{n-1}^{-1}])[X_n^{-1}]$. Then $M[X_1^{-1}, \dots, X_n^{-1}]$ is an $R[[X_1, X_2, \dots, X_n]]$-module. Since $R[X_1, X_2, \dots, X_n]$ is a subring of the ring $R[[X_1, X_2, \ldots, X_n]]$, it follows that $M[X_1^{-1}, \ldots, X_n^{-1}]$ is also an $R[X_1, X_2, \ldots, X_n]$-module.

Melkersson [5, Corollary 2.3(ii)] shows that if $M$ is a representable $R$-module, then $M[X^{-1}]$ is a representable $R[X]$-module. In view of Lemma 2.2(iv), the following result may be considered as a generalization of Melkersson's result. Also, it generalizes [9, Theorem 5] to semi co-Hopfian modules.

**Theorem 3.5.** *Let $X_1, \ldots, X_n$ be $n$ commutating indeterminates over $R$. For an $R$-module $M$, the following are equivalent:*

i) *$M$ is semi co-Hopfian $R$-module.*
ii) *$M[X_1^{-1}, \ldots, X_n^{-1}]$ is semi co-Hopfian $R[X_1, X_2, \ldots, X_n]$-module.*
iii) *$M[X_1^{-1}, \ldots, X_n^{-1}]$ is semi co-Hopfian $R[[X_1, X_2, \ldots, X_n]]$-module.*

*Proof.* Similar to Theorem 3.4, it is enough to treat only the case $n = 1$. The proof of (i)$\Longleftrightarrow$(ii) and (i)$\Longleftrightarrow$(iii) are similar and so we only prove (i)$\Longleftrightarrow$(ii).

Assume that $M[X^{-1}]$ is a semi co-Hopfian $R[X]$-module. Let $z$ be an $M$-regular element of $R$. Then one can check easily that $z \in R[X]$ is also $M[X^{-1}]$-regular. Thus $z$ is $M[X^{-1}]$-coregular, by the assumption on $M[X^{-1}]$. This yields that $z$ is also $M$-coregular.

Now, suppose that $M$ is a semi co-Hopfian $R$-module. We show that $M[X^{-1}]$ is a semi co-Hopfian $R[X]$-module. Let $f(X) = \sum_{i=0}^{m} a_i X^i$ be an $M[X^{-1}]$-regular element of $R[X]$. Clearly, this implies that $a_0$ is $M$-regular. Hence $a_0$ is $M$-coregular by the assumption on $M$. Let $\rho(X) = \sum_{j=0}^{n} b_j X^{-j}$ be an element of $M[X^{-1}]$. By induction on $n$, we show that $\rho(X) \in f(X)M[X^{-1}]$. There is $u \in M$ such that $a_0 u = b_n$. By induction hypothesis, there exists $\lambda(X) \in M[X^{-1}]$ such that $f(X)\lambda(X) = \rho(X) - f(X)(uX^{-n})$. Note that the last term of $f(X)(uX^{-n})$ is $b_n X^{-n}$. Hence $\rho(X) = f(X)(\lambda(X) + uX^{-n})$. ■

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