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On the Extension of Holomorphic Mappings Around Sets with Zero Hausdorff (2n- 1)-Measure

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Abstract. In this paper we give a Noguchi-type convergence-extension theorem for holomorphic mappings from the complement of a closed subset with zero Hausdorff $(2n-1)$-measure of a domain in $\mathbb{C}^n$ into a Caratheodory complete complex space.

1. Introduction

Our aim in this paper is to study the question concerning the extension of holomorphic mappings around closed sets with zero Hausdorff $(2n-1)$-measure into a complex space. Let D be a domain of $\mathbb{C}^n$ and $E \subset D$ a closed subset. If E is an analytic set with codim$E \geq 1$, Kwack [5] proved that all holomorphic mapping f from $D \setminus E$ to a compact complex hyperbolic space X can be extended holomorphically from D to X. Thai in [12] proved the same result but with X is a Caratheodory complete space. Note that if E is an analytic set then $\mathcal{H}_{2n-1}(E) = 0$.

We have generalized the above result of Thai and give a Noguchi-type convergence-extension theorem for holomorphic mappings. Precisely we proved the following:

Main Theorem. *Let D be a domain of $\mathbb{C}^n$ and $E \subset D$ a closed subset such that $\mathcal{H}_{2n-1}(E) = 0$. Then all holomorphic mapping f from $D \setminus E$ to a Caratheodory* *complete space X can be extended holomorphically from D to X, and if $\{f_n, n \in \mathbb{N}\} \subset Hol(D \setminus E, X)$ converges uniformly on compact subsets of $D \setminus E$ to f, then $\{\bar{f}_n, n \in \mathbb{N}\}$ converges uniformly to $\bar{f}$ on compact subsets of D, here $\bar{g} \in Hol(D, X)$ is the extension of $g \in Hol(D \setminus E, X)$.*

It is likely that this result can be obtained if X is a compact hyperbolic space, but as remarked in the last section, the techniques presented here cannot be used to study this problem.

2. Extension of Holomorphic Mappings

Let γ be a simply closed path in $\mathbb{C}$ and $\Omega(\gamma) := \text{Int } \gamma$. If $\Omega(\gamma) \subset \Omega(\sigma)$, we denote $R(\sigma, \gamma) := \Omega(\sigma) \setminus \bar{\Omega}(\gamma)$.

Let $E \subset \mathbb{C}$ be a closed set such that $\mathcal{H}_1(E) = 0$, then E is nowhere dense and hence for all $a \in E$ we can find a sequence of simply closed paths $\{\gamma_k\}$ in $\mathbb{D} \setminus E$ which converges to a.

Lemma 2.1. *Let X be a complex space and f be a holomorphic mapping from $\mathbb{D} \setminus E$ into X, with $\mathbb{D}$ is the unit disc of $\mathbb{C}$ and $E \subset \mathbb{D}$ a closed subset such that $\mathcal{H}_1(E) = 0$. Suppose the following is satisfied : for all a in E and for all sequence of simply closed paths $\{\gamma_k\}$ in $\mathbb{D} \setminus E$ converging to a, a subsequence of $\{f(\gamma_k)\}$ converges to a point of X. Then f extend holomorphically from $\mathbb{D}$ to X.*

Proof. The proof is essentially the same as the one given by Kwack in [5] for the case when $E = \{0\}$. Let $a \in E$ and $\{\gamma_k\}$ be a sequence of simply closed paths in $\mathbb{D} \setminus E$ converging to a. After taking a subsequence if necessary, we may assume that the sequence $f(\gamma_k)$ converges to a point p of X. Let V be an open neighborhood of p in X, then there is an open set U of $\mathbb{C}^n$ (which may be taken to be bounded) and a homeomorphism ψ from V to $\psi(V) \subset U$. Thus to prove the lemma it suffices to show that there exists k_o such that $f(\Omega(\gamma_{k_o}) \setminus E) \subset U$, then f can be extended holomorphically to a neighborhood of a (see [2, A1.4]). We can suppose that $p = (0, \cdots, 0) \in U$. Let $\varepsilon > 0$ such that $\bar{\Delta}_\varepsilon \subset U$ where $\Delta_\varepsilon = \{z \in \mathbb{C}^n / |z_i| < \varepsilon\}$, then there is an integer K such that $\forall k \geq K$ we have $f(\gamma_k) \subset \Delta_{\frac{\varepsilon}{2}}$. Suppose that for all k, $f(\Omega(\gamma_k) \setminus E)$ is not contained in $\Delta_{\frac{\varepsilon}{2}}$.

Let $k \geq K$, then there is a simply closed path γ in $\Omega(\gamma_k) \setminus E$ such that $a \in \Omega(\gamma)$ and $f(\gamma) \not\subset \Delta_{\frac{\varepsilon}{2}}$. Set $O_k := \{z \in \Omega(\gamma_k) \setminus E; f(z) \in \Delta_{\frac{\varepsilon}{2}}\}$. Then O_k is open and since $\{\gamma_k\}$ converges to a, there is an integer $k_o \geq k$ such that $\gamma_{k_o} \subset \Omega(\gamma)$, and we have $\gamma_{k_o} \subset O_k$.

Next, let Γ be a connected component of O_k containing γ_{k_o}. Set $\partial^+\Gamma := \partial\Gamma \cap \Omega(\gamma_{k_o})$ and $\partial^-\Gamma := \partial\Gamma \cap R(\gamma_k, \gamma_{k_o})$. We have $f(\partial^+\Gamma \setminus E) \subset S_\varepsilon$ and $f(\partial^-\Gamma \setminus E) \subset S_\varepsilon$ where S_ε is the boundary of $\bar{\Delta}_{\frac{\varepsilon}{2}}$. Let W_o be a doubly connected neighborhood of γ_{k_o} contained in Γ. Therefore there are $b^- \in \partial^-\Gamma \setminus E$, $b^+ \in \partial^+\Gamma \setminus E$, $b \in \gamma_{k_o}$ and two simply closed paths σ^- and σ^+ such that $b, b^- \in \sigma^-$, $b, b^+ \in \sigma^+$, $\Omega(\sigma^-) \subset \Gamma$ and $\Omega(\sigma^+) \subset \Gamma$. Then we can find two simply closed paths γ_k^+ and γ_k^- in $W_o \cup \Omega(\sigma^-) \cup \Omega(\sigma^+) \cup \{b^-, b^+\}$ with $b^- \in \gamma_k^-$ and $b^+ \in \gamma_k^+$

satisfying :
1- $a \in \Omega(\gamma_k^+) \cap \Omega(\gamma_k^-)$.
2- $\gamma_k^+ \subset \bar{\Gamma} \cap \Omega(\gamma_{k_o})$ and $\gamma_k^- \subset \bar{\Gamma} \cap [\mathbb{C} \setminus \bar{\Omega}(\gamma_{k_o})]$.
Hence,
i)- $\gamma_k^+ \cap E = \emptyset$ and $\gamma_k^- \cap E = \emptyset$.
ii)- $f(\gamma_k^+) \cap S_\varepsilon \neq \emptyset$ and $f(\gamma_k^-) \cap S_\varepsilon \neq \emptyset$.
iii)- $\Omega(\gamma_k^+) \subset \Omega(\gamma_{k_o}) \subset \Omega(\gamma_k^-)$ and $f(\overline{R(\gamma_k^-, \gamma_k^+)}) \subset \bar{\Delta}_{\frac{\varepsilon}{2}} \subset U$.

As $\overline{R(\gamma_k^-, \gamma_k^+)}$ is a compact subset of $\mathbb{D} \setminus E$, then there is a relatively compact open set W of $\mathbb{D} \setminus E$ which is a neighborhood of $\overline{R(\gamma_k^-, \gamma_k^+)}$ and such that $f(\bar{W}) \subset \Delta_\varepsilon$.

Let $z_k \in \gamma_{k_o}$, as $f \in Hol(W, U)$ then

$$\frac{1}{2\pi\sqrt{-1}} \int_{\gamma_k^-} \frac{f_i'(z)}{f_i(z) - f_i(z_k)} dz - \frac{1}{2\pi\sqrt{-1}} \int_{\gamma_k^+} \frac{f_i'(z)}{f_i(z) - f_i(z_k)} dz > 0$$

On the other hand, $f(z_k)$ converges to p, $\{\gamma_k^-\}$ and $\{\gamma_k^+\}$ converge to a. After taking subsequences if necessary we can suppose that $\{f(\gamma_k^-)\}$ and $\{f(\gamma_k^+)\}$ converge respectively to q' and q on X. Also we can assume that $q_1 \neq 0$ and $q_1' \neq 0$. Hence there is an integer K such that $f_1(z_k)$ is not contained in $f_1(\gamma_k^-) \cup f_1(\gamma_k^+)$ for all $k \geq K$, it follows that $\{f_1(\gamma_k^-)\}$ and $\{f_1(\gamma_k^+)\}$ are contained in simply connected domain in $\mathbb{C}$ which do not contain $f_1(z_k)$, then

$$\int_{\gamma_k^-} \frac{f_1'(z)}{f_1(z) - f_1(z_k)} dz = \int_{\gamma_k^+} \frac{f_1'(z)}{f_1(z) - f_i(z_k)} dz = 0.$$

This is a contradiction. ∎

In the proof of the main theorem we use the following two lemmas about Hausdorff measure [2]:

Lemma 2.2 *Let M, N be Riemanian manifolds of class $\mathcal{C}^1$, let $f : N \longrightarrow M$ be a smooth map, and let E be a subset in N such that $\mathcal{H}_\alpha(E) = 0$ for an $\alpha \geq m = dimM$. Then $\mathcal{H}_{\alpha - m}(E \cap f^{-1}(x)) = 0$ for almost all $x \in M$.*

Lemma 2.3. *Let E be a locally closed set in $\mathbb{C}^n$ such that $\mathcal{H}_{2p+1}(E) = 0$ for some integer $p < n$. If $a \in E$, then there are $r > 0$ and a unitary transformation $l : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ such that $B' = B'(a', r) \subset \mathbb{C}^p$, $B'' = B''(a'', r) \subset \mathbb{C}^{n-p}$, $l(E) \cap [\overline{B' \times B''}]$ is closed in $\overline{B' \times B''}$ and $l(E) \cap [\overline{B'} \times \partial B''] = \emptyset$.*

Proof of the main theorem.
(1) The case when $n = 1$: Let $a \in E$ and $r > 0$ such that $\mathbb{D}(a, r) \subset D$. Without loss of generality we may assume that $a = 0$ and $r = 1$. Let $b \in \mathbb{D} \cap E$, $\{\gamma_k\}$ be a sequence of simply closed paths $\{\gamma_k\}$ in $\mathbb{D} \setminus E$ converging to b and $(z_k)_{k \geq 1}$ a sequence such that $z_k \in \gamma_k$. Since z_k converge to b, then $(z_k)_{k \geq 1}$ is a $c_{\mathbb{D}}$-Cauchy sequence, $(z_k)_{k \geq 1}$ is also a $c_{\mathbb{D} \setminus E}$-Cauchy sequence. Indeed, let $g \in Hol(\mathbb{D} \setminus E, \mathbb{D})$ then g is extended holomorphically to $\tilde{g}$ from $\mathbb{D}$ into $\bar{\mathbb{D}}$ (see [2, A1.4]). It follows

from the maximum principle applied to $u(z) = |\tilde{g}(z)|$ that $\tilde{g}(\mathbb{D}) \subset \mathbb{D}$ and then for all $x, y \in \mathbb{D} \setminus E$ we have

$$c_{\mathbb{D}\setminus E}(x,y) = \sup_{g \in Hol(\mathbb{D}\setminus E, \mathbb{D})} \rho(x,y) = \sup_{g \in Hol(\mathbb{D},\mathbb{D})} \rho(x,y) = c_{\mathbb{D}}(x,y).$$

From this we deduce that $(f(z_k))_{k\geq 1}$ is a c_X-Cauchy sequence and since c_X is complete, $(f(z_k))_{k\geq 1}$ converges to a point $p \in X$.

Let W be an open neighborhood of p in X and $\varepsilon > 0$ such that the ball $B(p,\varepsilon) \subset W$. There is K_1 such that $c_X(p, f(z_k)) < \frac{\varepsilon}{2}$ for all $k \geq K_1$. We have $L_{c_{\mathbb{D}}}(\gamma_k) = L_{c_{\mathbb{D}\setminus E}}(\gamma_k)$, where $L_{c_{\mathbb{D}\setminus E}}(\gamma_k)$ (resp. $L_{c_{\mathbb{D}}}(\gamma_k)$) is the diameter of γ_k measured in terms of the Caratheodory distance $c_{\mathbb{D}\setminus E}$ (resp. $c_{\mathbb{D}}$), then $L_{c_{\mathbb{D}\setminus E}}(\gamma_k) \to 0$.

If we denote by $L_{c_X}(f(\gamma_k))$ the diameter of $f(\gamma_k)$ measured in terms of the Caratheodory distance c_X, then we have

$$L_{c_X}(f(\gamma_k)) \leq L_{c_{\mathbb{D}\setminus E}}(\gamma_k).$$

Therefore $L_{c_X}(f(\gamma_k))$ converges to 0.

Thus there is K_2 such that $L_{c_X}(f(\gamma_k)) < \frac{\varepsilon}{2}$ for all $k \geq K_2$, and then for $k \geq K := \max(K_1, K_2)$ we have

$$c_X(p, f(z)) \leq c_X(p, f(z_k)) + c_X(f(z_k), f(z)) < \varepsilon$$

for all $z \in \gamma_k$. Hence $f(\gamma_k) \subset W$ for $k \geq K$, and by Lemma 2.1, f can be extended holomorphically to $\mathbb{D}$.

(2) For the case $n > 0$, since $\mathcal{H}_{2n-1}(E) = 0$ and using Lemma 2.3. there are $r > 0$ and a unitary transformation $l : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ such that $\overline{B}(a', r) \times \overline{B}(a_n, r) \subset D$ and $l(E) \cap [\overline{B}(a', r) \times \partial B(a_n, r)] = \emptyset$. Moreover as $\partial B(a_n, r)$ is compact and E is closed we can find $0 < r_0 < r$ such that $l(E) \cap [\overline{B}(a', r) \times (B(a_n, r) \setminus \overline{B}(a_n, r_0))] = \emptyset$. We denote $B' = B'(a', r)$, $B_n = B(a_n, r)$ and $V_0 = B(a_n, r) \setminus \overline{B}(a_n, r_0)$. Without loss of generality we may assume that $l : z \longmapsto z$. Let $\xi \in E_r := E \cap [B' \times B_n]$. Put $E_\xi = \{z_n \in B_n / (\xi, z_n) \in E_r\}$.

Let $p : B' \times B_n \longrightarrow B'$ be the projection map, by Lemma 2.2. there is a subset $A \subset B'$ with Lebesgue measure equal to zero such that for all $\xi \in B' \setminus A$ the set $p^{-1}(\xi) \cap E_r = \{\xi\} \times E_\xi$ has zero Hausdorff $\mathcal{H}_{(2n-1)-(2n-2)}$-measure i.e. $\mathcal{H}_1(E_\xi) = 0$ for all $\xi \in B' \setminus A$.

The map f is holomorphic from $B' \times V_0$ to X and for all $\xi \in B' \setminus A$, $f_\xi = f(\xi, .)$ is holomorphic from $B_n \setminus E_\xi$ to X with $\mathcal{H}_1(E_\xi) = 0$. It follows from the case $n = 1$ that f_ξ can be extended holomorphically to B_n. On the other hand X is complete hyperbolic and then has Hartogs extension property. It follows from [1, 9] that f can be extended holomorphically to $(B' \setminus A)^* \times B_n$, where $(B' \setminus A)^*$ is the set of points when $B' \setminus A$ is locally pluriregular. As A has zero Lebesgue measure, then $(B' \setminus A)^* = B'$. Therefore f can be extended holomorphically to a neighborhood of a.

The last part of the main theorem is an immediate consequence of the following lemma:

Lemma 2.4. *Let X be a complex Kobayashi-hyperbolic space, $D \subset \mathbb{C}^n$ be a domain and $E \subset D$ be a closed subset with Lebesgue measure $\lambda(E) = 0$. Let*

$\{f_n, n \in \mathbb{N}\} \subset Hol(D, X)$ and $f \in Hol(D, X)$ such that $\{f_n, n \in \mathbb{N}\}$ converges uniformly on compact subsets of $D \setminus E$ to f. Then $\{f_n, n \in \mathbb{N}\}$ converges uniformly to f on compact subsets of D.

Proof. Let $a \in E$, $U \subset\subset D$ be a connected neighborhood of a and $\varepsilon > 0$. The set $D \setminus E$ is dense in D. Hence, let $b \in U \setminus E$ be such that $d_U(a, b) < \dfrac{\varepsilon}{3}$. There is $N > 0$ such that for all $n \geq N$, we have $d_X(f_n(b), f(b)) < \dfrac{\varepsilon}{3}$. It follows that:

$$\begin{aligned} d_X(f_n(a), f(a)) &\leq d_X(f_n(a), f_n(b)) + d_X(f_n(b), f(b)) + d_X(f(b), f(a)) \\ &< d_U(a, b) + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon. \end{aligned}$$

Consequently, for all $z \in D$ the set $\{f_n(z), n \in \mathbb{N}\}$ is relatively compact in X. Since X is hyperbolic, then $Hol(D, X)$ is equicontinuous. It follows by Ascoli-Arzelà theorem that the family $\{f_n, n \in \mathbb{N}\}$ is relatively compact in $Hol(D, X)$, then there is a subsequence $\{f_{n_p}, n \in \mathbb{N}\}$ which converges uniformly on compact subsets to a holomorphic mapping $g \in Hol(D, X)$. By hypotheses $g_{|D\setminus E} = f_{|D\setminus E}$, hence $g = f$. Remark that there are not subsequences which diverge and all subsequences converge to f, then the result follows. ■

Metric defined by plurisubharmonic functions (see [3] and [4]). Let X be a complex manifold. For $x_o \in X$, we denote by $P_X(x_o)$ the set of upper semi-continuous functions φ on X satisfying the following conditions:

(i). $0 \leq \varphi < 1$ (ii). $\varphi(x_o) = 0$ (iii). $\log \varphi$ is plurisubharmonic and (iv). In a local coordinates $z = (z_1, \cdots, z_n)$ centered in x_o, $\dfrac{\varphi}{||z||}$ is bounded in a neighborhood of x_o.

We consider the extremal function

$$\lambda_X(x, x_o) = \sup\{\varphi(x)\,;\, \varphi \in P_X(x_o)\}.$$

We then define $p'_X(x, x') = \max\{\rho(\lambda_X(x, x'), 0), \rho(\lambda_X(x', x), 0)\}$ for $x, x' \in X$, where ρ is the Poincare metric of $\mathbb{D}$.

Let $x = x_0, x_1, \cdots, x_k = x'$ be a chain and

$$p_X(x, x') = \inf \sum p'_X(x_{i-1}, x_i)$$

where the infimum is taken over all chains from x to x'. The pseudodistance p_X has the following properties:

(1). If $f : X \longrightarrow Y$ is a holomorphic map, then for all $x, x' \in X$ we have $p_Y(f(x), f(x')) \leq p_X(x, x')$.

(2). $p_{\mathbb{D}} = \rho$.

(3). $c_X \leq p_X \leq d_X$.

By definition we remark that if $F \subset \mathbb{D}$ is a closed polar subset then for all $r > 1$ we have $\lambda_{\mathbb{D}\setminus F}(x, y) \leq r\lambda_{\mathbb{D}}(x, y)$ and hence $\lambda_{\mathbb{D}\setminus F}(x, y) \leq \lambda_{\mathbb{D}}(x, y)$. It follows that $p_{\mathbb{D}\setminus F}(x, y) \leq p_{\mathbb{D}}(x, y)$. Therefore, we deduce from (1) that $p_{\mathbb{D}\setminus F}(x, y) = p_{\mathbb{D}}(x, y)$ for all $x, y \in \mathbb{D} \setminus F$.

By the same proof of the main theorem the following proposition is obtained.

Proposition 2.5. *Let D be a domain of $\mathbb{C}^n$ and $E \subset D$ a closed pluripolar subset. Let X be a complex manifold, if p_X is a complete distance then all holomorphic mapping f from $D \setminus E$ to X extend holomorphically from D to X.*

3. Remark on the Kwack Technique

Let $\mathbb{D}$ be the unit disc of $\mathbb{C}$, $E \subset \mathbb{D}$ a closed subset such that $\mathcal{H}_1(E) = 0$ and X a compact hyperbolic space.

Problem. *Does all holomorphic mapping f from $\mathbb{D} \setminus E$ to X can be extended holomorphically to $\mathbb{D}$?* For the case when E is polar see [6, 10, 11].
If we want to use Kwack technique we must have the following property:

Let $a \in E$ and $d_{\mathbb{D}\setminus E}$ be the Kobayashi distance. If $\{\gamma_k\}$ is a sequence of simply closed paths $\{\gamma_k\}$ in $\mathbb{D} \setminus E$ converging to a, then $\lim_{k\to+\infty} L_{d_{\mathbb{D}\setminus E}}(\gamma_k) = 0$.

We next show that, by a simple explicit example of E and γ_k, this is not the case in general.

Let Ω be a bounded domain of $\mathbb{C}^n$. A Kählerienne complete metric $g = \Sigma g_{i\bar{j}} dz_i \otimes d\bar{z}_j$ is called Einstein metric if there is $c \in \mathbb{R}_-$ such that $\mathrm{Ric}(g) = c\omega_g$ where $\omega_g = \Sigma g_{i\bar{j}} dz_i \wedge d\bar{z}_j$ and $\mathrm{Ric}(g) = -\sqrt{-1}\partial\bar{\partial}\log(det(g_{i\bar{j}}))$. Let

$$\omega_g^n = \underbrace{\omega_g \wedge \cdots \wedge \omega_g}_{\text{n foix}}$$

We denote by $\delta(x) = \delta(x, \partial\Omega)$ the euclidean distance.

Suppose that $-\log\delta(x) \geq 1$. Mok and Yau [7] prove that

$$\omega^n \geq \frac{C}{\delta^2(-\log\delta)^2}\sqrt{-1}dz_1 \wedge d\bar{z}_1 \cdots \sqrt{-1}dz_n \wedge d\bar{z}_n$$

where C is a constant depending only on n. By a result of Yau [13] we have $f^*\omega^n \leq \Theta$ for all holomorphic mapping $f : \mathbb{D}^n \longrightarrow \Omega$, where Θ is the Poincaré volume form of $\mathbb{D}^n$ given by $\Theta = \wedge_{i=1}^n \frac{2}{(1-|z_j|^2)^2}\sqrt{-1}dz_j \wedge d\bar{z}_j$.

On the other hand a holomorphic mapping $f : \mathbb{D}^n \longrightarrow \Omega$ is non-degenerate at $z \in \mathbb{D}^n$ if $f_* : T(\mathbb{D}^n)_z \longrightarrow T(\Omega)_{f(z)}$ is a linear isomorphism. The hyperbolic pseudo-volume form of Ω is defined as follows

$$\Psi_\Omega^n(x) = \inf f_*(\Theta(0)),$$

where the infimum is taken over all holomorphic mappings $f : \mathbb{D}^n \longrightarrow \Omega$ such that $f(0) = x$ and which are non-degenerate at 0. By [8, Proposition 2.3.5], we have the following inequality

$$\Psi_\Omega^n \geq \omega^n.$$

Next we consider the case $n = 1$ and $\Omega = \mathbb{D} \setminus E$. From above we have $\Psi_{\mathbb{D}\setminus E}^1 = \sqrt{-1}\lambda dz \wedge d\bar{z} \geq \dfrac{C}{\delta^2(-\log\delta)^2}\sqrt{-1}dz \wedge d\bar{z}$. We have

$$ds^2 = 2\lambda dz \wedge d\bar{z} \geq \frac{2C}{\delta^2(-\log\delta)^2} dz \wedge d\bar{z}$$

for ds^2 being the pseudo-metric associated to $\Psi^1_{\mathbb{D}\setminus E}$.

Let $r \in]0, \frac{1}{2}[$, for $n \geq 1$ we have $r^2 + r^n + r^{2n} < 1$, hence $r^{2n+2} < r_n < \frac{r^{2n}+r^{2n+2}}{2}$ where $r_n = r^{2n+2} + r^{4n}$. Let $\{a_i^n\}$ be a finite set of points in the circle $S(0, r^{2n+2})$ such that

$$\sup_{x \in S(0,r_n)} \delta(\{a_i^n\}, x) \leq r^{3n}.$$

Set $E = \{0\} \bigsqcup \bigcup_{n\geq 1}\{a_i^n\} \bigsqcup A$, where $A \subset \mathbb{D} \setminus \overline{\mathbb{D}}(0, r_1)$ is a closed discrete set such that $-\log \delta(x) \geq 1$ for $x \in \Omega = \mathbb{D} \setminus E$. We denote by $L(S(0, r_n))$ the length of $S(0, r_n)$ measured in terms of ds^2. Hence, we have

$$L(S(0, r_n)) = \int_0^1 \sqrt{2\lambda}|\phi_n^{'}(t)|dt$$

with $\phi_n(t) = r_n e^{2\pi it}$.

As $r^2 + r^n + r^{2n} < 1$, it is easy to see that $\delta(x, \partial\Omega) = \delta(x, \{a_i^n\})$ in $S(0, r_n)$ and as $-\log \delta(x) \geq 1$ then $\delta(x) \log \delta(x) \geq r^{3n} \log r^{3n}$ in $S(0, r_n)$, it follows that

$$\frac{1}{\delta(x) \log \frac{1}{\delta(x)}} \geq \frac{1}{r^{3n} \log \frac{1}{r^{3n}}}$$

Therefore

$$\begin{aligned} L(S(0, r_n)) &\geq \frac{2\sqrt{2C}\pi r_n}{r^{3n} \log \frac{1}{r^{3n}}} \\ &= \frac{2\sqrt{2C}\pi r^{2n+2}}{r^{3n} \log \frac{1}{r^{3n}}} + \frac{2\sqrt{2C}\pi r^{4n}}{r^{3n} \log \frac{1}{r^{3n}}} \\ &\longrightarrow +\infty \end{aligned}$$

To conclude we need the lemma below. To its exact statement, let M be a Riemann surface and $\Psi^1_M = a\sqrt{-1}dz \wedge d\bar{z}$ its hyperbolic volume form. We define the mapping $H_M : \mathrm{T}M \longrightarrow \mathbb{R}_+$ as follows. For any $v \in \mathrm{T}_zM \cong \mathbb{C}$ set $H_{M,z}(v) = \frac{1}{\sqrt{2}}\sqrt{< a(z)v, v >} = \frac{1}{\sqrt{2}}\sqrt{a(z)}|v|$, then we have:

Lemma 3.1. *We have $H_M \leq F_M$, for F_M being the Kobayashi differential metric on M..*

Proof. Let $f : \mathbb{D} \longrightarrow M$ be a holomorphic mapping, then

$$f^*\Psi^1_M \leq \Psi^1_{\mathbb{D}} = \frac{2\sqrt{-1}}{(1 - |z|^2)^2} dz \wedge d\bar{z}$$

and as $F_{\mathbb{D}}(v) = \frac{1}{(1 - |z|^2)}|v|$ for all $v \in \mathrm{T}_z\mathbb{D}$, it follows that $f^*H_M \leq F_{\mathbb{D}}$. On the other hand $H_M(0_z) = 0$ (where 0_z is the zero of T_zM) and $\forall v \in \mathrm{T}_zM$, $\forall t \in \mathbb{C}$ $H_M(tv) = |t|H_M(v)$, then by [8, Theorem 1.2.3] we have $H_M \leq F_M$. ∎

We deduce from above that the length of $S(0, r_n)$ measured in terms of Kobayashi metric of $\mathbb{D} \setminus E$ tends to $+\infty$.

Finally, let f be a holomorphic mapping from $\mathbb{D}\setminus E$ to a compact hyperbolic space X. As z is an isolated point for every $z \in \bigcup_{n\geq 1}\{a_i^n\} \bigsqcup A$ then f can be extended holomorphically to $\mathbb{D}\setminus\{0\}$ and hence to $\mathbb{D}$. We conclude that Kwack technique cannot be used in the study of the above problem.

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