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A Fractional Black-Scholes Model with Jumps*

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Abstract. In this paper,we introduce an approximate approach to a fractional Black-Scholes model with jumps perturbed by fractional noise. Based on a fundamental result on the L^2-approximation of this noise by semimartingales, we prove a convergence theorem concerning an approximate solution. A simulation example shows a significant reduction of error in a fractional jump model as compared to the classical jump model.

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1. Introduction

In some recent papers (see for examples [5, 6]), some fractional Black-Scholes model have been proposed as an improvement of the classical Black-Scholes. Common to these models is that they are driven by a fractional Brownian motion and that some stochastic calculus is created by using, for example, Malliavin calculus or Wick product analysis. Recently, an approximate approach to fractional Black-Scholes model is introduced and investigated in [10]. In this paper we use this approach to study a fractional Black-Scholes model with jumps.

Recall that a fractional Brownian motion B_t^H with Hurst index H, is a centered Gaussian process such that its covariance function $R(t, s) = EB_t^H B_s^H$ is given by

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$$R(t,s)=\frac{1}{2}(|t|^{\gamma}+|s|^{\gamma}-|t-s|^{\gamma}),\ \text{ where }\ \gamma=2H\ \text{ and }\ 0<H<1.$$

If $H=\frac{1}{2}$, $R(t,s)=\min(t,s)$ and B_t^H is the usual standard Brownian motion. In the case $\frac{1}{2}<H<1$ the fractional Brownian motion exhibits statistical long range dependency in the sense that $\rho_n := E[B_1^H(B_{n+1}^H-B_n^H]>0$ for all $n=1,2,3,...$ and $\Sigma_{n=1}^{\infty}\rho_n=\infty$ ([9, page 2]). Hence, in financial modelling, one usually assumes that $H\in(\frac{1}{2},1)$. Put $\alpha=H-\frac{1}{2}$. It is known that a fractional Brownian motion B_t^H can be decomposed as follows:

$$B_t^H=\frac{1}{\Gamma(1+\alpha)}[Z_t+\int_0^t(t-s)^{\alpha}dW_s],$$

where Γ is the gamma function,

$$Z_t=\int_{-\infty}^0[(t-s)^{\alpha}-(-s)^{\alpha}]dW_s,$$

and W_t is a standard Brownian motion. We suppose from now on $0<\alpha<\frac{1}{2}$. Then Z_t has absolutely continuous trajectories and it is the term $B_t:=\int_0^t(t-s)^{\alpha}dW_s$ that exhibits long range dependence. We will use B_t instead of B_t^H in fractional stochastic calculus. The fractional Black-Scholes model under our consideration is of the form

$$\begin{aligned} dS_t &= S_t(\mu dt+\sigma dB_t), 0\le t\le T, \\ S(0) &= S_0, \end{aligned} \tag{1}$$

where S_t is the price of a stock, μ, and σ are constants, and B_t as given above.

Now, consider the corresponding approximate model of (1)

$$\begin{aligned} dS_{\varepsilon}(t) &= S_{\varepsilon}(t)(\mu dt+\sigma dB_{\varepsilon}(t)), 0\le t\le T, \\ S_{\varepsilon}(0) &= S_0 \text{ (same initial condition as in (1))}, \end{aligned} \tag{2}$$

where $B_{\varepsilon}(t)=\int_0^t(t-s+\varepsilon)^{H-\frac{1}{2}}dW(s)$, $\frac{1}{2}<H<1$. Referring to the main result of Thao [10, Theorem 4.2], the solution $S_{\varepsilon}(t)$ of equation (2) converges to the solution S_t of (1) in $L^2(\Omega)$ as $\varepsilon\to 0$.

In this paper, we extend the main result of Thao [10] to a fractional Black-Scholes model with jumps. We also prove that the solution of our approximate models converges to the solution of the fractional Black-scholes model with jumps. In summary, this paper is organized as follows: In Sec. 2, we review the definition of the Poisson random measure and some preliminary notions of jump-diffusion processes which mostly come from [2]. In Sec. 3, we follow the general setting of [7, page 143] to consider the stock price model with jumps. In Sec. 4, we discuss an approximate model for a fractional stock-price model with jumps. Finally, we give some simulation examples to show the accuracy of approximations by the fractional Black-Scholes model with jumps as compared to the classical Black-Scholes model with jumps.

2. Poisson Random Measures

A Poisson process $(N(t), t \geq 0)$, with intensity λ, is defined as follows:

$$N(t) = \sum_{n \geq 1} 1_{\{T_n \leq t\}},$$

where $T_n = \sum_{i=1}^{n} \tau_i$ and $\tau_1, \tau_2, \ldots$ is a sequence of independent, identically exponentially distributed random variables (defined on some probability space $(\Omega, \boldsymbol{\mathcal{F}}, P)$) with parameter λ, that is, $P(\tau_1 > t) = e^{-\lambda t}$. $N(t)$ is simply the number of jumps between 0 and t, i.e.,

$$N(t) = \#\{n \geq 1, T_n \in [0, t]\}.$$

Similarly, if $t > s$ then

$$N(t) - N(s) = \#\{n \geq 1, T_n \in (s, t]\}.$$

The jump times $T_1, T_2, \ldots$, form a random configuration of points on $[0, \infty)$ and the Poisson process $N(t)$ counts the number of such points in the interval $[0, t]$. This counting procedure defines a measure N on $[0, \infty) := \mathbb{R}^+$ as follows: For any Borel measurable set $A \subset \mathbb{R}^+$,

$$N(\omega, A) = \#\{n \geq 1, T_n(\omega) \in A\} = \sum_{n \geq 1} 1_A(T_n(\omega)).$$

$N(\omega, \cdot)$ is a positive integer valued measure on Borel subsets of $\mathbb{R}^+$. We note that $N(\cdot, A)$ is finite with probability 1 for any bounded set $A \subset \mathbb{R}^+$. The measure $N(\omega, \cdot)$ depends on ω; it is thus a *random measure*. The intensity λ of the Poisson process determines the *average* value of the random measure $N(\cdot, A)$, that is

$$E[N(\cdot, A)] = \lambda |A|,$$

where $|A|$ is the Lebesgue measure of A.
$N(\omega, \cdot)$ is called a *Poisson random measure* associated with the Poisson process $N(t)$. The Poisson process $N(t)$ may be expressed in terms of the random measure N in the following way:

$$N(\omega, t) = N(\omega, [0, t]) = \int_{[0,t]} N(\omega, ds).$$

Conversely, the Poisson random measure N can also be viewed as the "derivative" of a Poisson process. Recall that each trajectory $t \mapsto N(\omega, t)$ of a Poisson process is an increasing step function. Hence its derivative (in the sense of distributions) is a positive measure on $\sigma-$algebra of Borel sets of $\mathbb{R}^+$. In fact, it is simply the superposition of Dirac masses located at the jump times:

$$\frac{d}{dt} N(\omega, t) = \sum_{n \geq 1} \delta_{T_n(\omega)}(\cdot) =: N(\omega, \cdot),$$

hence, for any predictable process $f(\omega, s)$, the stochastic integral with respect to the Poisson random measure N admits, for any $t \in \mathbb{R}^+$, the form

$$\int_0^t f(\cdot,s)N(\cdot,ds) = \sum_{n\geq 1} f(T_n)1_{\{T_n(\omega)\leq t\}}(\cdot) = \sum_{n=1}^{N(\cdot,t)} f(T_n),$$

or in a more compact form

$$\int\limits_0^t f(s)dN(s) = \sum_{n=1}^{N(t)} f(T_n). \tag{3}$$

We now assume that the T_n's correspond to the jump times of a Poisson process $N(t)$ and that Y_n is a sequence of indentically distributed random variables with values in $(-1,\infty)$. Let $S(t)$ be a predictable process. At time T_n the jump of the dynamics of $S(t)$ is given by

$$S(T_n) - S(T_n-) = S(T_n-)Y_n, \tag{4}$$

which, by the assumption $Y_n > -1$, leads always to positive values of the prices.

If $f(S,t)$ is a $C^{\{2,1\}}$-function (this means that f is C^2 in the first variable and C^1 in the second variable), then it follows from (3) that

$$\int_0^t [f(S(s-)(1+Y_s),s)-f(S(s-),s)]dN(s) = \sum_{n=1}^{N(t)} [f(S(T_n),T_n)-f(S(T_n-),T_n)] \tag{5}$$

where Y_t is obtained from Y_n by a piecewise constant and left continuous time interpolation. An application of equation (5) to the function $f(S,t) = S$ for $S \geq 0$ yields

$$\int_0^t [S(s-)(1+Y_s) - S(s-)]dN(s) = \sum_{n=1}^{N(t)} [S(T_n) - S(T_n-)]$$

or

$$\int_0^t S(s-)Y_s dN(s) = \sum_{n=1}^{N(t)} [S(T_n) - S(T_n-)]. \tag{6}$$

It then follows from equations (4) and (6) that

$$\int_0^t S(s-)Y_s dN(s) = \sum_{n=1}^{N(t)} S(T_n-)Y_n. \tag{7}$$

The following lemma is an Ito's formula for jump-diffusion process. Its proof can be found in [2, p. 275].

Lemma 1. *Let X be a diffusion process with jumps, defined as the sum of drift term, a Brownian stochastic integral and a compound Poisson process:*

$$X(t) = X(0) + \int_0^t b(s)ds + \int_0^t \sigma(s)dW(s) + \sum_{n=1}^{N(t)} \Delta X_n.$$

Here $b(t)$, $\sigma(t)$ are continuous nonanticipating processes with

$$E\left[\int_0^\tau \sigma^2(t)dt\right] < \infty,$$

and $\Delta X_n = X(T_n) - X(T_n-)$ *are the jump sizes. Then, for any* $C^{2,1}$ *function,* $f : \mathbb{R} \times [0, T] \to \mathbb{R}$*, the process* $Y(t) = f(X(t), t)$ *can be represented as:*

$$\begin{aligned} f(X(t), t) - f(X(0), 0) = & \int_0^t \left[\frac{\partial f}{\partial x}(X(s), s)b(s) + \frac{\partial f}{\partial s}(X(s), s)\right] ds \\ & + \frac{1}{2}\int_0^t \sigma^2(s)\frac{\partial^2 f}{\partial x^2}(X(s), s)ds + \int_0^t \frac{\partial f}{\partial x}(X(s), s)\sigma(s)dW(s) \\ & + \sum_{n=1}^{N(t)} \left[f(X(T_n), T_n) + f(X(T_n-), T_n)\right]. \end{aligned}$$

3. Stock Price Model with Jumps

Let us consider a probability space $(\Omega, \boldsymbol{\mathcal{F}}, \mathbb{P})$ on which we define a standard Brownian motion $(W(t), t \geq 0)$, a Poisson process $(N(t), t \geq 0)$ with intensity λ and a sequence $(Y_n, n \geq 1)$ of independent, identically distributed random variables taking values in $(-1, +\infty)$. We will assume that the σ-algebras generated respectively by $(W(t), t \geq 0)$, $(N(t), t \geq 0)$ and $(Y_n, n \geq 1)$ are independent.

The objective of this section is to model a financial market in which there is one riskless asset (with price $S^0(t) = e^{\mu t}$, at time t) and one risky asset whose price jumps at the proportions $Y_1, \ldots, Y_n, \ldots$, at some times $T_1, \ldots, T_n, \ldots$ and which, between any two jumps, follows the Black-Scholes model. Moreover, we will assume that the T_n's correspond to the jump times of a Poisson process.

The dynamics of $S(t)$, the price of the risky asset at time t, can now be described in the following manner. The process $(S(t), t \geq 0)$ is an adapted, right-continuous process such that on the time intervals $[T_n, T_{n+1})$,

$$dS(t) = S(t)(\mu dt + \sigma dW(t)), 0 \leq t \leq T \tag{8}$$

while at $t = T_n$, the jump of $S(t)$ is given by

$$\Delta S_n = S(T_n) - S(T_n-) = S(T_n-)Y_n.$$

Thus

$$S(T_n) = S(T_n-)(1 + Y_n).$$

By using the standard Itô formula, the solution of (8) on the interval $[0, T_1)$ is

$$S(t) = S(0)\exp\left((\mu - \frac{\sigma^2}{2})t + \sigma W(t)\right).$$

Consequently, the left-hand limit at T_1 is given by

$$S(T_1-) = \lim_{u \to T_1} S(u) = S(0)\exp\left((\mu - \frac{\sigma^2}{2})T_1 + \sigma W(T_1)\right)$$

and

$$S(T_1) = S(0)(1+Y_1)\exp\left(\Big(\mu - \frac{\sigma^2}{2}\Big)T_1 + \sigma W(T_1)\right).$$

Then, for $t \in [T_1, T_2)$,

$$\begin{aligned} S(t) &= S(T_1)\exp(\mu - \frac{\sigma^2}{2})(t - T_1) + \sigma(W(t) - W(T_1)) \\ &= S(0)(1+Y_1)\exp\left((\mu - \frac{\sigma^2}{2})t + \sigma W(t)\right). \end{aligned}$$

Repeating this scheme, we obtain

$$S(t) = S(0)\left[\prod_{N(t)} n = 1(1+Y_n)\right]\exp\left((\mu - \frac{\sigma^2}{2})t + \sigma W(t)\right) \tag{9}$$

with the convention $\prod_0 n = 1 = 1$. Using equation (3), $S(t)$ can be given in the following equivalent representations

$$\begin{aligned} S(t) &= S(0)\exp\left[(\mu - \frac{\sigma^2}{2})t + \sigma W(t) + \log\left(\prod_{n=1}^{N(t)}(1+Y_n)\right)\right] \\ &= S(0)\exp\left[(\mu - \frac{\sigma^2}{2})t + \sigma W(t) + \sum_{n=1}^{N(t)}\log(1+Y_n)\right] \\ &= S(0)\exp\left[(\mu - \frac{\sigma^2}{2})t + \sigma W(t) + \int_0^t \log(1+Y_s)dN(s)\right], \end{aligned}$$

where Y_t is obtained from Y_n by a piecewise constant and left continuous time interpolation.

The process $(S(t), t \geq 0)$ in equation (9) is right-continuous, adapted and has only finitely many discontinuities on each interval $[0, t]$. We can also prove the following.

Theorem 1. *For all $t \geq 0$, $(S(t), t \geq 0)$ in equation (9) satisfies:*

$$\mathbb{P} \quad a.s. \quad S(t) = S(0) + \int_0^t S(s)(\mu ds + \sigma dW(s)) + \sum_{n=1}^{N(t)} S(T_n-)Y_n \tag{10}$$

or, in differential form

$$\mathbb{P} \quad a.s. \quad dS(t) = S(t)(\mu dt + \sigma dW(t)) + S(t-)Y_t dN(t). \tag{11}$$

Proof. Let $\Delta S_n = S(T_n) - S(T_n-) = S(T_n-)Y_n$. Then (10) can be written in the following form:

$$\mathbb{P} \quad a.s. \quad S(t) = S(0) + \int_0^t S(s)\left(\mu ds + \sigma dW(s)\right) + \sum^{N(t)} n = 1\Delta S_n, \tag{12}$$

We choose the function $f(x, s) = \log x$. Direct calculation shows that

$$f_x = \frac{1}{x}, f_{xx} = -\frac{1}{x^2} \quad \text{and} \quad f_s = 0$$

We note that $f(x,t)$ is a $C^{2,1}$ function if $x > 0$. Assume that $S(t)$ in (10) is nonnegative. Applying the Itô formula for jump-diffusion processes (see Lemma 1) to $f(x,t) = \log x$, we obtain

$$\log S(t) = \log S(0)(\mu - \frac{\sigma^2}{2})t + \sigma W(s) + \sum_{n=1}^{N(t)} \log(1 + Y_n).$$

Thus,

$$S(t) = S(0) \left[\prod_{n=1}^{N(t)} (1 + Y_n) \right] \exp\left((\mu - \frac{\sigma^2}{2})t + \sigma W(t) \right).$$

Hence, we obtain (9) as asserted.

4. A Fractional Stock Price Model with Jumps

We use the same setting probability spaces as in Sec. 3. The objective of this section is to construct an approximate model for a financial market in which there is one riskless asset (with price $S^0(t) = e^{\mu t}$, at time t) and one risky asset whose price jumps in the proportions $Y_1, \ldots, Y_n, \ldots$ at some random times $T_1, T_2, ..., T_n, ...$and which, between two jumps, follows the fractional Black-Scholes model for a fractional process $B(t)$. These descriptions can be formalized on the intervals $[T_n, T_{n+1})$ by letting:

$$dS(t) = S(t)(\mu dt + \sigma dB(t)), 0 \le t \le T. \tag{13}$$

At $t = T_n$, the jump of $S(t)$ is given by

$$\Delta S_n = S(T_n) - S(T_n -) = S(T_n -)Y_n.$$

Now, we consider a *fractional Black-Scholes model with jumps* which is defined similarly to equation (11) by the following stochastic differential equation

$$\begin{aligned} dS(t) &= S(t)(\mu dt + \sigma dB(t)) + S(t-)Y_t dN(t), \\ S(t)|_{t=0} &= S(0). \end{aligned} \tag{14}$$

Here $B(t) = \int_0^t (t-s)^\alpha dW(s)$ where $0 < \alpha < \frac{1}{2}$.

The corresponding approximate model of (14) is defined for each $\varepsilon > 0$ by

$$\begin{aligned} dS_\varepsilon(t) &= S_\varepsilon(t)(\mu dt + \sigma dB_\varepsilon(t)) + S_\varepsilon(t-)Y_t dN(t), \\ S_\varepsilon(t)|_{t=0} &= S(0) \quad \text{(same intial conditon as in (14))}, \end{aligned} \tag{15}$$

where $B_\varepsilon(t) = \int_0^t (t-s+\varepsilon)^\alpha dW(s)$. One can prove that $B_\varepsilon(t)$ is a semimartingale and $B_\varepsilon(t)$ converges to $B(t)$ in $L^2(\Omega)$ when $\varepsilon \to 0$. This convergence is uniform with respect to $t \in [0,T]$ (see [10, Theorem 2.1]). We need the following lemma considered as a consequence of the L^2-convergence of $B_\varepsilon(t)$ to $B(t)$.

Lemma 2. *$B_\varepsilon(t)$ converges to $B(t)$ in $L^p(\Omega)$ for any $p \geq 2$, uniformly with respect to $t \in [0,T]$.*

Proof. The proof of this lemma is due to Nguyen Tien Dung [8].

Theorem 2. *Suppose that $S(0)$ is a random variable such that $E|S(0)|^{2+\delta}$ is finite for some $\delta > 0$. Then the solution of (15) is given by:*

$$S_\varepsilon(t) = S(0)\exp\left(-\frac{1}{2}\sigma^2\varepsilon^{2\alpha}t + \sigma\varepsilon^\alpha W(t) + \int_0^t H_\varepsilon(s)ds + \int_0^t \log(1+Y_s)dN(s)\right),$$

where $0 < \alpha < \frac{1}{2}$, and

$$H_\varepsilon(t) = \mu + \alpha\sigma\int_0^t (t-s+\varepsilon)^{\alpha-1}dW(s).$$

Furthermore, the stochastic process $S_*(t)$ defined by

$$S_*(t) = S(0)\exp\left(\mu t + \sigma B(t) + \int_0^t \log(1+Y_s)dN(s)\right)$$

is the limit in $L^2(\Omega)$ of $S_\varepsilon(t)$ as $\varepsilon \to 0$. This limit is uniform with respect to $t \in [0,T]$.

Proof. Letting $\varphi_\varepsilon(t) = \int_0^t (t-s+\varepsilon)^{\alpha-1}dW(s)$, and substituting $dB_\varepsilon(t) = \alpha\varphi_\varepsilon(t)dt + \varepsilon^\alpha dW(t)$ into equation (15), we obtain

$$dS_\varepsilon(t) = [\mu + \alpha\sigma\varphi_\varepsilon(t)]S_\varepsilon(t)dt + \sigma\varepsilon^\alpha S_\varepsilon(t)dW(t) + S_\varepsilon(t-)Y_t dN(t), \tag{16}$$

or,

$$\begin{aligned}\frac{dS_\varepsilon(t)}{S_\varepsilon(t)} &= [\mu + \alpha\sigma\varphi_\varepsilon(t)]dt + \sigma\varepsilon^\alpha dW(t) + \left(\frac{S_\varepsilon(t-)}{S_\varepsilon(t)}\right)Y_t dN(t) \\ &= H_\varepsilon(t)dt + \sigma\varepsilon^\alpha dW(t) + \left(\frac{S_\varepsilon(t-)}{S_\varepsilon(t)}\right)Y_t dN(t) \end{aligned} \tag{17}$$

where $H_\varepsilon(t) = \mu + \alpha\sigma\varphi_\varepsilon(t)$. Moreover, we can write equation (16) into an integral form as

$$\int_0^t dS_\varepsilon(t) = \int_0^t H_\varepsilon(s)S_\varepsilon(s)ds + \int_0^t \sigma\varepsilon^\alpha S_\varepsilon(s)dW(s) + \int_0^t S_\varepsilon(s-)Y_s dN(s).$$

Thus,

$$S_\varepsilon(t) = S(0) + \int_0^t H_\varepsilon(s)S_\varepsilon(s)ds + \int_0^t \sigma\varepsilon^\alpha S_\varepsilon(s)dW(s) + \int_0^t S_\varepsilon(s-)Y_s dN(s).$$

Using the formula (7), $S_\varepsilon(t)$ can be given in the following equivalent representations

$$S_\varepsilon(t) = S(0) + \int_0^t H_\varepsilon(s)S_\varepsilon(s)ds + \int_0^t \sigma\varepsilon^\alpha S_\varepsilon(s)dW(s) + \sum_{n=1}^{N(t)} S_\varepsilon(T_n-)Y_n. \quad (18)$$

Since $\Delta S_\varepsilon(T_n) = S_\varepsilon(T_n) - S_\varepsilon(T_n-) = S_\varepsilon(T_n-)Y_n$ then equation (18) becomes

$$S_\varepsilon(t) = S(0) + \int_0^t H_\varepsilon(s)S_\varepsilon(s)ds + \int_0^t \sigma\varepsilon^\alpha S_\varepsilon(s)dW(s) + \sum_{n=1}^{N(t)} \Delta S_\varepsilon(T_n).$$

Choosing the function $f(x,s) = \log x$ for $x = S_\varepsilon(t) > 0$, direct calculation shows that

$$f_x = \frac{1}{x}, f_{xx} = -\frac{1}{x^2} \quad \text{and} \quad f_s = 0$$

An application of the Itô formula for jump-diffusion processes (see Lemma 1) gives:

$$\begin{aligned}
\log S_\varepsilon(t) &= \log S(0) + \int_0^t \left(0 + \left(\frac{1}{S_\varepsilon(s)}\right) \cdot (H_\varepsilon(s)S_\varepsilon(s))\right) ds \\
&+ \frac{1}{2}\int_0^t (\sigma\varepsilon^\alpha)^2 S_\varepsilon^2(s) \left(-\frac{1}{S_\varepsilon(s)}\right)^2 ds \\
&+ \int_0^t \left(\frac{1}{S_\varepsilon(s)}\right)(\sigma\varepsilon^\alpha)S_\varepsilon(s)dW(s) \\
&+ \sum_{n=1}^{N(t)} [\log(S_\varepsilon(T_n-) + \Delta S_\varepsilon(T_n)) - \log(S_\varepsilon(T_n-))] \\
&= \log S(0) + \int_0^t H_\varepsilon(s)ds - \frac{1}{2}\int_0^t (\sigma\varepsilon^\alpha)^2 ds + \int_0^t \sigma\varepsilon^\alpha dW(s) \\
&+ \sum_{n=1}^{N(t)} \left[\log\left(\frac{S_\varepsilon(T_n-)(1+Y_n)}{S_\varepsilon(T_n-)}\right)\right] \\
&= \log S(0) + \int_0^t (H_\varepsilon(s)ds + \sigma\varepsilon^\alpha dW(s)) - \frac{1}{2}\int_0^t (\sigma\varepsilon^\alpha)^2 ds \qquad (19) \\
&+ \sum_{n=1}^{N(t)} \log(1+Y_n)
\end{aligned}$$

Using formulae (7) and (17), equation (19) can be given in the following equivalent representations

$$
\begin{aligned}
\log S_\varepsilon(t) &= \log S(0) + \int_0^t (H_\varepsilon(s)ds + \sigma\varepsilon^\alpha dW(s)) - \frac{1}{2}\int_0^t (\sigma\varepsilon^\alpha)^2 ds \\
&+ \int_0^t \log(1+Y_n)dN(s) \\
&= \log S(0) + \left(\int_0^t \frac{dS_\varepsilon(s)}{S_\varepsilon(s)} - \int_0^t \left(\frac{S_\varepsilon(s-)}{S_\varepsilon(s)}\right) Y_s dN(s)\right) - \frac{1}{2}\sigma^2\varepsilon^{2\alpha}t \\
&+ \int_0^t \log(1+Y_n)dN(s) \\
&= \log S(0) + \int_0^t \frac{dS_\varepsilon(s)}{S_\varepsilon(s)} - \frac{1}{2}\sigma^2\varepsilon^{2\alpha}t + \int_0^t \log(1+Y_n)dN(s) \\
&- \int_0^t \left(\frac{S_\varepsilon(s-)}{S_\varepsilon(s)}\right) Y_s dN(s).
\end{aligned}
$$

Here Y_t is obtained from Y_n by a piecewise constant and left continuous time interpolation. Thus

$$
\int_0^t \frac{dS_\varepsilon(s)}{S_\varepsilon(s)} = \log\frac{S_\varepsilon(t)}{S(0)} + \frac{1}{2}\sigma^2\varepsilon^{2\alpha}t - \int_0^t \log(1+Y_n)dN(s) + \int_0^t \left(\frac{S_\varepsilon(s-)}{S_\varepsilon(s)}\right) Y_s dN(s). \tag{20}
$$

Equating (20) and (17), we get

$$
\begin{aligned}
&\log\frac{S_\varepsilon(t)}{S(0)} + \frac{1}{2}\sigma^2\varepsilon^{2\alpha}t - \int_0^t \log(1+Y_n)dN(s) + \int_0^t \left(\frac{S_\varepsilon(s-)}{S_\varepsilon(s)}\right) Y_s dN(s) \\
&= \int_0^t H_\varepsilon(s)ds + \sigma\varepsilon^\alpha W(t) + \int_0^t \left(\frac{S_\varepsilon(s-)}{S_\varepsilon(s)}\right) Y_s dN(s).
\end{aligned}
$$

Hence, the solution of (15) is

$$
S_\varepsilon(t) = S(0)\exp\left(-\frac{1}{2}(\sigma\varepsilon^\alpha)^2 t + \sigma\varepsilon^\alpha W(t) + \int_0^t H_\varepsilon(s)ds + \int_0^t \log(1+Y_n)dN(s)\right). \tag{21}
$$

We note that

$$
\int_0^t H_\varepsilon(s)ds = \mu + \alpha\sigma\int_0^t \varphi_\varepsilon(s)ds.
$$

By application of the stochastic theorem of Fubini, we get

$$
\int_0^t \varphi_\varepsilon(s)ds = \frac{1}{\alpha}\left(B_\varepsilon(t) - \varepsilon^\alpha W(t)\right).
$$

Therefore

$$
\int_0^t H_\varepsilon(s)ds = \mu t + \sigma B_\varepsilon(t) - \sigma\varepsilon^\alpha W(t).
$$

Substituting the value of $\int_0^t H_\varepsilon(s)ds$ into equation (21), we get

$$S_\varepsilon(t) = S(0)\exp\left(\mu t - \frac{1}{2}(\sigma\varepsilon^\alpha)^2 t + \sigma B_\varepsilon(t) + \int_0^t \log(1+Y_n)dN(s)\right).$$

We note that $\frac{1}{2}(\sigma\varepsilon^\alpha)^2 t \to 0$ as $\varepsilon \to 0$ and $B_\varepsilon(t)$ converges uniformly to $B(t)$ in $L^2(\Omega)$ when $\varepsilon \to 0$. This motivates us to consider the process $S_*(t)$ defined by

$$S_*(t) = S(0)\exp\left(\mu t + \sigma B(t) + \int_0^t \log(1+Y_n)dN(s)\right).$$

We try to show that $S_*(t)$ is the limit of $S_\varepsilon(t)$ in $L^2(\Omega)$ as $\varepsilon \to 0$. We observe that

$$\begin{aligned} S_\varepsilon(t) - S_*(t) =& S(0)\exp\left(\mu t - \frac{1}{2}(\sigma\varepsilon^\alpha)^2 t + \sigma B_\varepsilon(t) + \int_0^t \log(1+Y_n)dN(s)\right) \\ & - S(0)\exp\left(\mu t + \sigma B(t) + \int_0^t \log(1+Y_n)dN(s)\right) \\ & = S(0)\exp\left(\mu t + \sigma B(t) + \int_0^t \log(1+Y_n)dN(s)\right) \\ & \left[\exp\left(-\frac{1}{2}(\sigma\varepsilon^\alpha)^2 t + \sigma(B_\varepsilon(t) - B(t))\right) - 1\right] \\ & = S(0)\exp\left(\mu t + \sigma B(t)\right) \cdot \exp\left(\int_0^t \log(1+Y_n)dN(s)\right) \\ & \left[\exp\left(-\frac{1}{2}(\sigma\varepsilon^\alpha)^2 t + \sigma(B_\varepsilon(t) - B(t))\right) - 1\right]. \end{aligned}$$

Put $p = 1 + \frac{\delta}{2}$ and $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. It follows from Holder's inequality that

$$\begin{aligned} ||S_\varepsilon(t) - S_*(t)||_2 \le & ||S(0)||_{2p} || \exp\left(\mu t + \sigma B(t)\right) \cdot \exp\left(\int_0^t \log(1+Y_n)dN(s)\right) \times \\ & \left[\exp\left(-\frac{1}{2}(\sigma\varepsilon^\alpha)^2 t + \sigma(B_\varepsilon(t) - B(t))\right) - 1\right] ||_{2q} \\ \le & ||S(0)||_{2+\delta} || \exp\left(\mu t + \sigma B(t)\right) \exp\left(\int_0^t \log(1+Y_n)dN(s)\right) ||_{4q} \times \\ & || \left[\exp\left(-\frac{1}{2}(\sigma\varepsilon^\alpha)^2 t + \sigma(B_\varepsilon(t) - B(t))\right) - 1\right] ||_{4q} \end{aligned} \tag{22}$$

In order to calculate the norm $||S_\varepsilon(t) - S_*(t)||_2$, we firstly note that

$$\begin{aligned} & || \exp\left(\mu t + \sigma B(t)\right) \exp\left(\int_0^t \log(1+Y_n)dN(s)\right) ||_{4q} \\ & \le || \exp\left(\mu t + \sigma B(t)\right) ||_{8q} || \exp\left(\int_0^t \log(1+Y_n)dN(s)\right) ||_{8q} < \infty. \end{aligned} \tag{23}$$

To see this we note that, for each t, B_t is a Gaussian random variable with zero mean and variance γ_t^2 for some real numbers γ_t. Then

$$|| \exp(\mu t + \sigma B(t)) ||_{8q} = \exp(\mu t)[Ee^{8q\sigma B(t)}]^{\frac{1}{8q}} = \exp(\mu t) e^{4q\sigma^2\gamma^2(t)} < \infty.$$

Moreover

$$\begin{aligned} &\left\| \exp\left(\int_0^t \log(1+Y_n) dN(s) \right) \right\|_{8q} \\ &= \left\| \exp\left(\sum_{n=1}^{N(t)} \log(1+Y_n) \right) \right\|_{8q} = \left\| \sum_{n=1}^{N(t)} (1+Y_n) \right\|_{8q} \leq K, \end{aligned}$$

where K is a constant. This is due to the fact that there is a finite number of jumps in the finite interval $[0, T]$.

Finally, we compute the last term on the right hand side of (22). It follows from the relation $e^A - 1 = A + o(A)$ that we have

$$\begin{aligned} &\left\| \left[\exp\left(-\frac{1}{2}(\sigma\varepsilon^\alpha)^2 t + \sigma(B_\varepsilon(t) - B(t)) \right) - 1 \right] \right\|_{4q} \\ &\leq \left\| -\frac{1}{2}(\sigma\varepsilon^\alpha)^2 t + \sigma(B_\varepsilon(t) - B(t)) \right\|_{4q} + \left\| o(-\frac{1}{2}(\sigma\varepsilon^\alpha)^2 t + \sigma(B_\varepsilon(t) - B(t))) \right\|_{4q} \\ &\leq \frac{1}{2}(\sigma\varepsilon^\alpha)^2 t + \sigma \left\| B_\varepsilon(t) - B(t) \right\|_{4q} + || o(-\frac{1}{2}(\sigma\varepsilon^\alpha)^2 t + \sigma(B_\varepsilon(t) - B(t))) ||_{4q} \end{aligned}$$

By application of Lemma 2, we have $||B_\varepsilon(t) - B(t)||_{4q} \to 0$ as $\epsilon \to 0$ (uniformly on $t \in [0, T]$). Hence

$$\begin{aligned} &\left\| \left[\exp\left(-\frac{1}{2}(\sigma\varepsilon^\alpha)^2 t + \sigma(B_\varepsilon(t) - B(t)) \right) - 1 \right] \right\|_{4q} \\ &\leq \frac{1}{2}(\sigma\varepsilon^\alpha)^2 T + \sigma ||B_\varepsilon(t) - B(t)||_{4q} + \left\| o(-\frac{1}{2}(\sigma\varepsilon^\alpha)^2 t + \sigma(B_\varepsilon(t) - B(t))) \right\|_{4q} \end{aligned}$$

The right hand side of the above inequality does not depend on t and approaches zero when $\varepsilon \to 0$. Therefore, one can see from (22) and (23), that $S_\varepsilon(t) \to S_*(t)$ in $L^2(\Omega)$ as $\varepsilon \to 0$ and the convergence is uniform with respect to t.

5. Simulation Examples

Let us consider the Thai stock market. Figure 1 shows the daily prices of a data set consisting of 150 open -prices of the Thai Petrochemical Industry (TPI) between June 9, 2004 and January 7, 2005. The empirical data for these stock prices were obtained from http://finance.yahoo.com.

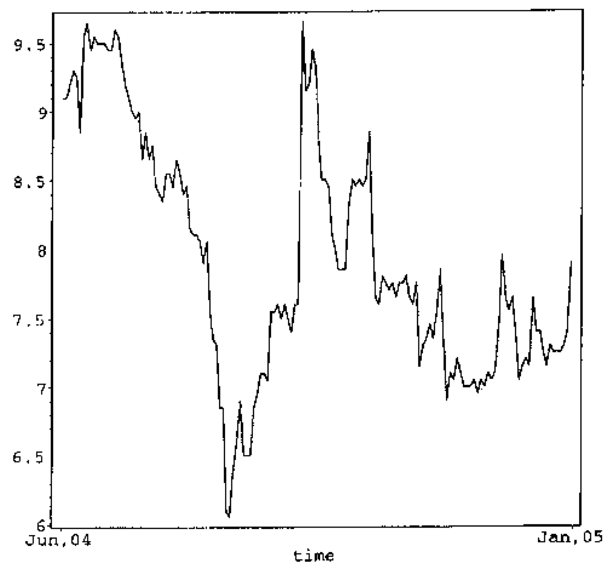

Fig. 1. Price behavior of *TPI, between June 4, 2004 and January 7, 2005*

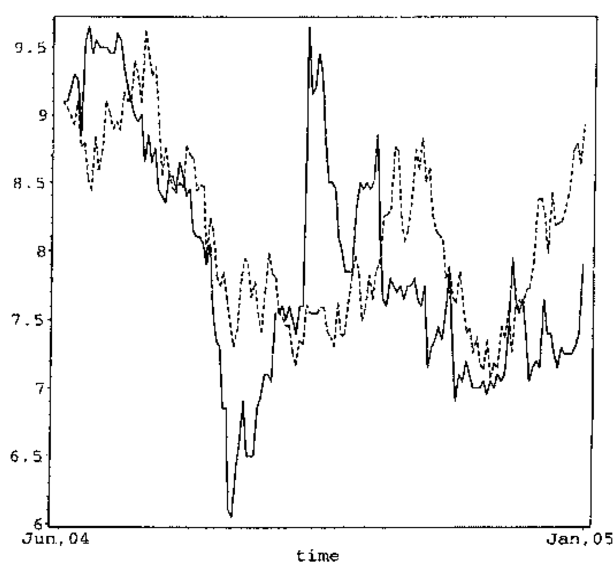

Fig. 2. Price behavior of *TPI, between June 4, 2004 and January 7, 2005,* compared with a scenario simulated from a Black-Scholes model with jumps (solid line:= empirical data, dashed line:= simulated by $S(t)=S(0)\exp((\mu-\frac{\sigma^2}{2})t+\sigma W(t)+\sum_{n=1}^{N(t)}(1+Y_n)))$, *ARPE(2) = 23.69%, and variance = 0.02656)*

Figure 2 shows the empirical data of TPI open-price as compared to the price that was simulated by a Black-Scholes pricing model with jump. In the simulation process, we use the algorithm that appeared in the paper of Cyganowski, Grunce and Kloeden [3]. The simulated model is $S(t) = S(0)\exp((\mu - \frac{\sigma^2}{2})t + \sigma W(t) + \sum_{n=1}^{N(t)}(1+Y_n)$. The model parameters $\mu = -0.0000725, \sigma = 0.3025$ and parameter for jumps as $\mu_j = 0.00007624$, $\sigma_j = 0.0003679$, $\lambda = 55.46$, $\gamma = 1$ are fixed. For comparative purposes, we compute the Average Relative Percentage Error(ARPE). By definition, ARPE= $(1/N)\sum_{k=1}^{N}\frac{|X_k - Y_k|}{X_k}.100$, where N is the number of price, $X = (X_k)_{k\geq 1}$ is the market prices and $Y = (Y_k)_{k\geq 1}$ is the model prices. We worked out 500 trails and computed ARPE. We denote the ARPE of Figure 2 and and Figure 3 by ARPE(2) and ARPE(3) respectively.

Figure 3 shows the empirical data of TPI open-price as compared to the price that was simulated by a fractional Black-Scholes pricing model with jumps. The simulated model is $S_{\in}(t) = S(0)\exp((\mu - \frac{1}{2}((\sigma\varepsilon^{\alpha})^2)t + \sigma B_{\in}(t) + \sum_{n=1}^{N(t)}(1+Y_n))$. The value of μ, σ and the parameters for jumps are the same as in Figure 2. For the remaining data, we choose $H = 0.50001$, $\varepsilon = 0.000001$.

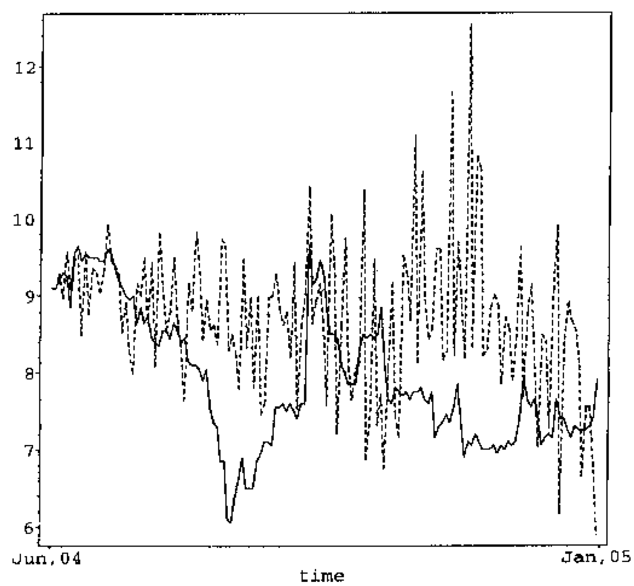

Fig. 3. Price behavior of *TPI, between June 4, 2004 and January 7, 2005,* compared with a scenario simulated from a fractional Black Scholes model with jumps (solid line := empirical data, dashed line := simulated by

$$S_{\varepsilon}(t)=S(0)\exp((\mu-\frac{1}{2}((\sigma\varepsilon^{\alpha})^2)t+\sigma B_{\in}(t)+\sum_{n=1}^{N(t)}(1+Y_n)).$$

ARPE(3) = 19.64%, and variance = 0.01546)

By comparing ARPE and variance of Figures 2 and 3, one can see that in case of TPI, the sample path from a fractional Black-Scholes pricing model with jumps gives a better fit with the data than Black-Scholes pricing model with jumps.

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