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Convexification by Duality for a Multiple Leontief Technology Production Design Problem

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Abstract. In this article we consider a multiple Leontief technology design problem that can be formulated as a nonconvex minimization problem. By quasiconvex duality we convert this problem into a less intractable problem that is a convex minimization problem.

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1. Introduction

In this article we are interested in an application of Duality Theory that enables us to convert an optimization problem into a less intractable one. In optimization problems the intractable structures often involve nonlinear and nonconvex factors. The well known Lagrange duality, Fenchel conjugate duality and their equivalences play a fundamental role in Convex Duality. To some extent the convex duality scheme can be applied to nonconvex problems where the objective function is a fractional function. By Charnes-Cooper's transformation a fractional problem can be converted into a convex problem (cf. Refs. [2, 4]). Therefore, the duality in fractional problems can be obtained from convex duality under suitable transformations (cf. Refs. [4, 1]). The duality with zero gap could be extended to a larger class that is the class of quasiconvex minimization problems (cf. Refs. [5 - 8]). For a quasiconvex problem there could be two alternative duality approaches: the duality by quasiconjugates and the duality by level sets. These two approaches have different interpretations, but they are basically equivalent (cf. Refs. [5, 6]). In certain applications the dual problem is less intractable than the primal problem w.r.t. the current solution methods. For instance, in Ref. [7] the nonlinear Leontief production problem can be reduced by duality to a quasiaffine problem. In this article we show further that the quasiconvex duality can be used to convert a nonconvex problem into a less intractable convex problem.

In Sec. 2 we consider a nonlinear program appeared in a multiple Leontief technology production design problem. In Sec. 3 we present an optimality criterion. In Sec. 4 we present a convexification by duality. Finally several concluding remarks will be drawn in Sec. 5.

2. Problem Setting

In the production problem under our consideration the final product can be produced from m materials. Denote by $x = (x_1, x_2, \ldots, x_m)^T \in R^m_+$ a material vector where x_i is the i-th material. A material vector can be produced from the operating budget M in a multi integrated technology production process. More concretely, if $c^j \in R^m_+$ is a characterized coefficient vector of the j-th integrated technology ($j = 1, 2, \ldots, n$) then the operating cost of producing the material vector x is defined by

$$c(x) \quad = \quad \min \left\{ \sum_{j=1}^{n} \mu_j : \ \sum_{j=1}^{n} \mu_j c^j \ \geq \ x, \ \mu \geq 0 \right\},$$

where $\mu = (\mu_1, \mu_2, \ldots, \mu_n)^T$. Asume that for any $i \in \{1, 2, \ldots, m\}$ there is an index $j \in \{1, 2, \ldots, n\}$ such that $c^j_i > 0$. Under this assumption $c(x)$ is finite for any $x \in R^m_+$. The operating cost function c is continuous, nondecreasing, homogeneous and convex on R^m_+. A material vector x is called feasible if the operating cost $c(x)$ is less than or equal to the operating budget M:

$$c(x) \leq M.$$

A material vector can be used to produce the final product in a multi Leontief technology production process. More concretely, if $a^k \in R^m_+$ is a characterized coefficient vector of the k-th Leontief technology ($k = 1, 2, \ldots, \ell$) then the final product obtained from a material vector x is defined by

$$p(x) \quad = \quad \max \left\{ \sum_{k=1}^{\ell} \theta_k : \ \sum_{k=1}^{\ell} \theta_k a^k \ \leq \ x, \ \theta \geq 0 \right\},$$

where $\theta = (\theta_1, \theta_2, \ldots, \theta_k)^T$. Assume that $a^k \neq 0$ for any $k \in \{1, 2, \ldots, \ell\}$. Under this assumption $p(x)$ is finite for any $x \in R^m_+$. The production function p is continuous, nondecreasing, homogeneous and concave on R^m_+.

Let Q be a given final product level. A material vector x is called currently capable if the product value $p(x)$ is greater than or equal to Q:

$$p(x) \geq Q.$$

In our problem we assume that there is no material vector that is both feasible and currently capable, i.e.,

$$\{x \geq 0 : \ c(x) \leq M, \ p(x) \geq Q\} = \emptyset.$$

In order to produce the product level Q from a feasible material vector we must upgrade the Leontief technologies. For an upgrade level γ ($\gamma \geq 0$) the upgraded k-th Leontief technology can produce $f_k(\gamma)$ times of that produced by the current k-th Leontief technology, i.e., the material vector $\theta_k a^k$ can produce the value $\theta_k f_k(\gamma)$ of the final product by the upgraded k-th Leontief technology, where f_k is an increasing convex upgrade function on R_+ such that

$$\begin{aligned} &f_k(0) = 1, \\ &f_k(\gamma) \ \to \ \infty \ \text{ as } \ \gamma \ \to \ \infty. \end{aligned}$$

An example of such a function f_k is the following

$$f_k(\gamma) = \alpha_k \gamma \ + \ 1 \quad (\alpha_k > 0).$$

Obviously, the higher the upgrade level γ is, the more efficient the upgraded k-th Leontief technology is. The final product produced from a material vector x in the upgraded multi Leontief technology production process with the upgrade level γ is defined by

$$p_\gamma(x) = \max\left\{ \sum_{k=1}^{\ell} \theta_k f_k(\gamma) : \sum_{k=1}^{\ell} \theta_k a^k \ \leq \ x, \ \theta \geq 0 \right\}.$$

For fixed x the value $p_\gamma(x)$ increases as γ increases, and for fixed γ the function p_γ is continuous, nondecreasing, homogeneous and concave on R^m_+. Moreover $p_\gamma(x)$ is continuous in $(x, \gamma) \in R^{m+1}_+$. A material vector x is called γ-capable if

$$p_\gamma(x) \geq Q.$$

An upgrade cost, denoted by $\Gamma(\gamma)$, by definition is an increasing function of γ. The problem now is to find the minimum value of $\Gamma(\gamma)$ such that there is a feasible material vector which is γ-capable. Since Γ is an increasing function of γ, this problem is equivalent to design the upgrade level $\overline{\gamma}$ that solves the following problem

$$\begin{aligned} &\min \ \gamma, \\ &\text{s.t. } c(x) \leq M, \ p_\gamma(x) \geq Q, \ x \geq 0. \end{aligned} \tag{1}$$

Since $p_\gamma(x)$ is nonconvex in (x, γ), this problem is a nonconvex program. By setting

$$q(x) = \inf\{\gamma : \ p_\gamma(x) \geq Q\},$$

we can rewrite problem (1) as follows

$$\begin{aligned} &\min \ q(x), \\ &\text{s.t. } c(x) \leq M, \ x \geq 0, \end{aligned}$$

where q can easily be checked quasiconvex in x. Thus our problem is a quasiconvex minimization program.

3. Optimality Criterion

Setting

$$\begin{aligned}\overline{c}^j &= \frac{1}{M}\ c^j \quad j = 1, 2, \ldots, n,\\ \overline{c}(x) &= \frac{1}{M}\ c(x),\\ \overline{a}^k &= \frac{1}{Q}\ a^k \quad k = 1, 2, \ldots, \ell,\\ \overline{p}_\gamma(x) &= \frac{1}{Q}\ p_\gamma(x),\end{aligned}$$

we convert Problem (1) into the following problem

$$\begin{aligned}&\min\ \gamma,\\ &\text{s.t. } \overline{c}(x) \le 1,\ \overline{p}_\gamma(x) \ge 1,\ x \ge 0,\end{aligned} \tag{2}$$

where

$$\begin{aligned}\overline{c}(x) &= \min\left\{\sum_{j=1}^{n} \mu_j :\ \sum_{j=1}^{n} \mu_j \overline{c}^j \ge x,\ \mu \ge 0\right\},\\ \overline{p}_\gamma(x) &= \frac{1}{Q}\ \max\left\{\sum_{k=1}^{\ell} \theta'_k f_k(\gamma) :\ \sum_{k=1}^{\ell} \theta'_k a^k \le x,\ \theta' \ge 0\right\}.\end{aligned} \tag{3}$$

By setting $\theta_k = \theta'_k f_k(\gamma)$ for any $k = 1, 2, \ldots, \ell$ we have

$$\overline{p}_\gamma(x) \ = \ \frac{1}{Q}\ \max\left\{\sum_{k=1}^{\ell} \theta_k :\ \sum_{k=1}^{\ell} \frac{\theta_k}{f_k(\gamma)}\ \overline{a}^k \le x,\ \theta \ge 0\right\}. \tag{4}$$

Since multiple fractions are involved in the definition (4) of $\overline{p}_\gamma(x)$, Problem (2) is a multiple fractional program. Since $f_k(\gamma) \ \to\ \infty$ as $\gamma \ \to\ \infty$, we can see that $\overline{p}_\gamma(x)$ is greater than 1 for $x > 0$ and for high enough γ. Therefore the constraint of (2) is consistent, hence (2) is solvable.

For optimality criterions in quasiconvex minimization problems we can use the quasisubdifferentials instead of the usual subdifferentials (cf. Refs. [6, 8]). Let x be a nonzero vector in R^m_+. A vector $u \in R^m_+$ is called a quasisubgradient of the convex function $\overline{c}$ at x if

$$\begin{aligned}&u^T x = 1,\\ &u^T y \le 1 \quad \forall y \ge 0 :\ \overline{c}(y) \le \overline{c}(x).\end{aligned}$$

The set of quasisubgradients of $\overline{c}$ at x is denoted by $\tilde{\partial}\overline{c}(x)$. A vector $u \in R^m_+$ is called a quasisupgradient of the concave function $\overline{p}_\gamma$ at x if

$$\begin{aligned}&u^T x = 1,\\ &u^T y \ge 1 \quad \forall y \ge 0 :\ \overline{p}_\gamma(y) \ge \overline{p}_\gamma(x).\end{aligned}$$

The set of quasisupgradients of $\overline{p}_\gamma$ at x is denoted by $\tilde{\partial}\overline{p}_\gamma(x)$.

Theorem 3.1. *Let $(\overline{x},\overline{\gamma})$ be a $(m+1)$-dimensional vector such that $\overline{x}\geq 0$ and $\overline{\gamma}\geq 0$. Then, $(\overline{x},\overline{\gamma})$ is optimal to Problem (2) if and only if*

$$\overline{c}(\overline{x})=1,\ \overline{p}_{\overline{\gamma}}(\overline{x})=1,\ \text{and}\ \ \tilde{\partial}\overline{c}(\overline{x})\cap\tilde{\partial}\overline{p}_{\overline{\gamma}}(\overline{x})\neq\emptyset. \tag{5}$$

Proof. Suppose that $(\overline{x},\overline{\gamma})$ is optimal to (2). Then, $\overline{c}(x)\leq 1$ and $\overline{p}_{\overline{\gamma}}(\overline{x})\geq 1$. If $\overline{c}(\overline{x})<1$ then there is $\beta>1$ such that $\overline{c}(\beta\overline{x})\leq 1$. We have $\overline{p}_{\overline{\gamma}}(\beta\overline{x})=\beta\overline{p}_{\overline{\gamma}}(\overline{x})>1$. So, there is $\gamma'<\overline{\gamma}$ such that $\overline{p}_{\gamma'}(\beta\overline{x})\geq 1$. Thus, $(\beta\overline{x},\gamma')$ is better than $(\overline{x},\overline{\gamma})$. This is contradictory with the optimality of $(\overline{x},\overline{\gamma})$. Similarly, if $\overline{p}_{\overline{\gamma}}(\overline{x})>1$ then we also arrive at the contradiction with the optimality of $(\overline{x},\overline{\gamma})$. Therefore

$$\overline{c}(\overline{x})=1\quad\text{and}\quad\overline{p}_{\overline{\gamma}}(\overline{x})\ =\ 1. \tag{6}$$

By the similar arguments we can prove that

$$\{x\geq 0:\ \overline{c}(x)<1\}\ \cap\ \{x\geq 0:\ \overline{p}_{\overline{\gamma}}(x)\geq 1\}=\emptyset.$$

Since the interior of the convex set $\{x\geq 0:\ \overline{c}(x)\leq 1\}$ is contained in $\{x\geq 0:\ \overline{c}(x)<1\}$, this implies that the closed convex set $\{x\geq 0:\ \overline{p}_{\overline{\gamma}}(x)\geq 1\}$ does not intersect with the interior of the closed convex set $\{x\geq 0:\ \overline{c}(x)\leq 1\}$. By the separation theorem there is a vector $u\in R^m$ such that

$$u^Tx\leq 1\ \ \forall x\geq 0:\ \overline{c}(x)\leq 1, \tag{7}$$
$$u^Tx\geq 1\ \ \forall x\geq 0:\ \overline{p}_{\overline{\gamma}}(x)\geq 1. \tag{8}$$

Since $\overline{p}_{\overline{\gamma}}$ is nondecreasing, the recession cone of the closed convex set $\{x\geq 0:\ \overline{p}_{\overline{\gamma}}(x)\geq 1\}$ is R^m_+. This together with (8) implies $u\geq 0$. From (6), (7), and (8) it follows that $u^T\overline{x}=1$. This together with (6) and (7) implies $u\in\tilde{\partial}\overline{c}(\overline{x})$. From $u^T\overline{x}=1$ and (8) it follows that $u\in\tilde{\partial}\overline{p}_\gamma(\overline{x})$. Thus we have obtained (8) from the optimality of $(\overline{x},\overline{\gamma})$.

Conversely suppose (8) holds at $(\overline{x},\overline{\gamma})$. If $(\overline{x},\overline{\gamma})$ is not optimal to (2) then there is (x',γ') such that

$$\begin{aligned}&x'\geq 0,\ \gamma'\geq 0,\\ &\overline{c}(x')\leq 1,\ \overline{p}_{\gamma'}(x')\geq 1\ \ \text{and}\ \ \gamma'<\overline{\gamma}.\end{aligned}$$

Since $\gamma'<\overline{\gamma}$, we have $\overline{p}_{\overline{\gamma}}(x')>\overline{p}_{\gamma'}(x')\geq 1$. Let $u\in\tilde{\partial}\overline{c}(\overline{x})\cap\tilde{\partial}\overline{p}_{\overline{\gamma}}(\overline{x})$, and $\beta<1$ such that $\overline{p}_{\overline{\gamma}}(\beta x')\geq 1$. Then, on one hand we have $u^Tx'\leq 1$ because $u\in\tilde{\partial}\overline{c}(\overline{x})$ and $\overline{c}(x')\leq 1=\overline{c}(\overline{x})$. However on the other hand we have $u^Tx'>1$ because $u^T\beta x'\geq 1$ (the last innequality is obtained from $u\in\tilde{\partial}\overline{p}_{\overline{\gamma}}(\overline{x})$ and $\overline{p}_{\overline{\gamma}}(\beta x')\geq 1=\overline{p}_{\overline{\gamma}}(\overline{x})$). So, we arrive at a contradiction. Thus $(\overline{x},\overline{\gamma})$ is optimal to (2). ∎

4. Convexification by Duality

Denote by X the convex set $\{x\geq 0:\ \overline{c}(x)\leq 1\}$. We have

$$X=\left\{x\geq 0:\ x\leq\sum_{j=1}^{n}\mu_j\overline{c}^j,\ \sum_{j=1}^{n}\mu_j\leq 1,\ \mu\geq 0\right\},$$

i.e., X is the convex hull of $\{0, \overline{c}^j\ j = 1, 2, \dots, n\}$. Denote by U the polar of X:

$$U = \{u \geq 0 :\ u^T x \leq 1\ \ \forall x \in X\}.$$

Since X is the convex hull of $\{0, \overline{c}^j\ j = 1, 2, \dots, n\}$, we have

$$U = \left\{u \geq 0 :\ \overline{c}^{j^T} u \leq 1\ \ j = 1, 2, \dots, n\right\}.$$

Set

$$c^*(u) = \max\left\{\overline{c}^{j^T} u\ \ j = 1, 2, \dots, n\right\}.$$

Then c^* is a continuous nondecreasing homogeneous and convex function on R^m_+. Moreover,

$$U = \{u \geq 0 :\ c^*(u) \leq 1\}.$$

Denote by Y_γ the convex set $\{x \geq 0 :\ \overline{p}_\gamma(x) \geq 1\}$. We have

$$Y_\gamma = \left\{x \geq 0 :\ x \geq \sum_{k=1}^{\ell} \frac{\theta_k}{f_k(\gamma)}\ \overline{a}^k,\ \sum_{k=1}^{\ell} \theta_k \geq 1,\ \theta \geq 0\right\},$$

i.e., Y_γ is the convex hull of the following subset

$$\left\{\frac{1}{f_k(\gamma)}\ \overline{a}^k + R^m_+\ \ k = 1, 2, \dots, \ell\right\}. \tag{9}$$

Denote by V_γ the conjugate of Y_γ:

$$V_\gamma = \{u \geq 0 :\ u^T x \geq 1\ \ \forall x \in Y_\gamma\}.$$

Since Y_γ is the convex hull of the subset given by (9), we have

$$V_\gamma = \left\{u \geq 0 :\ \frac{1}{f_k(\gamma)}\ \overline{a}^{k^T} u \geq 1\ \ \ k = 1, 2, \dots, \ell\right\}.$$

Set

$$p^*_\gamma(u) = \min\left\{\frac{1}{f_k(\gamma)}\ \overline{a}^{k^T} u\ \ \ k = 1, 2, \dots, \ell\right\}.$$

Then p^*_γ is a continuous nondecreasing homogenous and concave function on R^m_+. For fixed u the value $p^*_\gamma(u)$ decreases as γ increases and $p^*_\gamma(u)$ is continuous in $(u, \gamma) \in R^{m+1}_+$. Moreover,

$$V_\gamma = \{u \geq 0 :\ p^*_\gamma(u) \geq 1\}.$$

A dual Problem of problem (2) now can be stated as follows

$$\begin{aligned} &\max\ \gamma, \\ &\text{s.t. } c^*(u) \leq 1,\ p^*_\gamma(u) \geq 1,\ u \geq 0. \end{aligned} \tag{10}$$

Let $u \in R^m_+ \setminus \{0\}$. Similarly as in the last section we call a vector $x \in R^m_+$ a quasisubgradient of c^* at u if

$$\begin{aligned} &u^T x = 1, \\ &v^T x \leq 1\ \ \ \forall v \geq 0 :\ c^*(v) \leq c^*(u). \end{aligned}$$

The set of quasisubgradients of c^* at u is denoted by $\tilde{\partial} c^*(u)$. We call a vector $x \in R^m_+$ a quasisupgradient of p^*_γ at u if

$$u^T x = 1,$$
$$v^T x \geq 1 \quad \forall v \geq 0 : \; p^*_\gamma(v) \geq p^*_\gamma(u).$$

The set of quasisupgradients of p^*_γ at u is denoted by $\tilde{\partial} p^*_\gamma(u)$. Since $p^*_\gamma(u)$ decreases as γ increrases and γ is to be maximized in Problem (10), by the arguments quite similar to the proof of Therem 3.1 we obtain the following theorem.

Theorem 4.1. *Let $(\overline{u}, \overline{\gamma})$ be a $(m+1)$-dimensional vector such that $\overline{u} \geq 0$ and $\overline{\gamma} \geq 0$. Then, $(\overline{u}, \overline{\gamma})$ is optimal to Problem (10) if and only if*

$$c^*(\overline{u}) = 1, \; p^*_{\overline{\gamma}}(\overline{u}) \;=\; 1, \quad \text{and} \quad \tilde{\partial} c^*(\overline{u}) \cap \tilde{\partial} p^*_{\overline{\gamma}}(\overline{u}) \;\neq\; \emptyset.$$

For the duality relationship between Problem (2) and Problem (10) we have the following theorem.

Theorem 4.2. *Let $\overline{x} \geq 0$, $\overline{u} \geq 0$ and $\overline{\gamma} \geq 0$. Then, the three following assertions are equivalent.*

(i) $(\overline{x}, \overline{\gamma})$ *solves the primal Problem* (2) *and* $(\overline{u}, \overline{\gamma})$ *solves the dual Problem* (10)*;*

(ii) $\overline{c}(\overline{x}) \;=\; 1$, $\overline{p}_{\overline{\gamma}}(\overline{x}) \;=\; 1$, *and* $\overline{u} \in \tilde{\partial} \overline{c}(\overline{x}) \cap \tilde{\partial} \overline{p}_{\overline{\gamma}}(\overline{x})$*;*

(iii) $c^*(\overline{u}) \;=\; 1$, $p^*_{\overline{\gamma}}(\overline{u}) \;=\; 1$, *and* $\overline{x} \in \tilde{\partial} c^*(\overline{u}) \cap \tilde{\partial} p^*_{\overline{\gamma}}(\overline{u})$.

Proof. By Theorem 3.1 and Theorem 3.2 in order to prove this theorem we need only to show (ii) $\Leftrightarrow$ (iii). Suppose that (ii) holds for $(\overline{x}, \overline{u}, \overline{\gamma})$. Since

$$\overline{u}^T \overline{x} = 1 \;\geq\; \overline{u}^T x \quad \forall x \geq 0 : \; \overline{c}(x) \leq \overline{c}(\overline{x}) = 1,$$

it follows that $\overline{u} \in U$, i.e., $c^*(\overline{u}) \leq 1$. If $c^*(\overline{u}) < 1$ then

$$\overline{c}^{j^T} \overline{u} \;<\; 1 \quad j = 1, 2, \ldots, n$$

and since $\overline{x} \in X = \text{conv}\{0, \overline{c}^j \;\; j = 1, 2, \ldots, n\}$, this implies $\overline{u}^T \overline{x} < 1$. This is a contradiction. So, $c^*(\overline{u}) = 1$. Since

$$\overline{u}^T \overline{x} = 1 \;\leq\; \overline{u}^T x \quad \forall x \geq 0 : \; \overline{p}_{\overline{\gamma}}(x) \geq \overline{p}_{\gamma}(\overline{x}) = 1,$$

it follows that $\overline{u} \in V_{\overline{\gamma}}$, i.e., $p^*_{\overline{\gamma}}(\overline{u}) \geq 1$. If $p^*_{\overline{\gamma}}(\overline{u}) > 1$ then

$$\frac{1}{f_k(\overline{\gamma})} \, \overline{a}^{k^T} \overline{u} > 1, \quad k = 1, 2, \ldots, \ell,$$

and since $\overline{x} \in Y_{\overline{\gamma}}$ and $Y_{\overline{\gamma}}$ is the convex hull of the subset given in (9) where γ is replaced by $\overline{\gamma}$, this implies $\overline{u}^T \overline{x} > 1$. This is a contradiction. So $p^*_{\overline{\gamma}}(\overline{u}) = 1$. Since $\overline{x} \in X$, we have

$$\overline{u}^T \overline{x} \;=\; 1 \;\geq\; u^T \overline{x} \quad \forall u \geq 0 : \; c^*(u) \leq c^*(\overline{u}) = 1.$$

Therefore, $\overline{x} \in \tilde{\partial} c^*(\overline{u})$. Similarly since $\overline{x} \in Y_{\overline{\gamma}}$, we have

$$\overline{u}^T \overline{x} = 1 \;\leq\; u^T \overline{x} \quad \forall u \geq 0 : \; p^*_{\overline{\gamma}}(u) \geq p^*_{\overline{\gamma}}(\overline{u}) = 1.$$

Therefore, $\overline{x} \in \tilde{\partial} p^*_{\overline{\gamma}}(\overline{u})$. Thus, we have obtained (iii). Quite similarly we can obtain (ii) from (iii). ■

As a consequence of Theorem 4.2 we have

Corollary 4.1. *If* $(\overline{x}, \overline{\gamma})$ *is feasible to the primal Problem* (2) *and* $(\overline{u}, \overline{\gamma})$ *is feasible to the dual Problem* (10), *then* $(\overline{x}, \overline{\gamma})$ *is optimal to* (2) *and* $(\overline{u}, \overline{\gamma})$ *is optimal to* (10).

Proof. By the duality relationship given in Theorem 4.2, the optimal values in (2) and (10) exist and equal each other. Since in the primal Problem (2) γ is minimized and in the dual Problem (10) γ is maximized, the feasible value in (2) is always greater than or equal to the feasible value in (10). Therefore, if $(\overline{x}, \overline{\gamma})$ is feasible to (2) and $(\overline{u}, \overline{\gamma})$ is feasible to (10), then $\overline{\gamma}$ must be the minimum value of γ in (2) and the maximum value of γ in (10). ■

On the basis of the duality relationship we can solve the dual Problem (10) instead of the primal Problem (2). The dual problem can be rewritten as follows

$$\begin{aligned}
&\max\ \gamma,\\
&\text{s.t. } \overline{c}^{j^T} u \ \le\ 1 \quad j = 1, 2, \dots, n,\\
&\frac{1}{f_k(\gamma)}\ \overline{a}^{k^T} u \ \ge\ 1 \quad k = 1, 2, \dots, \ell,\\
&u \ \ge\ 0.
\end{aligned}$$

This problem is equivalent to the following

$$\begin{aligned}
&\max\ \gamma,\\
&\text{s.t. } \overline{c}^{j^T} u \ \le\ 1 \quad j = 1, 2, \dots, n,\\
&\overline{a}^{k^T} u \ \ge\ f_k(\gamma) \quad k = 1, 2, \dots, \ell,\\
&u \ \ge\ 0.
\end{aligned} \tag{11}$$

Since for any $k \in \{1, 2, \dots, \ell\}$ f_k is convex in γ, this is a convex program in (u, γ). Particularly if $f_k(\gamma) = \alpha_k \gamma + 1$ then Problem (11) is a linear program.

5. Concluding Remarks

The problem we have considered in the previous sections is to design the upgrade level γ such that the upgrade cost is minimum under the consistency condition between the feasibility and the γ-capability. The problem involves multiple fractions and it is a nonlinear quasiconvex minimization over a convex set. By dualtiy we convert the problem into a convex minimization problem over a convex set. Particularly, if the upgrade functions f_k $k = 1, 2, \dots, \ell$ are affine then the converted problem is a linear program. The duality presented in the previous section follows the level set approach. This approach is basically equivalent to the quasiconjugate approach. To be more concrete we set

$$\overline{q}(x) = \inf\{\gamma : \ \overline{p}_\gamma(x) \geq 1\},$$
$$q^*(u) = \sup\{\gamma : \ p^*_\gamma(u) \ \geq \ 1\}.$$

Then $\overline{q}$ is a quasiconvex function and q^* is a quasiconcave function. The primal Problem (2) can be represented as follows

$$\begin{aligned} &\min \ \overline{q}(x), \\ &\text{s.t. } \overline{c}(x) \ \leq \ 1, \ x \ \geq \ 0, \end{aligned} \tag{12}$$

and the dual Problem (10) can be represented as follows

$$\begin{aligned} &\max \ q^*(u), \\ &\text{s.t. } c^*(u) \ \leq \ 1, \ u \geq 0. \end{aligned} \tag{13}$$

It can be checked that the objective functions in (12) and (13) are the quasi-conjugates of each other, i.e.,

$$q^*(u) = \inf\{\overline{q}(x) : \ u^T x \leq 1, \ x \geq 0\},$$
$$\overline{q}(x) = \sup\{q^*(u) : \ u^T x \leq 1, \ u \geq 0\}.$$

Other applications of the level set duality approach and the quasiconjugate duality approach can be found in Refs. [5, 6].

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