Semicommutative and Reduced Rings

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Received June 13, 2006
Revised January 10, 2007

Abstract. A ring \( R \) is called semicommutative, if \( ab = 0 \) implies \( aRb = 0 \) for all \( a, b \in R \). It is well-known that the \( n \times n \) upper triangular matrix ring over any ring with identity is not semicommutative when \( n \geq 2 \). In the paper, a special semicommutative subring of upper triangular matrix ring over a reduced ring is obtained.

2000 Mathematics Subject Classification: 16U80, 16S50.

Keywords: semicommutative ring; Armendariz ring; reduced ring.

1. Introduction

Throughout this paper, all rings are associative with identity \( 1(\neq 0) \). For a ring \( R \), the notations \( \gamma_R(\cdot) \) and \( \iota_R(\cdot) \) are used for the right and left, respectively, annihilators over \( R \). A ring \( R \) is called semicommutative, if \( ab = 0 \) implies \( aRb = 0 \) for all \( a, b \in R \). According to Shin [1, Lemma 1.2], a ring \( R \) is semicommutative if and only if, for any \( a, b \in R, \) \( ab = 0 \) implies \( aRb = 0 \), if and only if any right annihilator over \( R \) is an ideal of \( R \), if and only if any left annihilator over \( R \) is an ideal of \( R \). Properties, examples and counterexamples of semicommutative rings are given in [2-4].

We fix some notations. Let \( R \) be a ring. We write \( M_n(R) \) and \( T_n(R) \) for the \( n \times n \) matrix ring and \( n \times n \) upper triangular matrix ring over \( R \), respectively. The \( n \times n \) identity matrix is denoted by \( I_n \). For any \( A \in M_n(R) \), let \( RA = \{ rA : r \in R \} \). For \( n \geq 2 \), let \( \{ E_{i,j} : 1 \leq i, j \leq n \} \) be the set of the matrix units.

Define a subring \( R_n \) of the \( n \times n \) matrix ring \( M_n(R) \) over \( R \) as follows:
It was proved in [2, Proposition 1.2 and Example 1.3] that if $R$ is reduced, then the ring $R_3$ is semicommutative but $R_n$ is not semicommutative for $n \geq 4$. In the paper we continue the study of semicommutative rings and try to find some bigger semicommutative subrings of $T_n(R)$ for $n \geq 2$ when $R$ is a reduced ring. Our method will also be used to give some Armendariz subrings of $T_n(R)$ for $n \geq 2$ when $R$ is a reduced ring. For this purpose, we introduce the following notation.

For an positive integer $n \geq 2$, we let

$$U_n(R) = \sum_{i=1}^{k} \sum_{j=k+1}^{n} RE_{i,j} + \sum_{j=k+2}^{n} RE_{k+1,j} + RI_n,$$

where $k = \lfloor n/2 \rfloor$, i.e., $k$ satisfies $n = 2k$ when $n$ is an even integer, and $n = 2k+1$ when $n$ is an odd integer.

For example,

$$U_4(R) = \left\{ \begin{pmatrix} a & 0 & b & e \\ 0 & a & d & f \\ 0 & 0 & a & f \\ 0 & 0 & 0 & a \end{pmatrix} : a, b, c, d, e, f \in R \right\},$$

$$U_5(R) = \left\{ \begin{pmatrix} a & 0 & a & b & c \\ 0 & a & d & e & f \\ 0 & 0 & a & g & h \\ 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & a \end{pmatrix} : a, b, c, d, e, f, g, h \in R \right\}.$$

Note that if $n = 3$, then the ring $U_3(R) = R_3$ is semicommutative [2, Proposition 1.2 and Example 1.3] when the ring $R$ is reduced.

2. Semicommutative and Reduced Rings

**Theorem 2.1.** Let $R$ be a reduced ring. Then the following hold.

1. $U_n(R)$ is a semicommutative ring for every $n = 2k + 1 \geq 3$;
2. $U_n(R)$ is a semicommutative ring for every $n = 2k \geq 2$.

**Proof.**

(1) Let
a = \begin{pmatrix}
  a & 0 & \cdots & 0 & a_{1,k+1} & a_{1,k+2} & \cdots & a_{1,2k+1} \\
  a & 0 & \cdots & 0 & a_{2,k+1} & a_{2,k+2} & \cdots & a_{2,2k+1} \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  a & a_{k,k+1} & a_{k,k+2} & \cdots & a_{k,2k+1} \\
  a & a_{k+1,k+2} & \cdots & a_{k+1,2k+1} \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  a & 0 & \cdots & 0 & a \\
  a & 0 & \cdots & 0 & a \\
  \end{pmatrix}

\beta = \begin{pmatrix}
  b & 0 & \cdots & 0 & b_{1,k+1} & b_{1,k+2} & \cdots & b_{1,2k+1} \\
  b & 0 & \cdots & 0 & b_{2,k+1} & b_{2,k+2} & \cdots & b_{2,2k+1} \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  b & a_{k,k+1} & b_{k,k+2} & \cdots & b_{k,2k+1} \\
  b & b_{k+1,k+2} & \cdots & b_{k+1,2k+1} \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  b & 0 & \cdots & 0 & b \\
  b & 0 & \cdots & 0 & b \\
  \end{pmatrix}

in U_n(R) for n = 2k + 1 \geq 3 satisfy \alpha \beta = (c_{p,q}) = 0. Then for any p, q \in \{1, 2, \cdots, k\} we have the following:

\begin{align}
  c_{p,q} &= ab = 0 \\
  c_{p,k+1} &= ab_{p,k+1} + a_{p,k+1}b = 0 \\
  c_{p,k+1+q} &= ab_{p,k+1+q} + a_{p,k+1}b_{k+1,k+1+q} + a_{p,k+1+q}b = 0 \\
  c_{k+1,k+1+q} &= ab_{k+1,k+1+q} + a_{k+1,k+1+q}b = 0
\end{align}

From (1), we have that ba = 0 since R is reduced. If we multiply (2) by b on the left side, then bab_{p,k+1} + ba_{p,k+1}b = 0 for p = 1, 2, \cdots, k. Thus we get that bab_{p,k+1}b = 0 and hence a_{p,k+1}b = 0 for p = 1, 2, \cdots, k. So ab_{p,k+1} = 0 for p = 1, 2, \cdots, k. From (4), continuing in the same manner, we can show that ab_{k+1,k+1+q}b = 0 for q = 1, 2, \cdots, k. If we multiply (3) on the left side by b, then we obtain that 0 = bab_{p,k+1+q} + ba_{p,k+1}b_{k+1,k+1+q} + ba_{p,k+1+q}b = ba_{p,k+1+q}b for any p, q \in \{1, 2, \cdots, k\}, thus a_{p,k+1+q}b = 0 for any p, q \in \{1, 2, \cdots, k\}. Thus

ab_{p,k+1+q} + a_{p,k+1}b_{k+1,k+1+q} = 0 \quad \text{for any } p, q \in \{1, 2, \cdots, k\} \tag{\ast}

Multiplying (\ast) on the right side by a, we obtain 0 = ab_{p,k+1+q}a + a_{p,k+1}b_{k+1,k+1+q}
\begin{align}
a &= ab_{p,k+1+q}a \quad \text{for any } p, q \in \{1, 2, \cdots, k\}. \quad \text{Thus } ab_{p,k+1+q} = 0 \quad \text{for any } p, q \in \{1, 2, \cdots, k\}. \quad \text{It also follows from (\ast) that } a_{p,k+1}b_{k+1,k+1+q} = 0 \quad \text{for any } p, q \in \{1, 2, \cdots, k\}. \quad \text{Now each}
\end{align}
\[ \gamma = \begin{pmatrix} c & 0 & \cdots & 0 & c_{1,k+1} & c_{1,k+2} & \cdots & c_{1,2k+1} \\ c & \cdots & 0 & c_{2,k+1} & c_{2,k+2} & \cdots & c_{2,2k+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c & a_{k,k+1} & c_{k,k+2} & \cdots & c_{k,2k+1} \\ c & \cdots & 0 \\ c & \cdots & 0 \\ c & \cdots & 0 \end{pmatrix} \in U_n(R). \]

We need only to prove \((\alpha \beta \gamma)_{ij} = d_{ij} = 0\) for each \((i, j) \in N \times N\), where \(N = \{1, 2, \ldots, n\}\). Since \(R\) is reduced, it is semicommutative. So for any \(x, y \in R\), \(xy = 0\) implies that \(xRy = 0\). Thus \(\alpha cb = 0\), \(d_{p,k+1} = (ac)b_{p,k+1} + (ac_{p,k+1} + a_{p,k+1})b = 0\) for \(p = 1, 2, \ldots, k\) and \(d_{k+1,k+1+q} = (ac)b_{k+1,k+1+q} + (ac_{k+1,k+1+q} + a_{k+1,k+1+q})b = 0\) for \(q = 1, 2, \ldots, k\). For any \(p, q \in \{1, 2, \ldots, k\}\), we have

\[
d_{p,k+1+q} = (ac)b_{p,k+1+q} + (ac_{p,k+1} + a_{p,k+1})b_{k+1,k+1+q} + (ac_{p,k+1+q} + a_{p,k+1+q})b_{k+1,k+1+q} = 0.
\]

By the above proof, we get that \((\alpha \gamma \beta)_{ij} = d_{ij} = 0\) for each \((i, j) \in N \times N\). Therefore \(\alpha \gamma \beta = 0\). Hence \(U_n(R)\) is semicommutative for \(n = 2k + 1 \geq 3\).

(2) It is similar to (1).

According to [5], a ring \(R\) is called an Armendariz ring if whenever polynomials \(f(x) = a_0 + a_1x + \cdots + a_m x^m, g(x) = b_0 + b_1x + \cdots + b_n x^n \in R[x]\) satisfy \(f(x)g(x) = 0\), then \(a_i b_j = 0\) for each \(i, j\). The name "Armendariz" was chosen because E. Armendariz [6, Lemma 1] had noted that a reduced ring satisfies this condition. Properties, examples and counterexamples of Armendariz rings are given in [3, 5 - 9].

Note that \(R_3\) is Armendariz when \(R\) is reduced, but \(R_n\) is not for any ring \(R\) when \(n \geq 4\) [7, Proposition 2 and Example 3]. By analogy with the proof of Theorem 2.1 we have the following result on Armendariz rings.

**Corollary 2.2.** Let \(R\) be a reduced ring. Then the following hold.

(1) \(U_n(R)\) is an Armendariz ring for every \(n = 2k + 1 \geq 3\);

(2) \(U_n(R)\) is an Armendariz ring for every \(n = 2k \geq 2\).

**Proof.**

(1) Let \(f(x) = \sum_{i=0}^s A_i x^i, g(x) = \sum_{j=0}^t B_j x^j \in U_n(R)[x]\) be such that \(f(x)g(x) = 0\). Suppose that...
A_i = \begin{pmatrix}
a^{(i)} & 0 & \cdots & 0 & a_1^{(i)} & a_2^{(i)} & \cdots & a_{1,2k+1}^{(i)} \\
a^{(i)} & 0 & \cdots & 0 & a_2^{(i)} & a_2^{(i)} & \cdots & a_{2,2k+1}^{(i)} \\
\vdots & & \ddots & & \vdots & & \vdots & \vdots \\
a^{(i)} & a_{k,k+1}^{(i)} & a_{k,k+2}^{(i)} & \cdots & a_{k,2k+1}^{(i)} & 0 & \cdots & 0 \\
a^{(i)} & a_{k+1,k+2}^{(i)} & a_{k+1,k+2}^{(i)} & \cdots & a_{k+1,2k+1}^{(i)} & 0 & \cdots & 0 \\
\cdots & & \cdots & & \cdots & & \cdots & \cdots \\
a^{(i)} & & & & & & & a^{(i)} \\
\end{pmatrix}, i = 0, 1, \cdots, s,

and

B_j = \begin{pmatrix}
b^{(j)} & 0 & \cdots & 0 & b_1^{(j)} & b_2^{(j)} & \cdots & b_{1,2k+1}^{(j)} \\
b^{(j)} & 0 & \cdots & 0 & b_2^{(j)} & b_2^{(j)} & \cdots & b_{2,2k+1}^{(j)} \\
\vdots & & \ddots & & \vdots & & \vdots & \vdots \\
b^{(j)} & b_{k,k+1}^{(j)} & b_{k,k+2}^{(j)} & \cdots & b_{k,2k+1}^{(j)} & 0 & \cdots & 0 \\
b^{(j)} & b_{k+1,k+2}^{(j)} & b_{k+1,k+2}^{(j)} & \cdots & b_{k+1,2k+1}^{(j)} & 0 & \cdots & 0 \\
\cdots & & \cdots & & \cdots & & \cdots & \cdots \\
b^{(j)} & & & & & & & b^{(j)} \\
\end{pmatrix}, j = 0, 1, \cdots, t.

Denote

f = \sum_{i=0}^{s} a^{(i)} x^i,

f_{1,k+1} = \sum_{i=0}^{s} a_{1,k+1}^{(i)} x^i, \cdots, f_{1,2k+1} = \sum_{i=0}^{s} a_{1,2k+1}^{(i)} x^i,

\cdots

f_{k,k+1} = \sum_{i=0}^{s} a_{k,k+1}^{(i)} x^i, \cdots, f_{k,2k+1} = \sum_{i=0}^{s} a_{k,2k+1}^{(i)} x^i,

f_{k+1,k+2} = \sum_{i=0}^{s} a_{k+1,k+2}^{(i)} x^i, \cdots, f_{k+1,2k+1} = \sum_{i=0}^{s} a_{k+1,2k+1}^{(i)} x^i,

and
\[ g = \sum_{j=0}^{t} b(j)x^j, \]

\[ g_{1,k+1} = \sum_{j=0}^{t} b_{1,k+1}^{(j)}x^j, \ldots, g_{1,2k+1} = \sum_{j=0}^{t} b_{1,2k+1}^{(j)}x^j, \]

\[ \ldots \]

\[ g_{k,k+1} = \sum_{j=0}^{t} b_{k,k+1}^{(j)}x^j, \ldots, g_{k,2k+1} = \sum_{j=0}^{t} b_{k,2k+1}^{(j)}x^j, \]

\[ g_{k+1,k+2} = \sum_{j=0}^{t} b_{k,k+2}^{(j)}x^j, \ldots, g_{k+1,2k+1} = \sum_{j=0}^{t} b_{k+1,2k+1}^{(j)}x^j. \]

Note that \( R[x] \) is reduced when \( R \) is reduced. So as in the proof of Theorem 2.1, we obtain that \( fg = 0, \ f g_{p,k+1} = 0, \ f g_{p,k+1+q} = 0, \ f g_{p,k+1+q+1} = 0, \ f g_{p,k+1+q+2} = 0, \ f g_{k+1,k+1+q} = 0, \ f g_{k+1,k+1+q+1} = 0 \) and \( f g_{k+1,k+1+q+2} = 0 \) for each \( p, q \in \{1, 2, \ldots, k\} \). Since reduced rings are Armendariz, it follows easily that each coefficient of \( f \) annihilators every coefficient of \( g \), each coefficient of \( f \) annihilators every coefficient of \( g_{p,k+1} \) for \( p = 1, 2, \cdots, n \), each coefficient of \( f \) annihilators every coefficient of \( g_{p,k+1} \) for \( p = 1, 2, \cdots, n \), etc. Now it is easy to prove that \( A_iB_j = 0 \) for any \( i = 0, 1, \cdots, s \) and \( j = 0, 1, \cdots, t \). Thus \( U_n(R) \) is an Armendariz ring for every \( n = 2k + 1 \geq 3 \).

(2) It is similar to (1).

\[ \begin{array}{l}
\textbf{Corollary 2.3.} \quad (\text{[2], Proposition 1.2}) \text{ Let } R \text{ be a reduced ring. Then }
\end{array} \]

\[ R_3 = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} : a, b, c, d \in R \right\} \]

is a semicommutative (and an Armendariz) ring.

Given a ring \( R \) and a bimodule \( _RM_R \), the trivial extension of \( R \) by \( M \) is the ring \( T(R, M) = R \oplus M \) with the usual addition and the following multiplication:

\[ (r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + m_1 r_2). \]

This is isomorphic to the ring of all matrices

\[ \begin{pmatrix} r & m \\ 0 & r \end{pmatrix}, \]

where \( r \in R \) and \( m \in M \) and the usual matrix operations are used.

\[ \begin{array}{l}
\textbf{Corollary 2.4.} \quad \text{Let } R \text{ be a reduced ring. Then } T(R, R) \text{ is a semicommutative (and an Armendariz) ring.}
\end{array} \]
References