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Semicommutative and Reduced Rings

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Abstract. A ring R is called semicommutative, if $ab = 0$ implies $aRb = 0$ for all $a, b \in R$. It is well-known that the n by n upper triangular matrix ring over any ring with identity is not semicommutative when $n \geq 2$. In the paper, a special semicommutative subring of upper triangular matrix ring over a reduced ring is obtained.

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1. Introduction

Throughout this paper, all rings are associative with identity $1(\neq 0)$. For a ring R, the notations $\gamma_R(-)$ and $\iota_R(-)$ are used for the right and left, respectively, annihilators over R. A ring R is called semicommutative, if $ab = 0$ implies $aRb = 0$ for all $a, b \in R$. According to Shin [1, Lemma 1.2], a ring R is semicommutative if and only if, for any $a, b \in R$, $ab = 0$ implies $aRb = 0$, if and only if any right annihilator over R is an ideal of R, if and only if any left annihilator over R is an ideal of R. Properties, examples and counterexamples of semicommutative rings are given in [2-4].

We fix some notations. Let R be a ring. We write $M_n(R)$ and $T_n(R)$ for the $n \times n$ matrix ring and $n \times n$ upper triangular matrix ring over R, respectively. The $n \times n$ identity matrix is denoted by I_n. For any $A \in M_n(R)$, let $RA = \{rA : r \in R\}$. For $n \geq 2$, let $\{E_{i,j} : 1 \leq i, j \leq n\}$ be the set of the matrix units.

Define a subring R_n of the $n \times n$ matrix ring $M_n(R)$ over R as follows:

$$R_n = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} : a, a_{ij} \in R \right\}$$

It was proved in [2, Proposition 1.2 and Example 1.3] that if R is reduced, then the ring R_3 is semicommutative but R_n is not semicommutative for $n \geq 4$. In the paper we continue the study of semicommutative rings and try to find some bigger semicommutative subrings of $T_n(R)$ for $n \geq 2$ when R is a reduced ring. Our method will also be used to give some Armendariz subrings of $T_n(R)$ for $n \geq 2$ when R is a reduced ring. For this purpose, we introduce the following notation.

For an positive integer $n \geq 2$, we let

$$U_n(R) = \sum_{i=1}^{k} \sum_{j=k+1}^{n} RE_{i,j} + \sum_{j=k+2}^{n} RE_{k+1,j} + RI_n,$$

where $k = [n/2]$, i.e., k satisfies $n = 2k$ when n is an even integer, and $n = 2k+1$ when n is an odd integer.

For example,

$$U_4(R) = \left\{ \begin{pmatrix} a & 0 & b & c \\ 0 & a & d & e \\ 0 & 0 & a & f \\ 0 & 0 & 0 & a \end{pmatrix} : a, b, c, d, e, f \in R \right\},$$

$$U_5(R) = \left\{ \begin{pmatrix} a & 0 & a & b & c \\ 0 & a & d & e & f \\ 0 & 0 & a & g & h \\ 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & a \end{pmatrix} : a, b, c, d, e, f, g, h \in R \right\}.$$

Note that if $n = 3$, then the ring $U_3(R) = R_3$ is semicommutative [2, Proposition 1.2 and Example 1.3] when the ring R is reduced.

2. Semicommutative and Reduced Rings

Theorem 2.1. *Let R be a reduced ring. Then the following hold.*
(1) *$U_n(R)$ is a semicommutative ring for every $n = 2k + 1 \geq 3$;*
(2) *$U_n(R)$ is a semicommutative ring for every $n = 2k \geq 2$.*

Proof.
(1) Let

$$\alpha = \begin{pmatrix} a & 0 & \cdots & 0 & a_{1,k+1} & a_{1,k+2} & \cdots & a_{1,2k+1} \\ & a & \cdots & 0 & a_{2,k+1} & a_{2,k+2} & \cdots & a_{2,2k+1} \\ & & \ddots & 0 & \vdots & \vdots & \vdots & \vdots \\ & & & a & a_{k,k+1} & a_{k,k+2} & \cdots & a_{k,2k+1} \\ & & & & a & a_{k+1,k+2} & \cdots & a_{k+1,2k+1} \\ & & & & & a & \cdots & 0 \\ & & & & & & a & 0 \\ & & & & & & & a \end{pmatrix}$$

and

$$\beta = \begin{pmatrix} b & 0 & \cdots & 0 & b_{1,k+1} & b_{1,k+2} & \cdots & b_{1,2k+1} \\ & b & \cdots & 0 & b_{2,k+1} & b_{2,k+2} & \cdots & b_{2,2k+1} \\ & & \ddots & 0 & \vdots & \vdots & \vdots & \vdots \\ & & & b & a_{k,k+1} & b_{k,k+2} & \cdots & b_{k,2k+1} \\ & & & & b & b_{k+1,k+2} & \cdots & b_{k+1,2k+1} \\ & & & & & b & \cdots & 0 \\ & & & & & & b & 0 \\ & & & & & & & b \end{pmatrix}$$

in $U_n(R)$ for $n = 2k+1 \geq 3$ satisfy $\alpha\beta = (c_{p,q}) = 0$. Then for any $p, q \in \{1, 2, \cdots, k\}$ we have the following:

$$c_{p,p} = ab = 0 \tag{1}$$

$$c_{p,k+1} = ab_{p,k+1} + a_{p,k+1}b = 0 \tag{2}$$

$$c_{p,k+1+q} = ab_{p,k+1+q} + a_{p,k+1}b_{k+1,k+1+q} + a_{p,k+1+q}b = 0 \tag{3}$$

$$c_{k+1,k+1+q} = ab_{k+1,k+1+q} + a_{k+1,k+1+q}b = 0 \tag{4}$$

From (1), we have that $ba = 0$ since R is reduced. If we multiply (2) by b on the left side, then $bab_{p,k+1} + ba_{p,k+1}b = 0$ for $p = 1, 2, \cdots, k$. Thus we get that $ba_{p,k+1}b = 0$ and hence $a_{p,k+1}b = 0$ for $p = 1, 2, \cdots, k$. So $ab_{p,k+1} = 0$ for $p = 1, 2, \cdots, k$. From (4), continuing in the same manner, we can show that $ab_{k+1,k+1+q} = a_{k+1,k+1+q}b = 0$ for $q = 1, 2, \cdots, k$. If we multiply (3) on the left side by b, then we obtain that $0 = bab_{p,k+1+q} + ba_{p,k+1}b_{k+1,k+1+q} + ba_{p,k+1+q}b = ba_{p,k+1+q}b$ for any $p, q \in \{1, 2, \cdots, k\}$, thus $a_{p,k+1+q}b = 0$ for any $p, q \in \{1, 2, \cdots, k\}$. Thus

$$ab_{p,k+1+q} + a_{p,k+1}b_{k+1,k+1+q} = 0 \quad \text{for any } p, q \in \{1, 2, \cdots, k\} \tag{$*$}$$

Multiplying $(*)$ on the right side by a, we obtain $0 = ab_{p,k+1+q}a + a_{p,k+1}b_{k+1,k+1+q}a = ab_{p,k+1+q}a$ for any $p, q \in \{1, 2, \cdots, k\}$. Thus $ab_{p,k+1+q} = 0$ for any $p, q \in \{1, 2, \cdots, k\}$. It also follows from $(*)$ that $a_{p,k+1}b_{k+1,k+1+q} = 0$ for any $p, q \in \{1, 2, \cdots, k\}$. Now each

$$\gamma = \begin{pmatrix} c & 0 & \cdots & 0 & c_{1,k+1} & c_{1,k+2} & \cdots & c_{1,2k+1} \\ & c & \cdots & 0 & c_{2,k+1} & c_{2,k+2} & \cdots & c_{2,2k+1} \\ & & \ddots & 0 & \vdots & \vdots & \vdots & \vdots \\ & & & c & a_{k,k+1} & c_{k,k+2} & \cdots & c_{k,2k+1} \\ & & & & c & c_{k+1,k+2} & \cdots & c_{k+1,2k+1} \\ & & & & & c & \cdots & 0 \\ & & & & & & c & 0 \\ & & & & & & & c \end{pmatrix} \in U_n(R).$$

We need only to prove $(\alpha\beta\gamma)_{ij} = d_{ij} = 0$ for each $(i,j) \in N \times N$, where $N = \{1, 2, \cdots, n\}$. Since R is reduced, it is semicommutative. So for any $x, y \in R, xy = 0$ implies that $xRy = 0$. Thus $acb = 0$, $d_{p,k+1} = (ac)b_{p,k+1} + (ac_{p,k+1} + a_{p,k+1}c)b = 0$ for $p = 1, 2, \cdots, k$ and $d_{k+1,k+1+q} = (ac)b_{k+1,k+1+q} + (ac_{k+1,k+1+q} + a_{k+1,k+1+q}c)b = 0$ for $q = 1, 2, \cdots, k$. For any $p, q \in \{1, 2, \cdots, k\}$, we have

$$\begin{aligned} d_{p,k+1+q} =& (ac)b_{p,k+1+q} + (ac_{p,k+1} + a_{p,k+1}c)b_{k+1,k+1+q} \\ &+ (ac_{p,k+1+q} + a_{p,k+1}c_{k+1,k+1+q} + a_{p,k+1+q}c)b \\ =& 0. \end{aligned}$$

By the above proof, we get that $(\alpha\gamma\beta)_{ij} = d_{ij} = 0$ for each $(i,j) \in N \times N$. Therefore $\alpha\gamma\beta = 0$. Hence $U_n(R)$ is semicommutative for $n = 2k+1 \geq 3$.

(2) It is similar to (1). ■

According to [5], a ring R is called an Armendariz ring if whenever polynomials $f(x) = a_0 + a_1x + \cdots + a_mx^m, g(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$ for each i, j. The name "Armendariz" was chosen because E. Armendariz [6, Lemma 1] had noted that a reduced ring satisfies this condition. Properties, examples and counterexamples of Armendariz rings are given in [3, 5-9].

Note that R_3 is Armendariz when R is reduced, but R_n is not for any ring R when $n \geq 4$ [7, Proposition 2 and Example 3]. By analogy with the proof of Theorem 2.1 we have the following result on Armendariz rings.

Corollary 2.2. *Let R be a reduced ring. Then the following hold.*

(1) *$U_n(R)$ is an Armendariz ring for every $n = 2k+1 \geq 3$;*

(2) *$U_n(R)$ is an Armendariz ring for every $n = 2k \geq 2$.*

Proof.

(1) Let $f(x) = \sum_{i=0}^{s} A_ix^i$, $g(x) = \sum_{j=0}^{t} B_jx^j \in U_n(R)[x]$ be such that $f(x)g(x) = 0$. Suppose that

$$A_i = \begin{pmatrix} a^{(i)} & 0 & \cdots & 0 & a^{(i)}_{1,k+1} & a^{(i)}_{1,k+2} & \cdots & a^{(i)}_{1,2k+1} \\ & a^{(i)} & \cdots & 0 & a^{(i)}_{2,k+1} & a^{(i)}_{2,k+2} & \cdots & a^{(i)}_{2,2k+1} \\ & & \ddots & 0 & \vdots & \vdots & \vdots & \vdots \\ & & & a^{(i)} & a^{(i)}_{k,k+1} & a^{(i)}_{k,k+2} & \cdots & a^{(i)}_{k,2k+1} \\ & & & & a^{(i)} & a^{(i)}_{k+1,k+2} & \cdots & a^{(i)}_{k+1,2k+1} \\ & & & & & a^{(i)} & \cdots & 0 \\ & & & & & & a^{(i)} & 0 \\ & & & & & & & a^{(i)} \end{pmatrix}, i = 0, 1, \cdots, s,$$

and

$$B_j = \begin{pmatrix} b^{(j)} & 0 & \cdots & 0 & b^{(j)}_{1,k+1} & b^{(j)}_{1,k+2} & \cdots & b^{(j)}_{1,2k+1} \\ & b^{(j)} & \cdots & 0 & b^{(j)}_{2,k+1} & b^{(j)}_{2,k+2} & \cdots & b^{(j)}_{2,2k+1} \\ & & \ddots & 0 & \vdots & \vdots & \vdots & \vdots \\ & & & b^{(j)} & a^{(j)}_{k,k+1} & b^{(j)}_{k,k+2} & \cdots & b^{(j)}_{k,2k+1} \\ & & & & b^{(j)} & b^{(j)}_{k+1,k+2} & \cdots & b^{(j)}_{k+1,2k+1} \\ & & & & & b^{(j)} & \cdots & 0 \\ & & & & & & b^{(j)} & 0 \\ & & & & & & & b^{(j)} \end{pmatrix}, j = 0, 1, \cdots, t.$$

Denote

$$\begin{aligned} f &= \sum_{i=0}^{s} a^{(i)} x^i, \\ f_{1,k+1} &= \sum_{i=0}^{s} a^{(i)}_{1,k+1} x^i, \cdots, f_{1,2k+1} = \sum_{i=0}^{s} a^{(i)}_{1,2k+1} x^i, \\ &\cdots \\ f_{k,k+1} &= \sum_{i=0}^{s} a^{(i)}_{k,k+1} x^i, \cdots, f_{k,2k+1} = \sum_{i=0}^{s} a^{(i)}_{k,2k+1} x^i, \\ f_{k+1,k+2} &= \sum_{i=0}^{s} a^{(i)}_{k,k+2} x^i, \cdots, f_{k+1,2k+1} = \sum_{i=0}^{s} a^{(i)}_{k+1,2k+1} x^i, \end{aligned}$$

and

$$g = \sum_{j=0}^{t} b^{(j)} x^j,$$

$$g_{1,k+1} = \sum_{j=0}^{t} b^{(j)}_{1,k+1} x^j, \cdots, g_{1,2k+1} = \sum_{j=0}^{t} b^{(j)}_{1,2k+1} x^j,$$

$$\cdots$$

$$g_{k,k+1} = \sum_{j=0}^{t} b^{(j)}_{k,k+1} x^j, \cdots, g_{k,2k+1} = \sum_{j=0}^{t} b^{(j)}_{k,2k+1} x^j,$$

$$g_{k+1,k+2} = \sum_{j=0}^{t} b^{(j)}_{k,k+2} x^j, \cdots, g_{k+1,2k+1} = \sum_{j=0}^{t} b^{(j)}_{k+1,2k+1} x^j.$$

Note that $R[x]$ is reduced when R is reduced. So as in the proof of Theorem 2.1, we obtain that $fg = 0$, $fg_{p,k+1} = 0$, $f_{p,k+1}g = 0$, $fg_{p,k+1+q} = 0$, $f_{p,k+1}g_{k+1,k+1+q} = 0$, $f_{p,k+1+q}g = 0$, $f_{k+1,k+1+q}g = 0$, and $fg_{k+1,k+1+q} = 0$ for each $p, q \in \{1, 2, \cdots, k\}$. Since reduced rings are Armendariz, it follows easily that each coefficient of f annihilators every coefficient of g, each coefficient of f annihilators every coefficient of $g_{p,k+1}$ for $p = 1, 2, \cdots, n$, each coefficient of $f_{p,k+1}$ annihilators every coefficient of g, $p = 1, 2, \cdots, n$, etc. Now it is easy to prove that $A_iB_j = 0$ for any $i = 0, 1, \cdots, s$ and $j = 0, 1, \cdots, t$. Thus $U_n(R)$ is an Armendariz ring for every $n = 2k + 1 \geq 3$.
(2) It is similar to (1). ∎

Corollary 2.3. ([2], Proposition 1.2) *Let R be a reduced ring. Then*

$$R_3 = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} : a, b, c, d \in R \right\}$$

is a semicommutative (and an Armendariz) ring.

Given a ring R and a bimodule ${}_RM_R$, the trivial extension of R by M is the ring $T(R, M) = R \bigoplus M$ with the usual addition and the following multiplication:

$$(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2).$$

This is isomorphic to the ring of all matrices

$$\begin{pmatrix} r & m \\ 0 & r \end{pmatrix},$$

where $r \in R$ and $m \in M$ and the usual matrix operations are used.

Corollary 2.4. *Let R be a reduced ring. Then $T(R, R)$ is a semicommutative (and an Armendariz) ring.*

References

1. G. Shin, Prime ideal and sheaf representation of a pseudo symmetric rings, *Trans. Amer. Math. Soc.* **184** (1973) 43–60.
2. N. K. Kim and Y. Lee, Extensions of reversible rings, *J. Pure Appl. Algebra* **185** (2003) 207–223.
3. C. Huh and Y. Lee, A. Smoktunowicz, Armendariz rings and semicommutative rings, *Comm. Algebra* **30** (2002) 751–761.
4. Z. K. Liu, Semicommutative Subrings of Matrix Rings, *J. Math. Reseach and Exposition* **26** (2006) 264–268.
5. M. B. Rege and S. Chhawchharia, Armendariz rings, *Proc. Japan Acad. Ser. A Math. Sci.* **73** (1997) 14–17.
6. E. P. Armendariz, A note on extensions of Baer and p.p.-rings, *J. Austral. Math. Soc.* **18** (1974) 470–473.
7. N. K. Kim and Y. Lee, Armendariz rings and reduced rings, *J. Algebra* **223** (2000) 477–488.
8. D. D. Anderson and V. Camillo, Armendariz rings and Gaussian rings, *Comm. Algebra* **26** (1998) 2265–2272.
9. T. K. Lee and Y. Q. Zhou, Armendariz and Reduced Rings, *Comm. Algebra* **32** (2004) 2287–2299.