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Weighted Inequalities for Commutators of Singular Integral Operators and Interior Estimates in Morrey Spaces for Solutions of Elliptic Equations

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Abstract. In this paper, the weighted inequalities for the commutators of some singular integral operators on the Morrey spaces $L^{p,\varphi}(\omega)$ are obtained. As an application, it is proved that, for the nondivergence elliptic equations $\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} = f$, if f belongs to $L^{p,\varphi}(\omega)$, then $u_{x_i x_j} \in L^{p,\varphi}(\omega)$, where u is the $W^{2,p}$-solution of the equations.

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1. Introduction and Notations

Throughout this paper, φ will denote a positive, increasing function on R^+ and there exists a constant $D > 0$ such that

$$\varphi(2t) \leqslant D\varphi(t), \quad \text{for all} \quad t \geq 0;$$

Let w be a non-negative weight function on R^n and Ω be an open set in R^n or $\Omega = R^n$. Suppose that f is a locally integrable function on R^n, we define, for $1 \leqslant p < \infty$,

$$||f||_{L^{p,\varphi}(\Omega,w)} = \sup_{\substack{x\in\Omega \\ r>0}} \Big(\frac{1}{\varphi(r)} \int_{B(x,r)} |f(y)|^p w(y) dy\Big)^{1/p},$$

where $B(x,r) = \{y \in R^n : |x-y| < r\}$. The weighted generalized Morrey spaces is defined by

$$L^{p,\varphi}(\Omega, w) = \{f \in L^1_{loc}(R^n) : ||f||_{L^{p,\varphi}(\Omega,w)} < \infty\};$$

If $\varphi(r) = r^\delta$, $\delta \geq 0$, $L^{p,\varphi} = L^{p,\delta}$, it is the classical Morrey spaces (see [1]).

The purpose of this paper is twofold: first, we establish the weighted inequalities of the commutators of some singular integral operators on generalized Morrey spaces, and second, we study the interior estimates in generalized Morrey spaces for the solutions of some nondivergence elliptic equations $\sum_{i,j=1}^n a_{ij}u_{x_ix_j} = f$, we prove if $f \in L^{p,\varphi}$, then $u_{x_ix_j} \in L^{p,\varphi}$, where u is the $W^{2,p}$-solution of the equation. The results extend the ones in [2, 4, 5].

First, let us introduce some notations. For a ball B and a locally integrable function f, let $f(B) = \int_B f(x)dx$ and $f_B = |B|^{-1}\int_B f(x)dx$. Let $M(f)$ be the Hardy-Littlewood maximal operator, we define, for $1 \leqslant p < \infty$,

$$M_p(f) = (M|f|^p)^{1/p} \text{ and } f^{\#}(x) = \sup_{B:\ x\in B} \frac{1}{|B|}\int_B |f(y) - f_B|dy.$$

Let Ω be an open set in R^n or $\Omega = R^n$, we define

$$||f||_{BMO} = \sup_B \frac{1}{|B|}\int_B |f(y) - f_B|dy$$

and

$$\delta(r) = \sup_{B\subset\Omega} \frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y) - f_B|dy.$$

The BMO and VMO spaces are defined by (see [6])

$$BMO(\Omega) = \{f \in L^1_{loc}(R^n) : ||f||_{BMO} < \infty\}$$

and

$$VMO(\Omega) = \left\{f \in L^1_{loc}(R^n) : \lim_{r\to 0}\delta(r) = 0\right\}.$$

The A_p weight is defined by (see [6])$(1 < p < \infty)$

$$A_p = \left\{0 < w \in L^1_{loc}(R^n) : \sup_B \Big(\frac{1}{|B|}\int_B w(x)dx\Big)\Big(\frac{1}{|B|}\int_B w(x)^{-1/(p-1)}dx\Big)^{p-1} < \infty\right\}$$

and

$$A_1 = \{0 < w \in L^1_{loc}(R^n) : M(w)(x) \leqslant Cw(x), a.e.\}.$$

Now, we define the singular integral operator and its commutator.

Definition. *Let T be a linear operator mapping the functions on R^n into ones on R^n, and*
(1) *T is bounded on $L^2(R^n)$;*
(2) *There exists a function K on $R^n \times R^n \setminus \{(x,x) : x \in R^n\}$ such that*

$$\int_B |K(x,y)|dy < \infty, \quad x \notin 2B$$

and for $f \in L^\infty(R^n)$ with supp$f \subseteq B = B(x', r)$, the following holds:

$$T(f)(x) = \int_{R^n} K(x, y) f(y) dy,$$

moreover, when $r > 0$, we have

$$\Big(\int_{r<|x-y|\leqslant 2r} |K(x, y) - K(x', y)|^{p_0'} dy\Big)^{1/p_0'} \leqslant c\theta(|x - x'|/r)|B|^{-1/p_0},$$

where $1 \leqslant p_0 < \infty$, $1/p_0 + 1/p_0' = 1$ and θ is an increasing function on R^+ and $\int_0^1 \theta(t)t^{-1}dt < \infty$. Then T is called the generalized singular integral operator.

When $\theta(t) = t^\delta$, $0 < \delta \leqslant 1$, T is non-convolution Calderon-Zygmund operator (see [6]). It has been proved that (see [8, 9]) T is weak type (1,1) and strong type (p, p) about weight $w \in A_p$ for $1 < p < \infty$.

Let T be the generalized singular integral operator and b be a locally integrable function, the commutator is defined by

$$[b, T](f) = bT(f) - T(bf).$$

2. Weighted Inequalities of Commutator

First, we prove the weighted inequalities of the generalized singular integral operators and maximal operator on Morrey spaces.

Theorem 1. *Let $1 < p < \infty$, $0 < D < 2^n$, $w \in A_1$, suppose that T and M are the generalized singular integral operator and maximal operator, respectively. Then, for $f \in L^{p,\varphi}(R^n, w)$,*
(a) $||T(f)||_{L^{p,\varphi}(R^n,w)} \leqslant C||f||_{L^{p,\varphi}(R^n,w)}$;
(b) $||M(f)||_{L^{p,\varphi}(R^n,w)} \leqslant C||f||_{L^{p,\varphi}(R^n,w)}$.

Proof. (a) Let $f \in L^{p,\varphi}(R^n, w)$, for a ball $B = B(x, r) \subset R^n$, since $M(w\chi_B) \in A_1$ (see [3]) and by the (p, p)-boundedness of T (see [8, 9]), we have

$$\begin{aligned}
&\int_{B(x,r)} |T(f)(y)|^p w(y)dy \leqslant \int_{R^n} |T(f)(y)|^p M(w\chi_B)(y)dy \\
&\leqslant C \int_{R^n} |f(y)|^p M(w\chi_B)(y)dy \\
&= C\Big[\int_{B(x,r)} |f(y)|^p M(w\chi_B)(y)dy + \sum_{k=0}^{\infty} \int_{2^{k+1}B\setminus 2^k B} |f(y)|^p M(w\chi_B)(y)dy\Big] \\
&\leqslant C\Big[\int_B |f(y)|^p w(y)dy + \sum_{k=0}^{\infty} \int_{2^{k+1}B\setminus 2^k B} |f(y)|^p \frac{w(B)}{|2^{k+1}B|} dy\Big] \\
&\leqslant C\Big[\int_B |f(y)|^p w(y)dy + \sum_{k=0}^{\infty} \int_{2^{k+1}B} |f(y)|^p \frac{Mw(y)}{2^{(k+1)n}} dy\Big]
\end{aligned}$$

$$\leqslant C\Big[\int_B |f(y)|^p w(y)dy + \sum_{k=0}^{\infty}\int_{2^{k+1}B}|f(y)|^p \frac{w(y)}{2^{kn}}dy\Big]$$
$$\leqslant C||f||^p_{L^{p,\varphi}(R^n,w)}\sum_{k=0}^{\infty}2^{-nk}\varphi(2^{k+1}r)$$
$$\leqslant C||f||^p_{L^{p,\varphi}(R^n,w)}\sum_{k=0}^{\infty}(2^{-n}D)^k\varphi(r)$$
$$\leqslant C||f||^p_{L^{p,\varphi}(R^n,w)}\varphi(r),$$

thus,

$$||T(f)||_{L^{p,\varphi}(R^n,w)} \leqslant C||f||_{L^{p,\varphi}(R^n,w)}.$$

The proof of (b) is similar to that of (a), we omit the details.

Now, we prove one of the main results in this paper.

Theorem 2. *Let $1 < p < \infty$, $0 < D < 2^n$, $w \in A_1$, $b \in BMO(R^n)$, suppose that T is the generalized singular integral operator, then, for $f \in L^{p,\varphi}(R^n, w)$,*

$$||[b,T](f)||_{L^{p,\varphi}(R^n,w)} \leqslant C||b||_{BMO}||f||_{L^{p,\varphi}(R^n,w)}.$$

Proof. First, we prove that, for $1 < t, s < \infty$,

$$([b,T](f))^{\#} \leqslant C||b||_{BMO}(M_s(Tf) + M_t(f)), \quad a.e..$$

In fact, choose a ball $B = B(x_0, r) \subset R^n$, we write

$$[b,T](f) = [b - b_B, T](f) = (b - b_B)T(f) - T((b - b_B)f\chi_{2B}) - T((b - b_B)f\chi_{(2B)^c})$$
$$= I + II + III.$$

By Hölder's inequality, we have

$$\frac{1}{|B|}\int_B |I|dx \leqslant \Big(\frac{1}{|B|}\int_B |b(x) - b_B|^{s'}dx\Big)^{1/s'}\Big(\frac{1}{|B|}\int_B |T(f)(x)|^s dx\Big)^{1/s}$$
$$\leqslant C||b||_{BMO}M_s(Tf).$$

By the L^q-boundedness of T (see [8],[9]) for $1 < q < \infty$, taking $k \in (1, \infty)$ such that $qk = t$, we have

$$\frac{1}{|B|}\int_B |II|dx \leqslant \Big(\frac{1}{|B|}\int_B |T((b - b_B)f\chi_{2B})(x)|^q dx\Big)^{1/q}$$
$$\leqslant C\Big(\frac{1}{|B|}\int_{2B}|(b(x) - b_B)f(y)|^q dx\Big)^{1/q}$$
$$\leqslant C\Big(\frac{1}{|B|}\int_{2B}|b(x) - b_B|^{qk'}dx\Big)^{1/qk'}\Big(\frac{1}{|B|}\int_B |f(x)|^{qk}dx\Big)^{1/qk}$$
$$\leqslant C||b||_{BMO}M_t(f);$$

For III, taking $l>1$ such that $p_0l=t$, we get

$$\begin{aligned}
&|T((b-b_B)f\chi_{(2B)^c})(x)-T((b-b_B)f\chi_{(2B)})(x_0)|\\
\leqslant&\int_{(2B)^c}|K(x,y)-K(x_0,y)||b(y)-b_B||f(y)|dy\\
=&\sum_{k=1}^{\infty}\int_{2^{k+1}B\setminus 2^kB}|K(x,y)-K(x_0,y)||b(y)-b_B||f(y)|dy\\
\leqslant&\sum_{k=1}^{\infty}\left(\int_{2^{k+1}B\setminus 2^kB}|K(x,y)-K(x_0,y)|^{p_0'}dy\right)^{1/p_0'}\\
&\times\left(\int_{2^{k+1}B}|b(y)-b_B|^{p_0}|f(y)|^{p_0}dy\right)^{1/p_0}\\
\leqslant&C\sum_{k=1}^{\infty}\theta(2^{-k})|2^{k+1}B|^{-1/p_0}\left(\int_{2^{k+1}B}|b(y)-b_B|^{p_0l'}dy\right)^{1/p_0l'}\\
&\times\left(\int_{2^{k+1}B}|f(y)|^{p_0l}dy\right)^{1/p_0l}\\
\leqslant&C\sum_{k=1}^{\infty}\theta(2^{-k})\left(\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}|b(y)-b_B|^{p_0l'}dy\right)^{1/p_0l'}\\
&\times\left(\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}|f(y)|^{p_0l}dy\right)^{1/p_0l}\\
\leqslant&C||b||_{BMO}M_t(f);
\end{aligned}$$

by the equivalence of $\sum_{k=1}^{\infty}\theta(2^{-k})$ and $\int_0^1\theta(t)t^{-1}dt$. Thus

$$\frac{1}{|B|}\int_B|III-T((b-b_B)f\chi_{(2B)^c})(x_0)|dx\leqslant C||b||_{BMO}M_t(f),$$

so that, we have

$$([b,T](f))^{\#}\leqslant C||b||_{BMO}(M_s(Tf)+M_t(f)).$$

Now, by the following inequality (see [6]): for $w\in A_p$,

$$\int_{R^n}(M(f)(x))^pw(x)dx\leqslant C\int_{R^n}|f^{\#}(x)|^pw(x)dx,$$

similarly by to the proof of Theorem 1, we get

$$||Mf||_{L^{p,\varphi}(R^n,w)}\leqslant C||f^{\#}||_{L^{p,\varphi}(R^n,w)}.$$

Next, taking $1<s,\ t<p$, by the boundedness of T and M on $L^{p,\varphi}(R^n,w)$ for $1<p<\infty$, we gain

$$\begin{aligned}
&||[b,T](f)||_{L^{p,\varphi}(R^n,w)}\leqslant C||M([b,T])f||_{L^{p,\varphi}(R^n,w)}\\
\leqslant&C||([b,T](f))^{\#}||_{L^{p,\varphi}(R^n,w)}\\
\leqslant&C||b||_{BMO}||M_s(Tf)+M_t(f)||_{L^{p,\varphi}(R^n,w)}\\
\leqslant&C||b||_{BMO}(||Tf||_{L^{p,\varphi}(R^n,w)}+||M_t(f)||_{L^{p,\varphi}(R^n,w)})\\
\leqslant&C||b||_{BMO}||f||_{L^{p,\varphi}(R^n,w)}.
\end{aligned}$$

This completes the proof of Theorem 2. ∎

Corollary. *Let T be the generalized singular integral operator, if $1 < p < \infty$, $w \in A_1$, $b \in VMO(R^n)$ and $0 < D < 2^n$. Then, for any $\varepsilon > 0$, there exists $r_0 > 0$, such that for any ball $B = B(x,r)$ in R^n, when $0 < r < r_0$ and $f \in L^{p,\varphi}(B,w)$, the following holds:*

$$||[b,T]f||_{L^{p,\varphi}(B,w)} \leqslant C\varepsilon ||f||_{L^{p,\varphi}(B,w)}.$$

Proof. Choose r_0 such that $\delta(r_0) < \varepsilon$, then the conclusion follows from Theorem 2.

3. Interior Estimate of Elliptic Equation

In this section, we will use Theorem 2 and Corollary in Sec. 2 to study the interior estimate of some elliptic equation.

Suppose $n > 2$ and Ω is an open set in R^n, let

$$L = \sum_{i,j=1}^{n} a_{ij}(x)(\partial^2/\partial x_i \partial x_j),$$

where $a_{ij} = a_{ji}$ for $i,j = 1,2,\cdots,n$, a.e. in Ω, and assume there exists $C > 0$ such that, for $y = (y_1,\cdots,y_n) \in R^n$,

$$C^{-1}|y^2| \leqslant \sum_{i,j=1}^{n} a_{ij}(x) y_i y_j \leqslant C|y|^2, \quad a.e. \text{ for } x \in \Omega;$$

denoting by $(A_{ij})_{n\times n}$ the inverse of the matrix $(a_{ij})_{n\times n}$, for $x \in \Omega$ and $y \in R^n$, let

$$K(x,y) = [(n-2)C_n(det(a_{ij}(x)))^{1/2}]^{-1} \left(\sum_{i,j=1}^{n} A_{ij}(x) y_i y_j \right)^{1-n/2}$$

and

$$K_i(x,y) = \frac{\partial}{\partial y_i} K(x,y), \qquad K_{ij}(x,y) = \frac{\partial^2}{\partial x_i \partial x_j} K(x,y).$$

From [2, 5], we have the interior representation formula, if $u \in W_0^{2,p}$ (see [7]),

$$\begin{aligned} u_{x_i x_j}(x) =& P.V. \int_B K_{ij}(x,x-y) \left[\sum_{k,l=1}^{n} (a_{kl}(x) - a_{kl}(y)) u_{x_k x_l}(y) + Lu(y) \right] dy \\ &+ Lu(x) \int_{|y|=1} K_i(x,y) y_j d\delta_y, \quad \text{a.e. for } x \in B \subset \Omega, \end{aligned}$$

where B is a ball in Ω. We also set

$$M = \max_{i,j=1,\cdots,n} \max_{|\alpha|\leqslant 2n} ||\partial^\alpha K_{ij}(x,y)/\partial y^\alpha||_{L^\infty} < \infty.$$

Theorem 3. *Let Ω be an open set in R^n, $w \in A_1$, $1 < p < \infty$, $0 < D < 2^n$, $a_{ij} \in VMO(\Omega)$, $i,j = 1,2,\cdots,n$. Then there exists a positive constant C such that for all balls $B \subset \Omega$ and $u \in W_0^{2,p}$, we have $u_{x_ix_j} \in L^{p,\varphi}(B,w)$ and*

$$||u_{x_ix_j}||_{L^{p,\varphi}(B,w)} \leqslant C||Lu||_{L^{p,\varphi}(B,w)}.$$

Proof. It is easy to verify that K_{ij} satisfy the condition of Corollary by the representation of $u_{x_ix_j}$ and the conditions of K_{ij}, thus, from Corollary, we have, for any $\varepsilon > 0$,

$$||u_{x_ix_j}||_{L^{p,\varphi}(B,w)} \leqslant C\varepsilon||u_{x_ix_j}||_{L^{p,\varphi}(B,w)} + C||Lu||_{L^{p,\varphi}(B,w)},$$

choosing ε small enough (e.g. $C\varepsilon < 1$), we get

$$||u_{x_ix_j}||_{L^{p,\varphi}(B,w)} \leqslant (C/(1-C\varepsilon))||Lu||_{L^{p,\varphi}(B,w)}.$$

This finishes the proof of Theorem 3.

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