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A Gagliardo–Nirenberg Inequality for Orlicz and Lorentz Spaces on $\mathbb{R}^n_+$ *

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Dedicated to Professor Hoang Tuy on the occasion of his 80th-birthday

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Abstract. In this paper, essentially developing the method of [1 – 4, 15], we give an extension of the Gagliardo-Nirenberg inequality to Orlicz and Lorentz spaces defined on $\mathbb{R}^n_+$.

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Let $\ell \geq 2$ and $b \geq 0$. Denote by $\mathbb{R}^n_{+,b} = \{x \in \mathbb{R}^n : x_j > b, j = 1, ..., n\}$, $\mathbb{R}^n_{+,0} = \mathbb{R}^n_+$ and $W^{\ell,\infty}(\mathbb{R}^n_{+,b})$ the set of all measurable on $\mathbb{R}^n_{+,b}$ functions f such that f and its generalized derivatives $D^\beta f$, $0 < |\beta| \leqslant \ell$, belong to $L_\infty(\mathbb{R}^n_{+,b})$. The following Gagliardo–Nirenberg theorem is well-known [10]: *Let $b \geq 0$. For fixed α, $0 < |\alpha| < \ell$, there is the best constant $C^+_{\alpha,\ell}$ not depending on b such that for any $f \in W^{\ell,\infty}(\mathbb{R}^n_{+,b})$,*

$$\|D^\alpha f\|_{\infty,b} \leqslant C^+_{\alpha,\ell} \|f\|_{\infty,b}^{1-\frac{|\alpha|}{\ell}} \Big(\sum_{|\beta|=\ell} \|D^\beta f\|_{\infty,b} \Big)^{\frac{|\alpha|}{\ell}},$$

where $\|\cdot\|_{\infty,b}$ is the norm of $L_\infty(\mathbb{R}^n_{+,b})$. By developing the methods of [1 - 4, 15], we extend the above Gagliardo–Nirenberg inequality to Orlicz spaces $L_\Phi(\mathbb{R}^n_+)$ and Lorentz spaces $N_\Psi(\mathbb{R}^n_+)$. The Gagliardo–Nirenberg inequality [7, 10]

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has applications to Partial differential equations and Interpolation theory. Note that the inequality was already proved in [4, 15] for the case $\mathbb{R}^n$, but it is more difficult for the case $\mathbb{R}^n_+$.

1. A Gagliardo–Nirenberg Inequality for Orlicz Space $L_\Phi(\mathbb{R}^n_+)$

Let G be a domain in $\mathbb{R}^n$, $\Phi : [0, +\infty) \to [0, +\infty]$ an arbitrary Young function (see [8, 11, 12]), i.e., $\Phi(0) = 0, \Phi(t) \geq 0, \Phi(t) \not\equiv 0$, and assume that Φ is convex. Denote by

$$\overline{\Phi}(t) = \sup_{s\geq 0} \big\{ts - \Phi(s)\big\}$$

the Young function conjugate to Φ, and by $L_\Phi(G)$ the space of measurable functions u such that

$$| \langle u, v\rangle | = | \int_G u(x)v(x)dx | < \infty$$

for all v with $\rho(v, \overline{\Phi}, G) < \infty$, where

$$\rho(v, \overline{\Phi}, G) = \int_G \overline{\Phi}(| v(x) |)dx.$$

Then $L_\Phi(G)$ is a Banach space with respect to the Orlicz norm

$$\| u \|_{\Phi,G} = \sup_{\rho(v,\overline{\Phi},G)\leqslant 1} | \int_G u(x)v(x)dx | \ ,$$

which is equivalent to the Luxemburg norm

$$||f||_{(\Phi,G)} = \inf\{\lambda > 0 : \int_G \Phi(|f(x)|/\lambda)dx \leqslant 1\} < \infty.$$

Recall that $|| \cdot ||_{(\Phi,G)} = || \cdot ||_{L_p(G)}$ where $\Phi(t) = t^p$ with $1 \leqslant p < \infty$, and $|| \cdot ||_{(\Phi,G)} = || \cdot ||_{L_\infty(G)}$ when $\Phi(t) = 0$ for $0 \leqslant t \leqslant 1$ and $\Phi(t) = \infty$ for $t > 1$. We have the following results (cf. [11, 12]):

Lemma 1. *Let $u \in L_\Phi(G)$ and $v \in L_{\overline{\Phi}}(G)$. Then*

$$\int_G |u(x)v(x)|dx \leqslant \|u\|_{\Phi,G}\|v\|_{(\overline{\Phi},G)} \ .$$

Lemma 2. *Let $u \in L_\Phi(\mathbb{R}^n)$ and $v \in L_1(\mathbb{R}^n)$. Then*

$$\|u * v\|_{\Phi,\mathbb{R}^n} \leqslant \|u\|_{\Phi,\mathbb{R}^n}\|v\|_1.$$

We have the following theorem.

Theorem 3. *Let $\ell \geq 2$ and let Φ be an arbitrary Young function, f and its generalized derivatives $D^\beta f$, $|\beta| = \ell$, be in $L_\Phi(\mathbb{R}^n_+)$. Then $D^\alpha f \in L_\Phi(\mathbb{R}^n_+)$ for* *all* α, $0<|\alpha|<\ell$ *and*

$$\|D^\alpha f\|_{\Phi,\mathbb{R}^n_+} \leqslant C^+_{\alpha,\ell}\|f\|^{1-\frac{|\alpha|}{\ell}}_{\Phi,\mathbb{R}^n_+}\Big(\sum_{|\beta|=\ell}\|D^\beta f\|_{\Phi,\mathbb{R}^n_+}\Big)^{\frac{|\alpha|}{\ell}}, \tag{1}$$

where the constant $C^+_{\alpha,\ell}$ *is defined as in the Gagliardo–Nirenberg inequality.*

Proof. Step 1. We begin to prove (1) with the assumption that $D^\alpha f \in L_\Phi(\mathbb{R}^n_+)$ for all $0\le|\alpha|\le\ell$. Fix α, $0<|\alpha|<\ell$. Let $\epsilon>0$ be given. We choose a function $v_\epsilon \in L_{\overline{\Phi}}(\mathbb{R}^n_+)$, $\rho(v_\epsilon,\overline{\Phi},\mathbb{R}^n_+)\leqslant 1$ such that

$$\Big|\int_{\mathbb{R}^n_+} D^\alpha f(x)v_\epsilon(x)dx\Big| \ge \|D^\alpha f\|_{\Phi,\mathbb{R}^n_+}-\varepsilon. \tag{2}$$

Put

$$F_\varepsilon(x)=\int_{\mathbb{R}^n_+} f(x+y)v_\epsilon(y)dy. \tag{3}$$

Then $F_\epsilon(x)\in L_\infty(\mathbb{R}^n_+)$ by virtue of Lemma 1, and it is easy to verify that

$$D^\beta F_\varepsilon(x)=\int_{\mathbb{R}^n_+} D^\beta f(x+y)v_\epsilon(y)dy, \quad 0\leqslant|\beta|\leqslant\ell,$$

in the distribution sense $\mathcal{D}'(\mathbb{R}^n_+)$. Since $\rho(v_\epsilon,\overline{\Phi},\mathbb{R}^n_+)\leqslant 1$, $\|v_\epsilon\|_{(\overline{\Phi},\mathbb{R}^n_+)}\leqslant 1$. So, for all $x\in\mathbb{R}^n_+$ and $0\leqslant|\beta|\leqslant\ell$, clearly,

$$|D^\beta F_\varepsilon(x)| \leqslant \|D^\beta f(x+\cdot)\|_{\Phi,\mathbb{R}^n_+}\|v_\epsilon\|_{(\overline{\Phi},\mathbb{R}^n_+)} \leqslant \|D^\beta f\|_{\Phi,\mathbb{R}^n_+}. \tag{4}$$

Now we prove the continuity of $D^\beta F_\varepsilon(x)$. We show this for $\beta=0$ by contradiction: Assume that for some $\delta>0$, a point $x_0\in\mathbb{R}^n_+$ and a sequence $\{t_m\}\subset\mathbb{R}^n : t_m\to 0$

$$\Big|\int_{\mathbb{R}^n_+}[f(x_0+t_m+y)-f(x_0+y)]v_\epsilon(y)dy\Big| \ge \delta\ ,\ m\ge 1. \tag{5}$$

Since $f\in L_\Phi(\mathbb{R}^n_+)$ we easily get $f\in L_{1,loc}(\mathbb{R}^n_+)$. Then for any $j=1,2,\dots$, $f(t_m+\cdot)\to f(\cdot)$ in $L_1([1/j,j]^n)$. Therefore, there exists a subsequence, denoted again by $\{t_m\}$, such that $f(t_m+y)\to f(y)$ a.e. in $[1/j,j]^n$. So, there exists a subsequence (for simplicity of notation we assume that it coincides with $\{t_m\}$) such that $f(x_0+t_m+y)\to f(x_0+y)$ a.e. in $\mathbb{R}^n_+$. For simplicity of notations we consider only the case $x_0=0$. Because inequality (1) holds for f if and only if it holds for f/C, where C is an arbitrary positive number, without loss of generality we may assume that $\rho(2f,\Phi,\mathbb{R}^n_+)<\infty$. As in [4], we have

$$|f(t_m+y)-f(y)||v_\epsilon(y)| \leqslant \frac{1}{2}\Phi(2|f(t_m+y)|)+\frac{1}{2}\Phi(2|f(y)|)+\overline{\Phi}(|v_\epsilon(y)|). \tag{6}$$

Since $\Phi(2|f|)$, $\overline{\Phi}(|v_\epsilon|)\in L_1(\mathbb{R}^n_+)$ and $t_m\to 0$, there are positive numbers M and h such that for all $m\ge 1$

$$\int_{\{|y|>M\}\cap\mathbb{R}^n_+}\Big(\Phi\big(2|f(y)|\big)+\Phi\big(2|f(t_m+y)|\big)+\overline{\Phi}\big(|v_\epsilon(y)|\big)\Big)dy < \frac{\delta}{2} \tag{7}$$

and

$$\begin{aligned}\int_G \Phi\big(2|f(y)|\big)dy &< \frac{\delta}{6} \quad \int_G \Phi\big(2|f(t_m+y)|\big)dy \\ &< \frac{\delta}{6} \quad \int_G \overline{\Phi}\big(|v_\epsilon(y)|\big)dy < \frac{\delta}{6}\end{aligned} \tag{8}$$

if $G \subset \mathbb{R}^n_+$, $\operatorname{mes}(G) < h$. On the other hand, by Egorov theorem, there is a set $A \subset \mathcal{B}_+(0, M)$, with $\operatorname{mes}(A) < h$, such that $f(t_m + y)v_\epsilon(y)$ uniformly converges to $f(y)v_\epsilon(y)$ on $\mathcal{B}_+(0, M) \setminus A$, where $\mathcal{B}_+(0, M)$ is the intersection of the ball of radius M centered at zero with $\mathbb{R}^n_+$. Therefore, applying (6) and (8), we have as in [4]

$$\overline{\lim_{m\to\infty}} \int_{\{|y|\leqslant M\}\cap\mathbb{R}^n_+} \big|f(t_m + y) - f(y)\big||v_\epsilon(y)|dy \leqslant \frac{\delta}{12} + \frac{\delta}{12} + \frac{\delta}{6} = \frac{\delta}{3}\cdot \tag{9}$$

Combining (7), (9) and using (6), we get for sufficiently large m

$$\int_{\mathbb{R}^n_+} \big|\big(f(t_m + y) - f(y)\big)v_\epsilon(y)\big|dy < \delta,$$

which contradicts (5). The cases $1 \leqslant |\beta| \leqslant \ell$ are proved similarly. The continuity of $D^\alpha F_\varepsilon$, $0 \leqslant |\beta| \leqslant \ell$ has been proved. The functions $D^\beta F_\varepsilon$, $0 \leqslant |\beta| \leqslant \ell$ are continuous and bounded on $\mathbb{R}^n_+$. Therefore, it follows from the Gagliardo–Nirenberg inequality and (2)–(3) that

$$\begin{aligned}\big(\|D^\alpha f\|_{\Phi,\mathbb{R}^n_+} - \epsilon\big) &\leq |D^\alpha F_\varepsilon(0)| \leq \|D^\alpha F_\varepsilon\|_{\infty,0} \\ &\leq C^+_{\alpha,\ell}\|F_\varepsilon\|^{1-\frac{|\alpha|}{\ell}}_{\infty,0}\Big(\sum_{|\beta|=\ell}\|D^\beta F_\varepsilon\|_{\infty,0}\Big)^{\frac{|\alpha|}{\ell}},\end{aligned}$$

which together with (4) implies

$$||D^\alpha f||_{\Phi,\mathbb{R}^n_+} - \epsilon \leqslant C^+_{\alpha,\ell}\|f\|^{1-\frac{|\alpha|}{\ell}}_{\Phi,\mathbb{R}^n_+}\Big(\sum_{|\beta|=\ell}\|D^\beta f\|_{\Phi,\mathbb{R}^n_+}\Big)^{\frac{|\alpha|}{\ell}}.$$

By letting $\epsilon \to 0$ we have (1).

Step 2. To complete the proof, it remains to show that $D^\alpha f \in L_\Phi(\mathbb{R}^n_+)$, $\forall\, \alpha$: $0 < |\alpha| < \ell$ if $f, D^\beta f \in L_\Phi(\mathbb{R}^n_+)$, $|\beta| = \ell$. Since f, $D^\beta f \in L_{1,loc}(\mathbb{R}^n_+)$, $|\beta| = \ell$, we get $D^\alpha f \in L_{1,loc}(\mathbb{R}^n_+)$, $0 < |\alpha| < \ell$ (see [9, p. 7]). We define for $0 \leqslant |\alpha| \leqslant \ell$,

$$f_{(\alpha)}(x) = \begin{cases} D^\alpha f(x), & x \in \mathbb{R}^n_+ \\ 0, & x \in \mathbb{R}^n \setminus \mathbb{R}^n_+. \end{cases}$$

Let $\psi(x) \in C^\infty_0(\mathbb{R}^n)$, $\psi(x) \geq 0$, $\operatorname{supp}\psi \subset \{x \in \mathbb{R}^n : 0 \leqslant x_j \leqslant 1, j = 1, 2, \ldots, n\}$ and $\int_{\mathbb{R}^n} \psi(x)dx = 1$. We put $\psi_\lambda(x) = \dfrac{1}{\lambda^n}\psi\big(\dfrac{x}{\lambda}\big)$, $\lambda > 0$ and $f_\lambda = f_{(0)} * \psi_\lambda$. Fix

$b > 0$. Then for all $\varphi \in C_0^\infty(\mathbb{R}^n_{+,b})$ we have for $0 < \lambda < b, 0 \leqslant |\alpha| \leqslant \ell$:

$$
\begin{aligned}
&< D^\alpha f_\lambda, \varphi >= (-1)^{|\alpha|} < f_\lambda, D^\alpha \varphi > \\
&= (-1)^{|\alpha|} \int_{\mathbb{R}^n_+} \left(\int_{\mathbb{R}^n_+} f_{(0)}(x-y)\psi_\lambda(y)dy \right) D^\alpha \varphi(x)dx \\
&= (-1)^{|\alpha|} \int_{\mathcal{B}_+(0,\lambda)} \left(\int_{\mathbb{R}^n_{+,b}} f_{(0)}(x-y) D^\alpha \varphi(x)dx \right) \psi_\lambda(y)dy \\
&= \int_{\mathcal{B}_+(0,\lambda)} \left(\int_{\mathbb{R}^n_{+,b}} D^\alpha f(x-y)\varphi(x)dx \right) \psi_\lambda(y)dy \\
&= \int_{\mathbb{R}^n_{+,b}} \left(\int_{\mathcal{B}_+(0,\lambda)} D^\alpha f(x-y)\psi_\lambda(y)dy \right) \varphi(x)dx \\
&= \int_{\mathbb{R}^n_{+,b}} \big(f_{(\alpha)} * \psi_\lambda \big)(x)\varphi(x)dx \\
&= \ \langle f_{(\alpha)} * \psi_\lambda, \ \varphi \rangle.
\end{aligned}
$$

So, we have proved for $0 < \lambda < b$ and $0 \leqslant |\alpha| \leqslant \ell$

$$
D^\alpha f_\lambda = f_{(\alpha)} * \psi_\lambda \tag{10}
$$

in the $\mathcal{D}'(\mathbb{R}^n_{+,b})$ sense. Therefore, for $0 < \lambda < b$ and $\alpha = 0$ or $|\alpha| = \ell$ we have

$$
\begin{aligned}
\|D^\alpha \big(f_{(0)} * \psi_\lambda \big)\|_{\Phi,\mathbb{R}^n_{+,b}} &= \|f_{(\alpha)} * \psi_\lambda\|_{\Phi,\mathbb{R}^n_{+,b}} \\
&\leqslant \|f_{(\alpha)} * \psi_\lambda\|_{\Phi,\mathbb{R}^n} \\
&\leqslant \|f_{(\alpha)}\|_{\Phi,\mathbb{R}^n} \\
&= \|f_{(\alpha)}\|_{\Phi,\mathbb{R}^n_+} \\
&= \|D^\alpha f\|_{\Phi,\mathbb{R}^n_+}.
\end{aligned} \tag{11}
$$

On the other hand, by using $D^\alpha \big(f_{(0)} * \psi_\lambda \big) = f_{(0)} * D^\alpha \psi_\lambda \in L_\Phi(\mathbb{R}^n)$, $\forall\, 0 \leqslant |\alpha| \leqslant \ell$ and the inequality proved in Step 1 for functions on $\mathbb{R}^n_{+,b}$, we get for $0 < |\alpha| < \ell$,

$$
||D^\alpha f_\lambda||_{\Phi,\mathbb{R}^n_{+,b}} \leqslant C^+_{\alpha,\ell} \|f_\lambda\|^{1-\frac{|\alpha|}{\ell}}_{\Phi,\mathbb{R}^n_{+,b}} \Big(\sum_{|\beta|=\ell} \|D^\beta f_\lambda\|_{\Phi,\mathbb{R}^n_{+,b}} \Big)^{\frac{|\alpha|}{\ell}}.
$$

Hence, by combining (10), (11) we obtain for all $0 < \lambda < b$, $0 < |\alpha| < \ell$,

$$
\begin{aligned}
||D^\alpha f_\lambda||_{\Phi,\mathbb{R}^n_{+,b}} &\leqslant C^+_{\alpha,\ell} \|f_\lambda\|^{1-\frac{|\alpha|}{\ell}}_{\Phi,\mathbb{R}^n_{+,b}} \Big(\sum_{|\beta|=\ell} \|D^\beta f_\lambda\|_{\Phi,\mathbb{R}^n_{+,b}} \Big)^{\frac{|\alpha|}{\ell}} \\
&\leqslant C^+_{\alpha,\ell} \|f_\lambda\|^{1-\frac{|\alpha|}{\ell}}_{\Phi,\mathbb{R}^n} \Big(\sum_{|\beta|=\ell} \|D^\beta f_\lambda\|_{\Phi,\mathbb{R}^n} \Big)^{\frac{|\alpha|}{\ell}} \\
&\leqslant C^+_{\alpha,\ell} \|f\|^{1-\frac{|\alpha|}{\ell}}_{\Phi,\mathbb{R}^n_+} \Big(\sum_{|\beta|=\ell} \|D^\beta f\|_{\Phi,\mathbb{R}^n_+} \Big)^{\frac{|\alpha|}{\ell}}.
\end{aligned} \tag{12}
$$

Fix α $(0 < |\alpha| < \ell)$. For $j = 1$, since $D^\alpha f \in L_{1,loc}(\mathbb{R}^n_+)$, there is a sequence of positive numbers $\{\lambda^1_m\}, \lambda^1_m \to 0$ such that

$$\lim_{m\to\infty} D^\alpha f_{\lambda^1_m}(x) = D^\alpha f(x) \ \text{ a.e. in } \mathbb{R}^n_{+,1}.$$

For $j = 2$, since $\lambda^1_m \to 0$, there exists a subsequence $\{\lambda^2_m\}$ of $\{\lambda^1_m\}$ such that

$$\lim_{m\to\infty} D^\alpha f_{\lambda^2_m}(x) = D^\alpha f(x) \ \text{ a.e. in } \mathbb{R}^n_{+,1/2}.$$

By repeating this argument for $j = 3, 4, \dots$ and by the diagonal process, we get a sequence of positive numbers $\{\lambda^*_m\} : \lambda^*_m \to 0$ such that

$$\lim_{m\to\infty} D^\alpha f_{\lambda^*_m}(x) = D^\alpha f(x) \ \text{ a.e. in } \mathbb{R}^n_+.$$

Hence,

$$\lim_{m\to\infty} f_{(\alpha)} * \psi_{\lambda^*_m}(x) = f_{(\alpha)}(x) = D^\alpha f(x) \ \text{ a.e. in } \mathbb{R}^n_+. \tag{13}$$

For each function $v \in L_{\overline{\Phi}}(\mathbb{R}^n_+)$, $\rho(v, \overline{\Phi}, \mathbb{R}^n_+) \leqslant 1$ and $m \geq 1$, by (12) - (13) and the definition of the Orlicz norm we get

$$\int_{\mathbb{R}^n_+} \big|(D^\alpha f_{\lambda^*_m})(x)v(x)\big|dx \leqslant C^+_{\alpha,\ell}\|f\|^{1-\frac{|\alpha|}{\ell}}_{\Phi,\mathbb{R}^n_+}\Big(\sum_{|\beta|=\ell}\|D^\beta f\|_{\Phi,\mathbb{R}^n_+}\Big)^{\frac{|\alpha|}{\ell}}. \tag{14}$$

Therefore, by using Fatou's lemma, (13) and (14), we obtain

$$\begin{aligned}\left|\int_{\mathbb{R}^n_+} D^\alpha f(x)v(x)dx\right| &\leqslant \int_{\mathbb{R}^n_+} \liminf_{m\to\infty}\big| D^\alpha f_{\lambda^*_m}(x)v(x)\big|dx \\ &\leqslant \liminf_{m\to\infty}\int_{\mathbb{R}^n_+}\big|(D^\alpha f_{\lambda^*_m})(x)v(x)\big|dx \\ &\leqslant C^+_{\alpha,\ell}\|f\|^{1-\frac{|\alpha|}{\ell}}_{\Phi,\mathbb{R}^n_+}\Big(\sum_{|\beta|=\ell}\|D^\beta f\|_{\Phi,\mathbb{R}^n_+}\Big)^{\frac{|\alpha|}{\ell}}. \end{aligned} \tag{15}$$

Because (15) is true for all $v \in L_{\overline{\Phi},}(\mathbb{R}^n_+)$, $\rho(v, \overline{\Phi}, \mathbb{R}^n_+) \leqslant 1$, by definition of the Orlicz norm we have

$$||D^\alpha f||_{\Phi,\mathbb{R}^n_+} \leqslant C^+_{\alpha,\ell}\|f\|^{1-\frac{|\alpha|}{\ell}}_{\Phi,\mathbb{R}^n_+}\Big(\sum_{|\beta|=\ell}\|D^\beta f\|_{\Phi,\mathbb{R}^n_+}\Big)^{\frac{|\alpha|}{\ell}} < \infty,\ 0 < |\alpha| < \ell.$$

The proof is complete. ∎

By Theorem 3, we have

Theorem 4. *Let Φ be an arbitrary Young function, $\ell \geq 2$, f and its generalized derivatives $D^\beta f$ be in $L_\Phi(\mathbb{R}^n_+)$, $|\beta| = \ell$. Then $D^\alpha f \in L_\Phi(\mathbb{R}^n_+)$ for all α, $0 < |\alpha| = r < \ell$ and*

$$\sum_{|\alpha|=r}\|D^\alpha f\|_{\Phi,\mathbb{R}^n_+} \leqslant C_{r,\ell}\|f\|^{1-\frac{r}{\ell}}_{\Phi,\mathbb{R}^n_+}\Big(\sum_{|\beta|=\ell}\|D^\beta f\|_{\Phi,\mathbb{R}^n_+}\Big)^{\frac{r}{\ell}}.$$

Corollary 5. *Let* Φ *be an arbitrary Young function,* $\ell \geq 2$*,* f *and its generalized derivatives* $D^\beta f$ *be in* $L_\Phi(\mathbb{R}^n_+)$*,* $|\beta| = \ell$*. Then* $D^\alpha f \in L_\Phi(\mathbb{R}^n_+)$ *for all* α*,* $0 < |\alpha| = r < \ell$ *and*

$$\sum_{|\alpha|=r} \|D^\alpha f\|_{\Phi,\mathbb{R}^n_+} \leqslant Ch^{-\frac{r}{\ell-r}} \|f\|_{\Phi,\mathbb{R}^n_+} + Ch \sum_{|\beta|=\ell} \|D^\beta f\|_{\Phi,\mathbb{R}^n_+},$$

for all $h > 0$ *and* C *does not depend on* f*.*

Remark 1. By the representation [12, 11]

$$\| u \|_{(\Phi,\mathbb{R}^n_+)} = \sup_{\|v\|_{\overline{\Phi}} \leqslant 1} \Big| \int_{\mathbb{R}^n_+} u(x)v(x)dx \Big|,$$

it is easy to see that Theorems 3, 4 still hold for any Luxemburg norm.

Remark 2. By the same method, it is easier to obtain similar results for $L_\Phi(G)$, where G is a product domain

$$-\infty < x_s < \infty,\ b_j < x_j < \infty, \quad b_j \in \mathbb{R}^1, s = 1, \dots, k,\ j = k+1, \dots, n.$$

2. A Gagliardo–Nirenberg Inequality for Lorentz Space $N_\Psi(\mathbb{R}^n_+)$

Let $\Psi : [0, \infty) \to [0, \infty)$ be a non-zero concave function which is non-decreasing and $\Psi(0+) = \Psi(0) = 0$. We put $\Psi(\infty) = \lim_{t\to\infty} \Psi(t)$. For an arbitrary measurable function f we define

$$\|f\|_{N_\Psi(G)} = \int_0^\infty \Psi\big(\lambda_f(y)\big)\,dy,$$

where $\lambda_f(y) = \text{mes}\{x \in G : |f(x)| > y\}$, $y \geq 0$. If the space $N_\Psi(G)$ consists of measurable functions f such that $\|f\|_{N_\Psi(G)} < \infty$ then $N_\Psi(G)$ is a Banach space. Denote by $M_\Psi(G)$ the space of measurable functions g such that

$$\|g\|_{M_\Psi(G)} = \sup \Big\{ \frac{1}{\Psi(\text{mes } \Delta)} \int_\Delta |g(x)|dx :\ \Delta \subset G,\ 0 < \text{mes } \Delta < \infty \Big\} < \infty.$$

Then $M_\Psi(G)$ is a Banach space, too [12–14].

We have the following results [13, 14]:

Lemma 6. *If* $f \in N_\Psi(G)$*,* $g \in M_\Psi(G)$ *then* $fg \in L_1(G)$ *and*

$$\int_G |f(x)g(x)|dx \leq \|f\|_{N_\Psi(G)} \|g\|_{M_\Psi(G)}.$$

Lemma 7. *If* $f \in N_\Psi(G)$ *then*

$$\|f\|_{N_\Psi(G)} = \sup_{\|g\|_{M_\Psi(G)} \leq 1} \Big| \int_G f(x)g(x)dx \Big|.$$

We have the following theorem.

Theorem 8. *Let $\ell \geq 2$ f and its generalized derivatives $D^\beta f$, $|\beta| = \ell$ be in $N_\Psi(\mathbb{R}^n_+)$. Then $D^\alpha f \in N_\Psi(\mathbb{R}^n_+)$ for all α, $0 < |\alpha| < \ell$ and*

$$\|D^\alpha f\|_{N_\Psi(\mathbb{R}^n_+)} \leq C^+_{\alpha,\ell} \|f\|^{1-\frac{|\alpha|}{\ell}}_{N_\Psi(\mathbb{R}^n_+)} \Big(\sum_{|\beta|=\ell} \|D^\beta f\|_{N_\Psi(\mathbb{R}^n_+)}\Big)^{\frac{|\alpha|}{\ell}}. \tag{16}$$

Proof. Step 1. We begin to prove (16) with the assumption that $D^\alpha f \in N_\Psi(\mathbb{R}^n_+), 0 \leq |\alpha| \leq \ell$. Fix $0 < |\alpha| < \ell$ and let $\epsilon > 0$. By Lemma 7 we have a function $v_\epsilon \in M_\Psi(\mathbb{R}^n_+)$ such that $\|v_\epsilon\|_{M_\Psi(\mathbb{R}^n_+)} = 1$ and

$$\Big| \int_{\mathbb{R}^n_+} f(x) v_\epsilon(x) dx \Big| \geq \|f\|_{N_\Psi(\mathbb{R}^n_+)} - \epsilon/2.$$

By Lemma 7, there is $\mathcal{H} := [0, H]^n$ such that

$$\Big| \int_{\mathbb{R}^n_+} f(x) v(x) dx \Big| \geq \|f\|_{N_\Psi(\mathbb{R}^n_+)} - \epsilon, \tag{17}$$

where $v = v(\mathcal{H}, \epsilon) := \chi_\mathcal{H} v_\epsilon$ and $\chi_\mathcal{H}$ is the characteristic function of $\mathcal{H}$. Put

$$F_\epsilon(x) = \int_{\mathbb{R}^n_+} f(x+y) v(y) dy.$$

Then $F_\epsilon \in L_\infty(\mathbb{R}^n_+)$ by virtue of Lemma 6, and it is easy to check that

$$D^\beta F_\varepsilon(x) = \int_{\mathbb{R}^n_+} D^\beta f(x+y) v(y) dy, \ \ 0 \leq |\beta| \leq \ell \tag{18}$$

in the distribution sense.

For all $x \in \mathbb{R}^n_+$, clearly,

$$|D^\beta F_\varepsilon(x)| \leq \|D^\beta f(x + \cdot)\|_{N_\Psi(\mathbb{R}^n_+)} \|v\|_{M_\Psi(\mathbb{R}^n_+)} \leq \|D^\beta f\|_{N_\Psi(\mathbb{R}^n_+)}. \tag{19}$$

Now we prove the continuity of $D^\beta F_\varepsilon$ on $\mathbb{R}^n_+$ $(0 \leq |\beta| \leq \ell)$. We show this for $\beta = 0$. Clearly, it suffices to prove that for any $x \in \mathbb{R}^n_+$,

$$\lim_{t\to 0} \|\chi_\mathcal{H}(\cdot)\big(f(x+t+\cdot) - f(x+\cdot)\big)\|_{N_\Psi(\mathbb{R}^n_+)} = 0.$$

Assume the contrary that for some $\delta > 0$, point x^0 and sequence $t_m \to 0$,

$$\|\chi_\mathcal{H}(\cdot)\big(f(x^0+t_m+\cdot) - f(x^0+\cdot)\big)\|_{N_\Psi(\mathbb{R}^n_+)} \geq \delta, \quad m \geq 1. \tag{20}$$

For simplicity of notation we suppose $x^0 = 0$. Since $f \in N_\Psi(\mathbb{R}^n_+)$, $f \in L_{1,\ell oc}(\mathbb{R}^n_+)$. It is known that

$$\int_\mathcal{H} |f(x+t_m) - f(x)| dx \to 0 \quad \text{as} \ \ m \to \infty.$$

Therefore, there exists a subsequence $\{t_{m_j}\}$, we still denote by $\{t_m\}$, such that $f(\cdot + t_m) \to f$ a.e. on $\mathcal{H}$. Define

$$g_n(x) = \inf_{m\geq n} |f(x+t_m)|, \ \ x \in \mathcal{H},$$

then $\{g_n\}$ is a non-decreasing sequence and $g_n \to |f|$ a.e. on $\mathcal{H}$. It is easy to see that

$$\lambda_{\chi_{\mathcal{H}} g_n}(t) \to \lambda_{\chi_{\mathcal{H}}|f|}(t) \;\text{ as } n \to \infty, \text{ for every } t > 0.$$

We have

$$\Psi(\lambda_{\chi_{\mathcal{H}}|f|}(t)) = \lim_{m\to\infty} \Psi(\lambda_{\chi_{\mathcal{H}}|g_m|}(t)) \le \varliminf_{m\to\infty} \Psi(\lambda_{\chi_{\mathcal{H}}|f(\cdot+t_m)|}(t)),\; t > 0. \tag{21}$$

It follows from the definition of Ψ that $\Psi(a+b) \le \Psi(a) + \Psi(b)$ for $a, b \ge 0$. Observe that, for any $f, g \in N_\Psi(\mathbb{R}^n_+)$ and $t > 0$, so we have $\lambda_{\chi_{\mathcal{H}}(f+g)}(2t) \le \lambda_{\chi_{\mathcal{H}} f}(t) + \lambda_{\chi_{\mathcal{H}} g}(t)$, then for all $m \ge 1$,

$$\Psi(\lambda_{\chi_{\mathcal{H}}|f(\cdot+t_m)-f|}(2t)) \le \Psi(\lambda_{\chi_{\mathcal{H}}|f(\cdot+t_m)|}(t)) + \Psi(\lambda_{\chi_{\mathcal{H}}|f|}(t)).$$

Hence

$$0 \le \Psi(\lambda_{\chi_{\mathcal{H}}|f(\cdot+t_m)|}(t)) + \Psi(\lambda_{\chi_{\mathcal{H}}|f|}(t)) - \Psi(\lambda_{\chi_{\mathcal{H}}|f(\cdot+t_m)-f|}(2t)),\; \forall t > 0.$$

It is easy to check that (see [5])

$$\lim_{m\to\infty} \|\chi_{\mathcal{H}} f(\cdot + t_m)\|_{N_\Psi(\mathbb{R}^n_+)} = \|\chi_{\mathcal{H}} f\|_{N_\Psi(\mathbb{R}^n_+)},\; \forall m \ge 1.$$

Applying Fatou's lemma to the sequence

$$\{\Psi(\lambda_{\chi_{\mathcal{H}}|f(\cdot+t_m)|}(t)) + \Psi(\lambda_{\chi_{\mathcal{H}}|f|}(t)) - \Psi(\lambda_{\chi_{\mathcal{H}}|f(\cdot+t_m)-f|}(2t))\},$$

we obtain

$$\begin{aligned}
&\int_0^\infty \varliminf_{m\to\infty} \big[\Psi(\lambda_{\chi_{\mathcal{H}}|f(\cdot+t_m)|}(t)) + \Psi(\lambda_{\chi_{\mathcal{H}}|f|}(t)) - \Psi(\lambda_{\chi_{\mathcal{H}}|f(\cdot+t_m)-f|}(2t))\big]\,dt \\
&\le \varliminf_{m\to\infty} \int_0^\infty \big[\Psi(\lambda_{\chi_{\mathcal{H}}|f(\cdot+t_m)|}(t)) + \Psi(\lambda_{\chi_{\mathcal{H}}|f|}(t)) - \Psi(\lambda_{\chi_{\mathcal{H}}|f(\cdot+t_m)-f|}(2t)\big]\,dt \\
&= 2\int_0^\infty \Psi(\lambda_{\chi_{\mathcal{H}}|f|}(t))dt - \frac{1}{2}\varlimsup_{m\to\infty} \int_0^\infty \Psi(\lambda_{\chi_{\mathcal{H}}|f(\cdot+t_m)-f|}(t))dt.
\end{aligned} \tag{22}$$

On the other hand,

$$\lambda_{\chi_{\mathcal{H}}|f(\cdot+t_m)-f|}(t) = \operatorname{mes}\{x \in \mathcal{H} : |f(x+t_m) - f(x)| > t\}.$$

Therefore, taking account of $f(\cdot + t_m) \to f$ a.e. on $\mathcal{H}$, we have

$$\lim_{m\to\infty} \lambda_{\chi_{\mathcal{H}}|f(\cdot+t_m)-f|}(t) = 0$$

and then

$$\lim_{m\to\infty} \Psi(\lambda_{\chi_{\mathcal{H}}|f(\cdot+t_m)-f|}(t)) = 0.$$

So, by (21) we get for any $t > 0$

$$2\Psi(\lambda_{\chi_{\mathcal{H}}|f|}(t)) = \lim_{m\to\infty} \Psi(\lambda_{\chi_{\mathcal{H}}|g_m|}(t)) + \Psi(\lambda_{\chi_{\mathcal{H}}|f|}(t)) - \lim_{m\to\infty} \Psi(\lambda_{\chi_{\mathcal{H}}|f(\cdot+t_m)-f|}(2t))$$
$$\leq \underline{\lim}_{m\to\infty} \big[\Psi(\lambda_{\chi_{\mathcal{H}}|f(\cdot+t_m)|}(t)) + \Psi(\lambda_{\chi_{\mathcal{H}}|f|}(t)) - \Psi(\lambda_{\chi_{\mathcal{H}}|f(\cdot+t_m)-f|}(2t))\big]. \tag{23}$$

From (22) and (23), we have

$$2\int_0^\infty \Psi(\lambda_{\chi_{\mathcal{H}}|f|}(t))dt \leq 2\int_0^\infty \Psi(\lambda_{\chi_{\mathcal{H}}|f|}(t))dt - \frac{1}{2}\overline{\lim_{m\to\infty}} \int_0^\infty \Psi(\lambda_{\chi_{\mathcal{H}}|f(\cdot+t_m)-f|}(t))dt.$$

Hence

$$\int_0^\infty \Psi(\lambda_{\chi_{\mathcal{H}}|f(\cdot+t_m)-f|}(t))dt \to 0 \quad \text{as } m \to \infty,$$

i.e.,

$$\lim_{m\to\infty} \|\chi_{\mathcal{H}}\big(f(\cdot+t_m) - f\big)\|_{N_\Psi(\mathbb{R}^n_+)} = 0,$$

which contradicts (20).

The cases $1 \leq |\beta| \leq \ell$ are proved similarly. The continuity of $D^\alpha F_\varepsilon,\ 0 \leq |\beta| \leq \ell$ has been proved.

The functions $D^\beta F_\varepsilon,\ 0 \leq |\beta| \leq \ell$ are continuous and bounded on $\mathbb{R}^n_+$. Therefore, it follows from the Gagliardo-Nirenberg inequality and (17)–(18) that

$$\big(\|D^\alpha f\|_{N_\Psi(\mathbb{R}^n_+)} - \epsilon\big) \leq |D^\alpha F_\varepsilon(0)| \leq \|D^\alpha F_\varepsilon\|_\infty \leq$$
$$\leq C^+_{\alpha,\ell}\|F_\varepsilon\|_\infty^{1-\frac{|\alpha|}{\ell}}\Big(\sum_{|\beta|=\ell}\|D^\beta F_\varepsilon\|_\infty\Big)^{\frac{|\alpha|}{\ell}},$$

which together with (19) implies

$$\|D^\alpha f\|_{N_\Psi(\mathbb{R}^n_+)} - \epsilon \leq C^+_{\alpha,\ell}\|f\|^{1-\frac{|\alpha|}{\ell}}_{N_\Psi(\mathbb{R}^n_+)}\Big(\sum_{|\beta|=\ell}\|D^\beta f\|_{N_\Psi(\mathbb{R}^n_+)}\Big)^{\frac{|\alpha|}{\ell}}.$$

By letting $\epsilon \to 0$ we have (16).

Step 2. To complete the proof, it remains to show that $D^\alpha f \in N_\Psi(\mathbb{R}^n_+),\ \forall\alpha : 0 < |\alpha| < \ell$ if $f, D^\alpha f \in N_\Psi(\mathbb{R}^n_+),\ |\alpha| = \ell$. Fix $b > 0$. With notations as in the proof of Theorem 3, we have for $0 < \lambda < b, 0 \leqslant |\alpha| \leqslant \ell :$

$$D^\alpha f_\lambda = f_{(\alpha)} * \psi_\lambda \tag{24}$$

in the $\mathcal{D}'(\mathbb{R}^n_{+,b})$ sense.

Taking Lemma 6 into account, we get easily $D^\beta f_\lambda = f * D^\beta\psi_\lambda \in N_\Psi(\mathbb{R}^n),\ 0 \leq$

$|\beta| \le \ell$ and

$$\begin{aligned} \|f_\lambda\|_{N_\Psi(\mathbb{R}^n)} &= \|f_{(0)} * \psi_\lambda\|_{N_\Psi(\mathbb{R}^n)} \le \|f_{(0)}\|_{N_\Psi(\mathbb{R}^n)} \|\psi_\lambda(x-\cdot)\|_1 \\ &= \|f_{(0)}\|_{N_\Psi(\mathbb{R}^n)}, \qquad (25) \\ \|D^\alpha f_\lambda\|_{N_\Psi(\mathbb{R}^n)} &= \|f_{(\alpha)} * \psi_\lambda\|_{N_\Psi(\mathbb{R}^n)} \le \|f_{(\alpha)}\|_{N_\Psi(\mathbb{R}^n)} \|\psi_\lambda(x-\cdot)\|_1 \\ &= \|f_{(\alpha)}\|_{N_\Psi(\mathbb{R}^n)}. \qquad (26) \end{aligned}$$

Therefore, for $0 < \lambda < b$ and $\alpha = 0$ or $|\alpha| = \ell$ we have

$$\begin{aligned} \|D^\alpha\big(f_{(0)} * \psi_\lambda\big)\|_{N_\Psi(\mathbb{R}^n_{+,b})} &= \|f_{(\alpha)} * \psi_\lambda\|_{N_\Psi(\mathbb{R}^n_{+,b})} \\ &\leqslant \|f_{(\alpha)} * \psi_\lambda\|_{N_\Psi(\mathbb{R}^n)} \\ &\leqslant \|f_{(\alpha)}\|_{N_\Psi(\mathbb{R}^n)} \\ &= \|f_{(\alpha)}\|_{N_\Phi(\mathbb{R}^n_+)} \\ &= \|D^\alpha f\|_{N_\Phi(\mathbb{R}^n_+)}. \qquad (27) \end{aligned}$$

On the other hand, using $D^\alpha\big(f_{(0)} * \psi_\lambda\big) = f_{(0)} * D^\alpha \psi_\lambda \in N_\Psi(\mathbb{R}^n)$, $\forall\ 0 \leqslant |\alpha| \leqslant \ell$, (25)–(27) and the inequality proved in Step 1 for functions on $\mathbb{R}^n_{+,b}$, we get for all $0 < \lambda < b$, $0 < |\alpha| < \ell$,

$$\begin{aligned} ||D^\alpha f_\lambda||_{N_\Psi(\mathbb{R}^n_{+,b})} &\leqslant C^+_{\alpha,\ell} \|f_\lambda\|^{1-\frac{|\alpha|}{\ell}}_{N_\Psi(\mathbb{R}^n_{+,b})} \Big(\sum_{|\beta|=\ell} \|D^\beta f_\lambda\|_{N_\Psi(\mathbb{R}^n_{+,b})}\Big)^{\frac{|\alpha|}{\ell}} \\ &\leqslant C^+_{\alpha,\ell} \|f_\lambda\|^{1-\frac{|\alpha|}{\ell}}_{N_\Psi(\mathbb{R}^n)} \Big(\sum_{|\beta|=\ell} \|D^\beta f_\lambda\|_{N_\Psi(\mathbb{R}^n)}\Big)^{\frac{|\alpha|}{\ell}} \\ &\leqslant C^+_{\alpha,\ell} \|f\|^{1-\frac{|\alpha|}{\ell}}_{N_\Psi(\mathbb{R}^n_+)} \Big(\sum_{|\beta|=\ell} \|D^\beta f\|_{N_\Psi(\mathbb{R}^n_+)}\Big)^{\frac{|\alpha|}{\ell}}. \qquad (28) \end{aligned}$$

Fix α $(0 < |\alpha| < \ell)$. Repeating the arguments used in the proof of Theorem 3 , we get a sequence of positive numbers $\{\lambda^*_m\} : \lambda^*_m \to 0$ such that

$$\lim_{m\to\infty} D^\alpha f_{\lambda^*_m}(x) = D^\alpha f(x) \ \text{ a.e. in } \mathbb{R}^n_+.$$

Hence,

$$\lim_{m\to\infty} f_{(\alpha)} * \psi_{\lambda^*_m}(x) = f_{(\alpha)}(x) = D^\alpha f(x) \ \text{ a.e. in } \mathbb{R}^n_+. \qquad (29)$$

For each function $v \in M_\Psi(\mathbb{R}^n_+)$, $\|v\|_{M_\Psi((\mathbb{R}^n_+)} \leqslant 1$ and $m \ge 1$, by (18) - (19) and the definition of the Lorentz norm we get

$$\int_{\mathbb{R}^n_+} \big|(D^\alpha f_{\lambda^*_m})(x)v(x)\big|dx \leqslant C^+_{\alpha,\ell} \|f\|^{1-\frac{|\alpha|}{\ell}}_{N_\Psi(\mathbb{R}^n_+)} \big(\sum_{|\beta|=\ell} \|D^\beta f\|_{N_\Psi(\mathbb{R}^n_+)}\big)^{\frac{|\alpha|}{\ell}}. \qquad (30)$$

Therefore, by using Fatou's lemma, (29) and (30), we obtain

$$\begin{aligned}\left|\int_{\mathbb{R}^n_+} D^\alpha f(x)v(x)dx\right| &\leqslant \int_{\mathbb{R}^n_+} \liminf_{m\to\infty}\left| D^\alpha f_{\lambda^*_m}(x)v(x)\right|dx\\ &\leqslant \liminf_{m\to\infty}\int_{\mathbb{R}^n_+}\left|(D^\alpha f_{\lambda^*_m})(x)v(x)\right|dx\\ &\leqslant C^+_{\alpha,\ell}\|f\|^{1-\frac{|\alpha|}{\ell}}_{N_\Psi(\mathbb{R}^n_+)}\Big(\sum_{|\beta|=\ell}\|D^\beta f\|_{N_\Psi(\mathbb{R}^n_+)}\Big)^{\frac{|\alpha|}{\ell}}. \qquad (31)\end{aligned}$$

Because (31) is true for all $v \in M_\Psi(\mathbb{R}^n_+)$, $\|v\|_{M_\Psi((\mathbb{R}^n_+)} \leqslant 1$, by definition of the Lorentz norm we have

$$||D^\alpha f||_{N_\Psi(\mathbb{R}^n_+)} \leqslant C^+_{\alpha,\ell}\|f\|^{1-\frac{|\alpha|}{\ell}}_{N_\Psi(\mathbb{R}^n_+)}\Big(\sum_{|\beta|=\ell}\|D^\beta f\|_{N_\Psi(\mathbb{R}^n_+)}\Big)^{\frac{|\alpha|}{\ell}} < \infty,\ 0<|\alpha|<\ell.$$

The proof is complete. ∎

By Theorem 8, we have

Theorem 9. *Let $\ell \geq 2$, f and its generalized derivatives $D^\beta f$, $|\beta| = \ell$ be in $N_\Psi(\mathbb{R}^n_+)$. Then $D^\alpha f \in N_\Psi(\mathbb{R}^n_+)$ for all α, $0 < |\alpha| = r < \ell$ and*

$$\sum_{|\alpha|=r}\|D^\alpha f\|_{N_\Psi(\mathbb{R}^n_+)} \leq C_{r,\ell}\|f\|^{1-\frac{r}{\ell}}_{N_\Psi(\mathbb{R}^n_+)}\big(\sum_{|\beta|=\ell}\|D^\beta f\|_{N_\Psi(\mathbb{R}^n_+)}\big)^{\frac{r}{\ell}}.$$

Corollary 10. *Let $\ell \geq 2$, f and its generalized derivatives $D^\beta f$, $|\beta| = \ell$ be in $N_\Psi(\mathbb{R}^n_+)$. Then $D^\alpha f \in N_\Psi(\mathbb{R}^n_+)$ for all α, $0 < |\alpha| = r < \ell$ and*

$$\sum_{|\alpha|=r}\|D^\alpha f\|_{N_\Psi(\mathbb{R}^n_+)} \leq Ch^{-\frac{r}{\ell-r}}\|f\|_{N_\Psi(\mathbb{R}^n_+)} + Ch\sum_{|\beta|=\ell}\|D^\beta f\|_{N_\Psi(\mathbb{R}^n_+)},$$

for all $h > 0$ and C does not depend on f.

Remark 3. Note that the techniques applied in the proof of Theorem 3 for Orlicz spaces $L_\Psi(\mathbb{R}^n_+)$ cannot be used for Lorentz spaces $N_\Psi(\mathbb{R}^n_+)$.

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