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# The Existence of $\varepsilon$-Solutions to General Quasiequilibrium Problems

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**Abstract.** We consider three types of approximate solutions of multivalued quasiequilibrium vector problems. Sufficient conditions for the $\varepsilon$-solution existence are established for variants of such problems. Several applications are provided as examples to show that our results can imply consequences about approximate solutions of many optimization-related problems.

*Keywords:* Quasiequilibrium problems, $\varepsilon$-solutions, $W$-quasiconvexity relative to a set, quasivariational inequalities, quasioptimization problems.

## 1. Introduction and Preliminaries

The equilibrium problem was proposed by Blum and Oettli [4] as a generalization of variational inequalities and optimization problems and includes also other problems such as the complementarity problem, the Nash equilibrium, the fixed point and coincidence point problems, the traffic network problem, etc. On the other hand Bensoussan, Goursat and Lions [3], considering random impulse control problems, observed the necessity to investigate constraint sets depending on the state variable. This paper led to the birth of the quasivariational inequality, and later, of the quasiequilibrium problem. The solution existence was often of interest first, see e.g. recent papers [2, 5 -18, 21 - 23] and references therein. However, the conditions for the existence of exact solutions are often rather strict. Moreover, some problems in practice do not have exact solutions, but possess $\varepsilon$-solutions (approximate solutions with $\varepsilon$-tolerance), see e.g. Examples 1.1 and 1.2. Such solutions make sense in practical situations, since the data of problems under consideration are obtained approximately by measurements or statistical ways and hence the mathematically exact solutions are in fact also approximate ones. Therefore, demands on the existence of exact solutions may be too costly.

To the best of our knowledge, there are not papers dealing with the existence of approximate solutions of equilibrium or quasiequilibrium problems in the literature (the only paper [1] considers the semicontinuity of approximate solution sets). This motivates our aim in this note: to establish sufficient conditions for the existence of $\varepsilon$-solutions to quasiequilibrium problems in general spaces. It appears that for $\varepsilon = 0$, i.e. for exact solutions, our results are also new, and shown by examples to be more applicable than existing ones in some cases.

We now outline the remainder of the paper. The rest of this section is devoted to the problem setting and some preliminaries. The main results are presented in Sec. 2. In the final Sec. 3, some applications are provided.

Throughout the paper, unless otherwise stated, let $X$ and $Z$ be Hausdorff topological vector spaces and $Y$ be a linear metric space with invariant metric $d(.,.)$. Let $A \subseteq X$ and $B \subseteq Z$ be nonempty compact convex sets. Let $C \subseteq Y$ be closed with the interior $\operatorname{int} C \neq \emptyset$ and $C \neq Y$. Let the multifunctions $K : A \to 2^X$, $T : A \to 2^B$ and $F : T(A) \times X \times A \to 2^Y$ have nonempty values. For subsets $U, V$ and points $x, y$ under consideration we adopt the notations

$$
\begin{aligned}
&r_1(U,V) \text{ means } U \cap V \neq \emptyset;\ \ \bar{r}_1(U,V) \text{ means } U \cap V = \emptyset;\\
&r_2(U,V) \text{ means } U \subseteq V;\ \ \bar{r}_2(U,V) \text{ means } U \not\subseteq V;\\
&\alpha_1(x,y,U,V) \text{ means } \forall x \in U, \exists y \in V;\\
&\alpha_2(x,y,U,V) \text{ means } \exists y \in V, \forall x \in U.
\end{aligned}
$$

For each $r \in \{r_1, r_2\}$ and each $\alpha \in \{\alpha_1, \alpha_2\}$, we consider the following quasiequilibrium problem

$$
(\mathrm{QEP}_{r,\alpha}) \qquad \text{Find } (\bar{x} \in A\cap)\ \mathrm{cl}(K(\bar{x})) \text{ such that } \alpha(y, \bar{t}, K(\bar{x}), T(\bar{x})),\\
r(F(\bar{t}, y, \bar{x}), Y\backslash -\mathrm{int}C).
$$

Let us use the notations

$$
\begin{aligned}
&\mathrm{comp}\ (-\mathrm{int}C)_1^{\varepsilon} = \{y \in Y \mid d(y, Y\backslash -\mathrm{int}C) \leq \varepsilon\},\\
&\mathrm{comp}(-\mathrm{int}C)_2^{\varepsilon} = (Y\backslash -\mathrm{int}C) + \bar{B}_Y^{\varepsilon},\\
&\mathrm{comp}(-\mathrm{int}C)_3^{\varepsilon} = \{y \in Y \mid d(y, Y\backslash -\mathrm{int}C) < \varepsilon\},
\end{aligned}
$$

where $d(y,V) := \inf_{v \in V} d(y,v)$ is the distance between the point $y$ and the set $V$, $\bar{B}_Y^{\varepsilon} := \{y \in Y \mid d(0,y) \leq \varepsilon\}$ and $B_Y^{\varepsilon} := \{y \in Y \mid d(0,y) < \varepsilon\}$. The notation "comp(.)" is related to the word "complement".

*Remark 1.* (i) We have, for $\varepsilon > 0$,

$$\{y \in Y \mid d(y, Y \setminus -\text{int}C) < \varepsilon\} = (Y \setminus -\text{int}C) + B_Y^\varepsilon.$$

Indeed, we prove more generally that $\{y \in Y \mid d(y, Q) < \varepsilon\} := Q^\varepsilon$ is equal to $Q + B_Y^\varepsilon$, for any $\emptyset \neq Q \subseteq Y$. To see "$\supseteq$" let $y = q + z$ for some $q \in Q$ and $z \in B_Y^\varepsilon$. Then, $d(y,q) = d(y-q, 0) = d(z, 0) < \varepsilon$, i.e. $y \in Q^\varepsilon$. For the inverse inclusion "$\subseteq$", let $y \in Q^\varepsilon$. Then $d(y, Q) := d_y < \varepsilon$. Hence, there is $q \in Q$ with $d_y < d(y,q) < \varepsilon$. Consequently, $y - q \in B_Y^\varepsilon$ and $y \in Q + B_Y^\varepsilon$.

(ii) For $\emptyset \neq Q \subseteq Y$, denote $\bar{Q}^\varepsilon = \{y \in Y \mid d(y,Q) \leq \varepsilon\}$. Then, following Remark 1.1 of [1], $Q + \bar{B}_Y^\varepsilon \subseteq \bar{Q}^\varepsilon$ and one has an equality if $Y$ is finite dimensional and $Q$ is closed. However, while $Y$ is infinite dimensional $Q + \bar{B}_Y^\varepsilon$ may be properly contained in $\bar{Q}^\varepsilon$ since $Q + \bar{B}_Y^\varepsilon$ may be not closed even for a closed set $Q$, see Examples 1.1 and 1.2 of [1].

According to Remark 1 we have the following definition of three kinds of $\varepsilon$-solutions.

**Definition 1.1.** *The problem* $(QEP_{r,\alpha})$ *has three kinds of ε-solutions corresponding to the above three sets* $\text{comp}\,(-\text{int}\,C)_k^\varepsilon$. *For instance* $\bar{x} \in A \cap\ clK(\bar{x})$ *is said to be an ε-solution of type* $k, k = 1, 2, 3$, *of problem* $(QEP_{r,\alpha})$ *if* $\alpha(y, \bar{t}, K(\bar{x}), T(\bar{x})),\ r(F(\bar{t}, y, \bar{x}),\ \ \text{comp}\,(-\text{int}\,C)_k^\varepsilon)$.

Note that the $\varepsilon$-solutions of types 1 and 2 were proposed in [1]. Following Remark 1 (ii) each $\varepsilon$-solution of type 2 is an $\varepsilon$-solution of type 1, but the converse is not true if $Y$ is infinite dimensional.

The following example shows a case where problem $(\text{QEP}_{r_1,\alpha_1})$ is unsolvable (in the exact sense) but its $\varepsilon$-solutions exist.

*Example 1.* Let $X = Y = Z = R$, $A = [0,1]$, $K(x) \equiv [0,1]$, $C = R_+$, $T(x) = [0, x]$ and $F(t, y, x) = [-0.1, -0.1 + 0.05x]$. Then it is clear that the exact solution of $(\text{QEP}_{r_1,\alpha_1})$ does not exist. However, for $\varepsilon \geq 0.1$, each $\bar{x} \in [0,1]$ is an $\varepsilon$-solution of type 1.

$\varepsilon$-solution sets depend, in general, on $\varepsilon$ as shown in the following example.

*Example 2.* Let $X, Y, Z, A, K$ and $C$ be as in Example 1. Let $T(x) = \{x\}$ and $F(t, y, x) = [-0.1 + x, 1]$. Then it is easy to check that the $\varepsilon$-solution set of $(\text{QEP}_{r_1,\alpha_1})$ is $[0.1 - \varepsilon, 1]$ for $0 \leq \varepsilon < 0.1$ and [0,1] for $\varepsilon \geq 0.1$.

Our main tool in this paper is the following fixed point theorem, which is a slightly weaker version (suitable for our use) of the corresponding theorem in [20].

**Theorem 1.1.** *Let $X$ be a Hausdorff topological vector space, $A \subseteq X$ be nonempty compact convex and $\varphi : A \to 2^X$ be a multifunction with nonempty convex values. Assume that, for each $x \in A, \varphi^{-1}(x)$ is open in $A$. Then there is a fixed point $\hat{x} \in A$ of $\varphi$, i.e. $\hat{x} \in \varphi(\hat{x})$.*

We recall now semicontinuity notions of multifunctions needed in the sequel. Let $X$ and $Y$ be topological spaces and $H : X \to 2^Y$ be a multifunction. $H$

is called lower semicontinuous (lsc) at $x_0 \in X$ if, for any open subset $U$ such that $U \cap H(x_0) \neq \emptyset$, there exists a neighborhood $N$ of $x_0$ such that, $\forall x \in N$, $U \cap H(x) \neq \emptyset$. $H$ is termed upper semicontinuous (usc) at $x_0 \in X$ if, for any open subset $U$ such that $U \supseteq H(x_0)$, there exists a neighborhood $N$ of $x_0$ such that $U \supseteq H(N)$. $H$ is called lsc (or usc) if $H$ is lsc (usc, respectively) at every point $x \in \mathrm{dom}H := \{x \in X : H(x) \neq \emptyset\}$. $H$ is said to be closed if the graph $\mathrm{gr}H := \{(x,y) \in X \times Y \mid y \in H(x)\}$ is closed.

The convexity assumptions imposed in our theorems are the following relaxed property. Let $X$ be a vector space and $D \subseteq X$ be nonempty and convex. Let $P, Q, V$ and $W \subseteq V$ be nonempty sets. Let $T : P \to 2^Q$ and $F : Q \times D \to 2^V$ be multifunctions. For $x \in P$, $F$ is said to be $W$-quasiconvex relative to $T(x)$ of type 1 if, $\forall \xi, \eta \in D, \forall \lambda \in [0,1]$,

$$[F(t,\xi) \cap W = \emptyset \text{ and } F(t,\eta) \cap W = \emptyset,\ \forall t \in T(x)]$$
$$\Rightarrow [F(t,(1-\lambda)\xi + \lambda\eta) \cap W = \emptyset,\ \forall t \in T(x)]. \quad (1)$$

$F$ is called $W$-quasiconvex relative to $T(x)$ of type 2 if (1) is replaced by

$$[F(t,\xi) \not\subseteq W \text{ and } F(t,\eta) \not\subseteq W,\ \forall t \in T(x)]$$
$$\Rightarrow [F(t,(1-\lambda)\xi + \lambda\eta) \not\subseteq W,\ \forall t \in T(x)].$$

To see the nature of these definitions, consider the simplest case, where $X = D = V = P = Q = R$, $T(x) \equiv \{x_0\}, W = R_+$ and $F : \{x_0\} \times X \to R$ is single-valued, depending only on $x \in X$. Then the above two types of relaxed convexity coincide and become:

$$\forall \xi, \eta \in R, \forall \lambda \in [0,1],\ [F(\xi) < 0 \text{ and } F(\eta) < 0] \Rightarrow [F((1-\lambda)\xi + \lambda\eta) < 0].$$

This property is a relaxed 0-level quasiconvexity, since $F$ is called quasiconvex if $\forall \xi, \eta \in R, \forall \lambda \in [0,1], F((1-\lambda)\xi + \lambda\eta) \leq \max\{F(\xi), F(\eta)\}$.

For the special case of the above general quasiconvexity, where $T(x) \equiv \{x_0\}$, i.e. $F$ depends on only one variable x, we simply say that $F$ is $W$-quasiconvex of type 1 or type 2.

## 2. Main Results

**Theorem 2.1.** *With fixed $i \in \{1,2\}, k \in \{1,2,3\}$ and $\varepsilon \geq 0$, assume for problem* $(\mathrm{QEP}_{r_i,\alpha_1})$ *that*

(i) *for each $x \in A, F(.,.,x)$ is comp$(-intC)^{\varepsilon}_k$-quasiconvex relative to $T(x)$ of type $i$ and $r_i(F(t,x,x), \mathrm{comp}\,(-int\,C)^{\varepsilon}_k)$ for some $t \in T(x)$;*

(ii) *for each $y \in A$, the set $\{x \in A \mid \exists t \in T(x), r_i(F(t,y,x), \mathrm{comp}\,(-int\,C)^{\varepsilon}_k)\}$ is closed;*

(iii) *$clK(.)$ is usc; for each $x \in A, A \cap K(x) \neq \emptyset$ and $K(x)$ is convex; for each $y \in A, K^{-1}(y)$ is open in $A$.*

*Then problem* $(\mathrm{QEP}_{r_i,\alpha_1})$ *has $\varepsilon$-solutions of type $k$.*

*Proof.* For $x \in A$ set

$$P(x) = \{z \in A \mid \forall t \in T(x), \bar{r}_i(F(t,z,x),\ \mathrm{comp}\,(-\mathrm{int}\,C)^{\varepsilon}_k)\},$$
$$E = \{z \in A \mid z \in\ \mathrm{cl}K(z)\}.$$

By virtue of (i), $P(x)$ is convex for all $x \in A$. By (iii), $\mathrm{cl}K(.)$ is closed and hence $E$ is a closed set.

One has, $\forall y \in A$,

$$\begin{aligned} A \backslash P^{-1}(y) &= \{x \in A \mid y \notin P(x)\} \\ &= \{x \in A \mid \exists t \in T(x), r_i(F(t,y,x),\ \mathrm{comp}\,(-\mathrm{int}\,C)^{\varepsilon}_{k})\}. \end{aligned}$$

In view of (ii), $P^{-1}(y)$ is open in $A, \forall y \in A$.

We define multifunction $Q : A \to 2^A$ by

$$Q(x) = \begin{cases} K(x) \cap P(x) & \text{if } x \in E, \\ A \cap K(x) & \text{if } x \in A \backslash E. \end{cases}$$

Then, $\forall x \in A, Q(x)$ is convex. We have, for $y \in A$,

$$\begin{aligned} Q^{-1}(y) &= \{x \in E \mid x \in K^{-1}(y) \cap P^{-1}(y)\} \cup \{x \in A \backslash E \mid x \in K^{-1}(y)\} \\ &= K^{-1}(y) \cap [P^{-1}(y) \cup (A \backslash E)]. \end{aligned}$$

Therefore,

$$A \backslash Q^{-1}(y) = [A \backslash K^{-1}(y)] \cup [(A \backslash P^{-1}(y)) \cap E].$$

Since $K^{-1}(y)$ and $P^{-1}(y)$ are open in $A$, this implies the openness of $Q^{-1}(y)$ in $A$, for all $y \in A$. From (i), $x \notin P(x)$ and then $x \notin Q(x)$, for all $x \in A$. Applying Theorem 1.1 to multifunction $Q$, one gets $\hat{x} \in A$ such that $Q(\hat{x}) = \emptyset$. Since $A \cap K(\hat{x}) \neq \emptyset$, $\hat{x} \in E$ and $K(\hat{x}) \cap P(\hat{x}) = \emptyset$. Thus, $\hat{x} \in A \cap \mathrm{cl}K(\hat{x})$ and, $\forall y \in K(\hat{x})$, $y \notin P(\hat{x})$, i.e., one has $\hat{t} \in T(\hat{x})$ such that $r_i(F(\hat{t}, y, \hat{x}),\ \mathrm{comp}\,(-\mathrm{int}\,C)^{\varepsilon}_{k})$ and hence $\hat{x}$ is an $\varepsilon$-solution of type $k$ of $(\mathrm{QEP}_{r_i,\alpha_1})$. ■

*Remark 2.* Now, we consider as an example the special case of Theorem 2.1 where $r_i = r_1$ and $k = 1$.

(a) If, for each $y \in A, F(., y, .)$ and $T(.)$ are usc and map compact sets to compact sets, then the assumption (ii) in Theorem 2.1 is satisfied.

(b) If the set $C$ in problem $(\mathrm{QEP}_{r_1,\alpha_1})$ depends on $x \in A$, i.e. $C : A \to 2^Y$ is a multifunction, then it is not hard to check that Theorem 2.1 is still valid with $C$ replaced by $C(x)$, and with the additional assumption that $Y \backslash -\mathrm{int}C(.)$ is usc.

(c) If $A$ is not compact but the following coercivity assumption is additionally imposed:

(iv) there exists a nonempty compact subset $D \subseteq A$ such that for each finite subset $M \subseteq A$, there is a compact convex subset $L_M$ of $A$, containing $M$, such that $\forall x \in L_M \backslash D, \exists y \in L_M \cap K(x)$, $\forall t \in T(x)$,

$$F(t, y, x) \cap\ \mathrm{comp}\,(-\mathrm{int}\,C)^{\varepsilon}_{1} = \emptyset;$$

then Theorem 2.1 is still valid.

(d) The special case of Theorem 2.1 with $r_i = r_1, k = 1$, $\varepsilon = 0$ and the assumptions mentioned in (a) and (b) is a result stronger than Theorem 4.13 of

[16], since our quasiconvexity assumption is more relaxed than the corresponding assumption there.

The next example indicates that an $\varepsilon$-solution of type 1 may exist even though $\varepsilon$-solutions of type 3 do not exist.

*Example 3.* Let $X = Y = Z = R$, $A = [0,1]$, $K(x) \equiv [0,1]$, $C = R_+$, $T(x) = \{x\}$ and $F(t,y,x) \equiv \{-0.1\}$. Then it is evident that problem $(\mathrm{QEP}_{r_1,\alpha_1})$ does not have $\varepsilon$-solutions of type 3 for $\varepsilon = 0.1$. However, all assumptions of Theorem 2.1 are fulfilled for $k = 1$ and hence $\varepsilon$-solutions of type 1 exist.

**Theorem 2.2.** *With fixed* $i \in \{1,2\}, k \in \{1,2,3\}$ *and* $\varepsilon \geq 0$*, assume for problem* $(\mathrm{QEP}_{r_i,\alpha_2})$ *that*

(i) *for each* $(t,x) \in B \times A, F(t,.,x)$ *is* $\mathrm{comp}\,(-\mathrm{int}\,C)^{\varepsilon}_{k}$*-quasiconvex relative to* $T(x)$ *of type* $i$ *and* $r_i(F(t,x,x), \mathrm{comp}\,(-\mathrm{int}\,C)^{\varepsilon}_{k})$, for $k = 1,2,3$;

(ii) $T$ *has nonempty convex values and* $T^{-1}(z)$ *is open in* $A$ *for all* $z \in B$; $\{(t,x) \in B \times A \mid \exists y \in K(x), \bar{r}_i(F(t,y,x), \mathrm{comp}\,(-\mathrm{int}\,C)^{\varepsilon}_{k})\}$ *and* $\{(t,x) \in B \times A \mid r_i(F(t,y,x), \mathrm{comp}\,(-\mathrm{int}\,C)^{\varepsilon}_{k})\}$ *are closed,* $\forall y \in A$;

(iii) $\forall x \in A, A \cap K(x) \neq \emptyset$ *and* $K(x)$ *is nonempty convex;* $\forall y \in A, K^{-1}(y)$ *is open in* $A$.

*Then there exist* $\varepsilon$*-solutions of type* $k$ *of problem* $(\mathrm{QEP}_{r_i,\alpha_2})$.

*Proof.* For $(t, x \in B \times A)$ and for fixed $i, k, \varepsilon$ set

$$
\begin{aligned}
P(t,x) &= \{z \in A \mid \bar{r}_i(F(t,z,x),\ \mathrm{comp}(-\mathrm{int}C)^{\varepsilon}_{k})\},\\
E &= \{(t,x) \in B \times A \mid K(x) \cap P(t,x) \neq \emptyset\},\\
S(t,x) &= \begin{cases} K(x) \cap P(t,x) & \text{if}\,(t,x) \in E,\\ A \cap K(x) & \text{if}\,(t,x) \in (B \times A)\backslash E,\end{cases}\\
Q(t,x) &= (T(x), S(t,x)).
\end{aligned}
$$

By (i), $P(t,x)$ is convex and so is $Q(t,x)$ for all $(t,x) \in B \times A$. We claim that $E$ is closed. Indeed,

$$
E = \big\{(t,x) \in B \times A \mid \exists y \in K(x), \bar{r}_i(F(t,y,x),\ \mathrm{comp}\,(-\mathrm{int}\,C)^{\varepsilon}_{k})\big\},
$$

which is closed by (ii).

Now we have, for $y \in A$ and $z \in B$,

$$
\begin{aligned}
S^{-1}(y) &= \{(t,x) \in E \mid x \in K^{-1}(y), (t,x) \in P^{-1}(y)\}\\
&\quad \cup \{(t,x) \in (B \times A)\backslash E \mid x \in K^{-1}(y)\}\\
&= [E \cap P^{-1}(y) \cap (B \times K^{-1}(y))] \cup [((B \times A)\backslash E) \cap (B \times K^{-1}(y))]\\
&= [((B \times A)\backslash E) \cup P^{-1}(y)] \cap [B \times K^{-1}(y)],
\end{aligned}
$$

$$
\begin{aligned}
Q^{-1}(z,y) &= \{(t,x) \in B \times A \mid x \in T^{-1}(z), (t,x) \in S^{-1}(y)\}\\
&= S^{-1}(y) \cap (B \times T^{-1}(z)).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
(B \times A)\backslash Q^{-1}(z,y) &= [(B \times A)\backslash S^{-1}(y)] \cup [(B \times A)\backslash(B \times T^{-1}(z))] \\
&= [(B \times A)\backslash S^{-1}(y)] \cup [B \times (A\backslash T^{-1}(z))] \\
&= [E \cap ((B \times A)\backslash P^{-1}(y))] \cup [B \times (A\backslash K^{-1}(y))] \\
&\quad \cup [B \times (A\backslash T^{-1}(z))].
\end{aligned} \tag{3}
$$

We see also that

$$(B \times A)\backslash P^{-1}(y) = \{(t,x) \in B \times A \mid r_i(F(t,y,x), \text{comp}\,(-\text{int}\, C)_k^{\varepsilon})\}$$

is closed, by (ii). This together with the openness in $A$ of $K^{-1}(y)$ and $T^{-1}(z)$ and with (3) show that $Q^{-1}(z,y)$ is open in $B \times A, \forall (z,y) \in B \times A$. By virtue of Theorem 1.1, there exists a fixed point $(\bar{t}, \bar{x})$ of $Q(.,.)$. Suppose that $(\bar{t}, \bar{x}) \in E$. Then $\bar{x} \in P(\bar{t}, \bar{x})$ contradicting (i). Thus $(\bar{t}, \bar{x}) \notin E$. Consequently, by the definitions of $E, S$ and $Q$,

$$\bar{x} \in K(\bar{x}), \bar{t} \in T(\bar{x}), K(\bar{x}) \cap P(\bar{t}, \bar{x}) = \emptyset.$$

Therefore, for all $y \in K(\bar{x})$, we have $y \notin P(\bar{t}, \bar{x})$, i.e.,

$$r_i(F(\bar{t}, y, \bar{x}), \;\; \text{comp}\; (-\text{int} C)_k^{\varepsilon}),$$

which means that $\bar{x}$ is an $\varepsilon$-solution of type $k$ of $(\text{QEP}_{r_i,\alpha_2})$. ■

*Remark 3.* For the special case where $r_i = r_2$ and $\varepsilon = 0$, i.e. we are concerned with the (exact) solution of problem $(\text{QEP}_{r_2,\alpha_2})$. [7, 18] contain results different from Theorem 2.2. The following example gives a case where Theorem 2.2 is applicable but the mentioned results are not.

*Example 4.* Let $X = Y = Z = R, A = B = [0,1], C = R_+$, $T(x) = [0,x]$ and $F(t,y,x) \equiv [0.5, 1]$ and

$$K(x) = \begin{cases} 0 & \text{if } x = 0 \text{ or } x = 1, \\ [0,x) & \text{if } 0 < x < 1. \end{cases}$$

Then all the assumptions of Theorem 2.2 are satisfied for $\varepsilon = 0, k = 1$ and hence solutions of problem $(\text{QEP}_{r_2,\alpha_2})$ exist. However, since $K(.)$ is not continuous, Theorem 3.1 of [18] and Theorem 1 of [7] cannot be employed.

## 3. Applications

It is well-known that quasiequilibrium problems include as special cases many optimization-related problems (see Sec. 1). Therefore, our results in Sec. 2 have direct consequences for these special cases. Here we provide as examples only two such consequences.

### 3.1. Approximate Quasivariational Inequalities

Let $X, Y, A, B$ and $K$ be as in Sec. 1. Let $C$ be a closed cone with nonempty interior. Let $Z = L(X, Y)$, the space of the bounded linear mappings from $X$ into $Y$. Let $g : X \to X$ be a continuous mapping. The following quasivariational inequality has been considered by many authors, see e.g. [10, 12]:

$$\text{(QVI)} \qquad \text{Find } \bar{x} \in A \cap \operatorname{cl} K(\bar{x}) \text{ such that, } \forall y \in K(\bar{x}),$$
$$(T(\bar{x}), y - g(\bar{x})) \cap (Y \setminus -\operatorname{int} C) \neq \emptyset,$$

where $(t, y)$ stands for the value of linear mapping $t \in L(X, Y)$ at $y \in X$.

Now we investigate the following approximate quasivariational inequality:

$$(\text{QVI}^{\varepsilon}) \qquad \text{Find } \bar{x} \in A \cap \operatorname{cl} K(\bar{x}) \text{ such that, } \forall y \in K(\bar{x}),$$
$$(T(\bar{x}), y - g(\bar{x})) \cap \operatorname{comp}(-\operatorname{int} C)_1^{\varepsilon} \neq \emptyset.$$

Applying Theorem 2.1 with $r_i = r_1, k = 1, F(t, y, x) = (t, y - g(x))$ one obtains the following new existence result.

**Corollary 3.1.** *For problem* $(\text{QVI}^{\varepsilon})$ *assume that* $T$ *is usc and compact-valued and* $\operatorname{comp}(-\operatorname{int} C)_1^{\varepsilon}$ *is convex. Then problem* $(\text{QVI}^{\varepsilon})$ *has solutions.*

3.2. Approximate Quasioptimization Problems

Let $X, Y, A, B, K$ and $C$ be as in Subsection 3.1. Let $K$ have closed values. Let $G : A \to B$ be a mapping. The following quasioptimization problem has been studied, e.g. in [9, 19]:

$$\text{(QOP)} \qquad \text{Find } \bar{x} \in K(\bar{x}) \text{ such that, } \forall y \in K(\bar{x}),$$
$$G(y) - G(\bar{x}) \in Y \setminus -\operatorname{int} C.$$

(Then $\bar{x}$ is called a weakly efficient solution).

Let us consider the following approximate quasioptimization problem

$$(\text{QOP}^{\varepsilon}) \qquad \text{Find } \bar{x} \in K(\bar{x}) \text{ such that, } \forall y \in K(\bar{x}),$$
$$G(y) - G(\bar{x}) \in \operatorname{comp}(-\operatorname{int} C)_3^{\varepsilon}.$$

**Corollary 3.2.** *For* $(\text{QOP}^{\varepsilon})$ *assume that*

(i) $\forall b \in B, G(.) - b$ *is* $\operatorname{comp}(-\operatorname{int} C)_3^{\varepsilon}$*-quasiconvex of type 1;*

(ii) $\forall y \in A, \{(t, x) \in B \times A \mid \exists y \in K(x), G(y) - G(x) \notin \operatorname{comp}(-\operatorname{int} C)_3^{\varepsilon})\}$ *and* $\{x \in A \mid G(y) - G(x) \in \operatorname{comp}(-\operatorname{int} C)_3^{\varepsilon}\}$ *are closed;*

(iii) *assume* (iii) *of Theorem* 2.2.

*Then problem* $(\text{QOP}^{\varepsilon})$ *has solutions.*

*Proof.* The corollary is derived from Theorem 2.2 with $r_i = r_1$, $k = 3$ and

$$T(x) \equiv B,$$
$$F(t, y, x) = G(y) - G(x).$$

■

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