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Finite Semigroups with Infinite Product and Languages of Infinite Words

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Abstract. A semigroup (monoid) S, which can be equipped with an infinite product compatible with the given multiplication in S, is called a semigroup (monoid, resp.) with infinite product (abbreviated by SWIP and MWIP, resp.). In this paper, necessary and sufficient conditions for a finite semigroup to be a SWIP are established. Relationships between MWIPs and varieties of finite monoids, and also between MWIPs and regular languages of infinite words are considered.

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1. Introduction

Multiplicative semigroups, on which can be defined an infinite product compatible with the given muptiplication, have been considered in [7] in the viewpoint of universal algebra, where a sufficient condition has been established for a class of infinite semigroups. In this paper, we restrict ourselves to consider this topic for the case of finite semigroups.

In Sec. 2, necessary and sufficient conditions for a finite semigroup to have an infinite product are given, by means of which we show that there exist algorithms to verify, for any finite semigroup S, whether S can be provided with an infinite product or not. To do this, several algebraic results, obtained in studying syntactic semigroups of languges of infinite words [4], have been used.

Varieties of finite semigroups (S-varieties) and of finite monoids (M-varieties) play an important role in the algebraic theory of formal languages (see for example [3]). In Sec. 3, relationships between monoids with infinite product and M-varieties are considered. Namely, we show that the family of finite monoids with strict infinite product constitutes an M-variety consisting of all finite monoids whose Green relation R is trivial. Also, we show that an M-variety can be generated by a family of finite monoids with infinite product if and only if this variety contains the two element monoid $U_1 = \{0, 1\}$ with 1 as its unit element and 0 as zero element.

A fundamental result, due to Arnold [1], says that a language of infinite words (ω-language, for short) is regular if and only if it is recognized by a finite monoid. In Sec. 4, a new form of recognizing ω-languages by finite monoids with infinite product is considered. We show that an ω-language is regular if and only if it is recognized in this form by a finite monoid with infinite product. Also, the family of finite monoids with infinite product and that of syntactic monoids of regular ω-languages (ω-syntactic monoids, for short) are shown to be different.

2. Finite Semigroups with Infinite Product

Definition 2.1. *Given a semigroup S. we denote by S^ω the set of all infinite sequences of elements of S. We say that S is a semigroup with infinite product (SWIP, for short), or S has an infinite product, if there exists a mapping $\alpha : S^\omega \to S$ such that, for any $\mathbf{s} = (s_1, s_2, \dots)$ in S^ω, and any increasing sequence of positive integers $i_1 < i_2 < \dots$, the following two conditions hold true.*

(i) $\alpha(s_1, s_2, \dots) = s_1\alpha(s_2, s_3, \dots)$,

(ii) $\alpha(s_1, s_2, \dots) = \alpha(s_1 \dots s_{i_1}, s_{i_1+1} \dots s_{i_2}, \dots)$.

Then α is called an infinite product in S.

If α satisfies, moreover, the condition

(iii) $\alpha(e, e, \dots) = e$ *for all idempotent e in S,*

then S is called a semigroup with strict infinite product (SWSIP, for short).

Example 1. The set R of all real numbers with the binary operation Max constitutes a semigroup. It is easy to check immediately that the operation Sup is a strict infinite product on R which is compatible with Max.

Example 2. Recall that a semigroup S is a nilpotent semigroup with zero 0 if there exists a natural number n such that $S^n = \{0\}$. It is easy to see that 0 is the unique idempotent of S. We now show that $\alpha : S^\omega \to S$, defined as $\alpha(\mathbf{s}) = 0$ for all $\mathbf{s} \in S^\omega$, is the unique infinite product on S, which is also a strict infinite product. The conditions (i)–(iii) in Definition 2.1, therefore α is a strict infinite product on S. Suppose β is an arbitrary infinite product on S. Let $(s_1, s_2, \dots)$ be an arbitrary element of S^ω. We have

$$\beta(s_1, s_2, \dots) = s_1 \dots s_n\beta(s_{n+1}, s_{n+2}, \dots) = 0.\beta(s_{n+1}, s_{n+2}, \dots) = 0$$

which implies $\beta = \alpha$. Thus α is the unique infinite product on S.

As usual we denote by A^* and A^ω the sets of all words and all infinite words over an alphabet A, respectively, and $A^\infty = A^* \cup A^\omega$. Then, A^∞ becomes a monoid with the multiplication defined as (see [8]):

$$u.v = \begin{cases} u \text{ if } u \in A^\omega, \\ uv \text{ otherwise.} \end{cases}$$

Example 3. It is easy to check that in the monoid A^∞ there is a strict infinite product α defined as:

$$\alpha(u_1, u_2, \dots) = \begin{cases} u_1 u_2 \dots \text{ if all } u_i s \text{ are in } A^*, \\ u_1 u_2 \dots u_k \text{ if } u_1, u_2, \dots, u_{k-1} \in A^*, \text{ and } u_k \in A^\omega. \end{cases}$$

Let S be a semigroup, let $P(S) = \{(e,f) \in S \times S \mid ef = e, ff = f\}$, and let (e,f) and (g,h) be in $P(S)$. We say that (e,f) and (g,h) are *conjugate*, denoted by $(e,f) \simeq (g,h)$, if there exist $p, q \in S$ such that $f = pq$, $h = qp$, and $g = ep$ (hence $e = gq$).

This conjugacy relation is reflexive, symmetric but not transitive in general. We denote by $\equiv$ the transitive closure of $\simeq$, which is an equivalence relation on $P(S)$. The quotient $P(S)/\equiv$ is denoted by $I(S)$. For any (e,f) in $P(S)$, the equivalence class of (e,f) with respect to $\equiv$ is denoted by $[e,f]$.

Denote by S^ω the set of all infinite sequences of elements of S. Let $\mathbf{s} = (s_1, s_2, \dots)$ be in S^ω, let $(e,f) \in P(S)$. We say that (e,f) and $\mathbf{s}$ are *compatible* each with other if there exists an increasing infinite sequence $\{i_j\}_{j\geq 1}$ of positive integers, $i_1 < i_2 < \dots$, such that $s_1 s_2 \dots s_{i_1} = e$, $s_{i_j+1} \dots s_{i_{j+1}} = f$ for all $j \geq 1$. The following fact is well-known as a folklore.

Lemma 2.2. *If S is a finite semigroup then, for every $\mathbf{s} \in S^\omega$, there exists $(e,f) \in P(S)$ which is compatible with $\mathbf{s}$.*

The following has been proved in [4].

Lemma 2.3. [4] *Let S be a finite semigroup, then for any (e,f) and (g,h) in $P(S)$, (e,f) and (g,h) are conjugate if and only if they are both compatible with the same sequence $\mathbf{s}$ in S^ω.*

Corollary 2.4. *Strict infinite product on a finite semigroup S, if exists, is unique.*

Proof. Let α and α' be two arbitrary strict infinite products on S. Let $\mathbf{s} = (s_1, s_2, \dots)$ be an arbitrary element in S^ω. By Lemma 2.3 there exists $(e,f) \in P(S)$ which is compatible with $\mathbf{s}$. By (i)–(iii) in Definition 2.1 we have $\alpha(\mathbf{s}) = \alpha(s_1, s_2, \dots) = \alpha(e, f, f, \dots) = e\alpha(f, f, \dots) = ef = e$. Similarly, we have also $\alpha'(\mathbf{s}) = e$. Hence $\alpha = \alpha'$. ■

The following facts, which are easily verified, are useful in the sequel:

Lemma 2.5. *Let S be a semigroup, $(e,f) \in P(S)$. Let $\mathbf{s} = (s_1, s_2, \dots)$ be an element of S^ω and $\{i_j\}_{j\geq 1}$ an increasing infinite sequence of positive integers.*

Let $s'_1 = s_1 s_2 \dots s_{i_1}$, $s'_{j+1} = s_{i_j+1} \dots s_{i_{j+1}}$ *for all* $j \geq 1$, *and* $\mathbf{s}' = (s'_1, s'_2, \dots)$. *Then*

(i) *If* (e, f) *is compatible with* $\mathbf{s}$, *then, for any* $t \in S$, (te, f) *is compatible with* $(ts_1, s_2, \dots)$.

(ii) *If* (e, f) *is compatible with* $\mathbf{s}'$ *then it is also compatible with* $\mathbf{s}$.

Now we state a necessary and sufficient criterion for a finite semigroup to be a SWIP.

Theorem 2.6. *Let* S *be a finite semigroup. Then*

(i) S *is a SWIP if and only if there exist a left ideal* I *of* S *and a surjection* $h : I(S) \to I$ *such that*

$$s.h([e, f]) = h([se, f]) \text{ for all } s \in S, [e, f] \in I(S). \tag{1}$$

(ii) *If there exist* I *and* h *satisfying* (1) *then an infinite product* α *on* S *can be defined by*

$$\alpha(\mathbf{s}) = h([e, f]), \tag{2}$$

where $\mathbf{s}$ *is in* S^ω *and* (e, f) *is any element in* $P(S)$ *which is compatible with* $\mathbf{s}$.

Proof. Suppose S has an infinite product $\alpha : S^\omega \to S$. We define I and $h : I(S) \to I$ as follows

$$I = \alpha(S^\omega), \tag{3}$$

$$h([e, f]) = \alpha(\mathbf{s}), \tag{4}$$

with $(e, f) \in P(S)$ and $\mathbf{s}$ any sequence compatible with (e, f).

The fact that I, defined by (3), is a left ideal of S is due to (i) in Definition 2.1. Now we show that h is well-defined by (4). For this, it suffices to show that, for any $(e, f), (e', f') \in P(S)$ and for any $\mathbf{s}, \mathbf{s}' \in S^\omega$, which are compatible with (e, f) and (e', f') respectively, $(e, f) \simeq (e', f')$ implies $\alpha(\mathbf{s}) = \alpha(\mathbf{s}')$. Indeed, by (ii) in Definition 2.1, we have $\alpha(\mathbf{s}) = \alpha(e, f, f, \dots)$, $\alpha(\mathbf{s}') = \alpha(e', f', f', \dots)$. By Lemma 2.3, there exists $\mathbf{t} \in S^\omega$ which is compatible with both (e, f) and (e', f'). Therefore, again by (ii) in Definition 2.1, we have $\alpha(\mathbf{t}) = \alpha(e, f, f, \dots)$ and $\alpha(\mathbf{t}) = \alpha(e', f', f', \dots)$. It follows that $\alpha(e, f, f, \dots) = \alpha(e', f', f', \dots)$, hence $\alpha(\mathbf{s}) = \alpha(\mathbf{s}')$. By Lemma 2.2, h is surjective. From (4), (i) in Definition 2.1 and Lemma 2.5(i) it follows that

$$s.h([e, f]) = s.\alpha(s_1, s_2, \dots) = \alpha(s, s_1, s_2, \dots) = h([se, f])$$

which means h satisfies (1).

Conversely, suppose there are a left ideal I of S and a surjection $h : I(S) \to I$ satisfying (1). Define α as in (2). By the condition (1), h satisfies (i) in Definition 2.1. Let $\mathbf{s} = (s_1, s_2, \dots)$ be any element of S^ω, and $\{i_j\}_{j \geq 1}$ be an increasing infinite sequence of positive integers. Let $(e, f) \in P(S)$ is compatible with $\mathbf{s}$. By the definition of α and Lemma 2.5(ii) we have

$$\alpha((s_1 \dots s_{i_1}), (s_{i_1+1} \dots s_{i_2}), \dots) = h([e, f]) = \alpha(s_1, s_2, \dots),$$

which means that h satisfies (ii) in Definition 2.1. Thus α is an infinite product on S, i.e. S is a $SWIP$. This completes the proof. ∎

From the above theorem we get

Corollary 2.7. *One can decide, for any given finite semigroup S, whether S is a SWIP or not.*

Proof. The deciding algorithm consists of the following steps:
1. Find all possible left ideals I of S;
2. Compute $I(S) = P(S)/\equiv$;
3. For every I, find all possible surjections $h : I(S) \to I$;
4. For every such a surjection h, verify whether the condition (1) holds. If yes, S is a $SWIP$, otherwise, it isn't. ∎

For strict infinite product we have the following similar result.

Theorem 2.8. *Let S be a finite semigroup. Then, S is a SWSIP if and only if there exist a left ideal I of S and a surjection $h : I(S) \to I$ satisfying* (1) *in Theorem* 2.6 *and also the following condition*

$$h([e, f]) = e \textit{ for all } (e, f) \in P(S). \tag{5}$$

Proof. The proof is similar to that of Theorem 2.6 except for verifying (5) for "only part", and (iii) in Definition 2.1 for "if part". But this is immediate from the following derivations, according to the case:

$$h([e, f]) = \alpha(e, f, f, \ldots) = e.\alpha(f, f, \ldots) = ef = e',$$
$$\alpha(g, g, \ldots) = h([g, g]) = g,$$

where $(e, f) \in P(S)$ and g is any idempotent. ∎

Corollary 2.9. *One can decide whether, for any given finite semigroup S, S is a SWSIP or not.*

Proof. It suffices to apply the following verifying algorithm:
1. Find all possible left ideals I of S;
2. Compute the quotient $I(S) = P(S)/\equiv$;
3. For every I find all possible surjections $h : I(S) \to I$;
4. For every such a surjection h, check whether the conditions (1) and (5) hold true. If yes then S is a $SWSIP$, otherwise it isn't.

The following corollaries give examples of SWIPs and SWSIPs. ∎

Corrolay 2.10. *Let S be a finite cyclic semigroup without unit, generated by a, with i and p as its index and period, respectively. Then, S is a SWIP if and only if $p = 1$. In that case, S has a unique infinite product, which is also a strict infinite product, α, defined as $\alpha(\mathbf{s}) = a^i$ for all $\mathbf{s} \in S^\omega$.*

Proof. By the definition of S we have $a^{i+p} = a^i$, and $S = \{a, a^2, \dots, a^i, a^{i+1}, \dots, a^{i+p-1}\}$. Denote the unique idempotent of S by e. The minimality in defining i implies $a^j.e = a^j$ iff $j \geq i$, i.e. $P(S) = \{(a^j, e) \mid j \geq i\}$. Being compatible with the same sequence $(a, a, \dots)$, all the elements of $P(S)$ are conjugate. Therefore $I(S)$ consists of only one element, $I(S) = \{[a^i, e]\}$. Note that S has exactly two left ideals which are $I_1 = \{a^j \mid i \leq j \leq i+p-1\}$ and $I_2 = S$. Therefore, there exists a surjection h from $I(S)$ onto a left ideal of S iff either $p = 1$ or $|S| = 1$, which, in turn, implies $p = 1$ too. In both cases the surjection h is unique and $h([a^i, e]) = a^i$ ($i = 1$ when $|S| = 1$). Since $p = 1$, $a^i = e$. Again by $p = 1$, for any j, $a^j.a^i = a^i$, which means $a^j.h([a^i, e]) = h([a^j.a^i, e])$. Thus, by Theorem 2.6, S is a $SWIP$ iff $p = 1$. In such a case, by Theorem 2.6(ii), an infinite product on S can be defined as $\alpha(s_1, s_2, \dots) = h([a^i, e]) = a^i$. Obviously, $\alpha(e, e, \dots) = h([a^i, e]) = a^i = e$. So, α is also a strict infinite product. The uniqueness of the surjection h implies the uniqueness of infinite product on S. ■

Corollary 2.13. *A finite multiplicative group G has an infinite product (strict infinite product) if and only if G is a trivial group. In that case, on G there is a unique infinite product which is also a strict infinite product.*

Proof. The unit element 1 is the unique idempotent of G, and G is the unique left ideal of itself. Evidently $P(G) = \{(p, 1) \mid p \in G\}$. Any two elements $(p, 1)$ and $(q, 1)$ are compatible with the same sequence $(p, p^{-1}q, q^{-1}p, p^{-1}q, \dots)$, hence they are conjugate. Therefore $I(G)$ consists of only one element, $I(G) = \{[1, 1]\}$. Thus there is a surjection h from $I(G)$ onto G iff $|G| = 1$, i.e. G is a trivial group, $G = \{1\}$. In that case, the unique surjection h, given by $h([1, 1]) = 1$, determines a unique infinite product α with $\alpha(1, 1, \dots) = 1$, which is evidently also a strict infinite product. Thus G is a SWIP (SWSIP, resp.) iff G is a trivial group. ■

3. Finite $MWIPs$ and M-Varieties

An *M-variety* is a family of finite monoids which is closed under finite direct product, homomorphism and taking submonoid. The Green relation R in a monoid M is defined as: mRm' iff $mM = m'M$ or, equivalently, mRm' iff $\exists x, y \in M$ such that $m = m'x, m' = my$. It is well-known that the family of finite monoids with R trivial constitutes an M-variety.

Theorem 3.1. *The family of all finite monoids with strict infinite product coincides with the M-variety of the finite monoids whose Green relation R is trivial.*

Proof. Let M be a finite monoid with strict infinite product. By Theorem 2.8, there exist a left ideal I of M and a surjection $h : I(M) \to I$ satisfying (3) and (5). Let $e, e' \in M$ be such that eRe'. There exist then $x, y \in M$ such that $e' = ex, e = e'y$. Since M is finite, there exist natural numbers $i, j \geq 1$ such that $(xy)^i$ and $(yx)^j$ are idempotents of M. Put $f = (xy)^i, f' = (yx)^j$, we have $(e,f),(e',f') \in P(M)$. Because the sequence $(e,x,y,x,y\dots)$ is obviously compatible with both (e,f) and (e',f'), we have $(e,f) \simeq (e',f')$, hence $[e,f] = [e',f']$. From (5) it follows that $e = e'$, which means R is trivial.

Conversely, suppose M is a finite monoid with R trivial. By Lemma 2.2, for every sequence $\mathbf{s} = (s_1, s_2, \dots)$ in M^ω, there exists $(e,f) \in P(M)$ compatible with $\mathbf{s}$. Define

$$\alpha(\mathbf{s}) = e.$$

If $(e,f) \simeq (e',f')$ then $e' = ex, e = e'y$ for some $x,y \in M$, therefore, since R is trivial, $e = e'$. Thus, α is well defined. By virtue of Lemma 2.5, it is easy to check that α is an infinite product on M. For any idempotent e in M, the sequence $(e,e,\dots)$ is compatible with (e,e), therefore $\alpha(e,e,\dots) = e$. Thus α is a strict infinite product on M, i.e. M is a finite monoid with strict infinite product. ∎

Corollary 3.2. *The class of finite MWSIPs is stricly included in the class of finite MWIPs.*

Proof. Let us take a finite monoid M with non-trivial Green relation R. Put $I = I(M)$. Without loss of generality we may assume $I \cap M = \emptyset$. We provide I with a multiplication $\circ$ such that I becomes a semigroup of left zeros: $u \circ v = u$ for all $u, v \in I$. Put $M' = M \cup I$ and provide M' with the multiplication $*$ given as

$$u * v = u \circ v, u * m = m * u = u, m * m' = mm\text{'}, \forall\, u, v \in I, \text{ and } \forall\, m, m' \in M.$$

It is easy to check that M' is a finite monoid containing M as a submonoid. We define $\alpha : M'^\omega \to M'$ as follows

$$\alpha(s_1, s_2, \dots) = \begin{cases} s_i \in I, \text{ if } \{s_1, s_2, \dots\} \cap I \neq \emptyset, i \text{ is the smallest index} \\ [e,f], \text{ otherwise }, (e,f) \text{ is compatible with } (s_1, s_2, \dots). \end{cases}$$

It is easy to check that α is an infinite product on M'. Since the relation R is not trivial in M, it is not trivial in M' either. By Theorem 3.1, M' has no strict infinite product. ∎

It appears that every M-variety V can be generated by a family of MWIPs, except for when V is a variety of groups. More precisely we have

Theorem 3.3. *An M-variety V can be generated by a family of finite MWIPs if and only if V contains the monoid U_1. If V does not contain U_1 then the trivial monoid* 1 *is the unique monoid in V which has an infinite product.*

Proof. Suppose $\mathbf{V}$ contains U_1. For any monoid M in $\mathbf{V}$ we denote by M^0 the monoid obtained from M by adding a new element 0 as zero. Since M^0 is isomorphic to the Rees quotient of $M \times U_1$ by $M \times \{0\}$, $M^0 \in \mathbf{V}$. It is easy to see that α, defined as $\alpha(\mathbf{s}) = 0$ for all $\mathbf{s} \in {M^0}^\omega$, is an infinite product on M^0. It follows that $\mathbf{V}$ is generated by a family of monoids with infinite product. Conversely, if $\mathbf{V}$ does not contain U_1, then $\mathbf{V}$ consists of only finite groups (see

[3, 5]). Therefore, as known in Corollary 2.11, in **V** the unique monoid having infinite product is the trivial group. ■

4. Finite *MWIPs* and Regular ω-Languages

Given an alphabet A. Any subset of A^ω is called an ω-language over A. Let $h : A^* \to M$ be a monoid morphism from the free monois A^* into a monoid M. We say that the morphism h *saturates* an ω-languge L if, for any $(p,q) \in M \times M$,

$$h^{-1}(p)[h^{-1}(q)]^\omega \cap L \neq \emptyset \Rightarrow h^{-1}(p)[h^{-1}(q)]^\omega \subseteq L.$$

If L is saturated by $h : A^* \to M$ we say also that L is recognized by the morphism h or by the monoid M. A fundamental result, due to Arnold [1] (see also [4]), says that

Lemma 4.1. [1] *ω-language L over a finite alphabet A is regular if and only if there exist a finite monoid M and a morphism $h : A^* \to M$ saturating L.*

The following result has been proved in [4].

Lemma 4.2. [4] *Let L be an ω-language over a finite alphabet A. If L is saturated by the morphism $h : A^* \to M$ from A^* into a finite monoid M then L can be represented in the form*

$$L = \cup_{(e,f)\in J} h^{-1}[e,f],$$

where

$$h^{-1}[e,f] = \cup_{(p,q)\in P(M),(p,q)\equiv(e,f)} h^{-1}(p)[h^{-1}(q)]^\omega$$

and

$$J = \{(p,q) \in P(M) : h^{-1}[p,q] \cap L \neq \emptyset\}.$$

Defintion 4.3. *Let L be an ω-language over a finite alphabet A. Let M be a finite monoid having an infinite product $\alpha : M^\omega \to M$, and let $f : A^* \to M$ be a monoid morphism. Define the mapping $f_\alpha : A^\omega \to M$ like as: for $w \in A^\omega$, say $w = a_1a_2\ldots$ with $a_i \in A$, $f_\alpha(w) = \alpha(f(a_1), f(a_2), \ldots)$. Then, L is said to be ω-recognized by f, if there exists a subset B of M such that $L = \cup_{b\in B} f_\alpha(b)$. An ω-language is called ω-recognizable if it is ω-recognized by some morphism f.*

Theorem 4.4. *Let L be an ω-language over a finite alphabet A. Then, L is regular if and only if it is ω-recognizable.*

Proof. Suppose L is ω-recognizable. Then there exist a finite monoid M having an infinite product α, a morphism $f : A^* \to M$ and a subset B of M such that $L = {f_\alpha}^{-1}(B)$. By Theorem 2.6, there exist a left ideal I of M and a surjection $h : I(M) \to I$ satisfying (1) and such that, for any $\mathbf{s} \in M^\omega$, $\alpha(\mathbf{s}) = h([p,q])$, where (p, q) is any couple compatible with $\mathbf{s}$. Then, for any $b \in B$ we have

$$\begin{aligned} f_\alpha^{-1}(b) &= \{w = a_1 a_2 \cdots \in A^\omega \mid f_\alpha(w) = b\} \\ &= \{w = a_1 a_2 \cdots \in A^\omega \mid \alpha(f(a_1), f(a_2), \dots) = b\} \\ &= \{w = a_1 a_2 \cdots \in A^\omega \mid h([p,q]) = b,\ (p,q) \in P(M) \text{ and compatible} \\ &\quad \text{with } (f(a_1), f(a_2), \dots)\} \\ &= \{w \in f^{-1}(p)[f^{-1}(q)]^\omega \mid (p,q) \in P(M), h([p,q]) = b\} \\ &= \{w \in f^{-1}(p)[f^{-1}(q)]^\omega \mid (p,q) \in P(M), \alpha(p,q,q,\dots) = b\} \\ &= \cup_{(p,q)\in P(M) \& \alpha(p,q,q,\dots)=b} f^{-1}(p)[f^{-1}(q)]^\omega. \end{aligned}$$

It follows that L is a finite union of ω-languages of the form $f^{-1}(p)[f^{-1}(q)]^\omega$ which are all regular (see for example [3]). Namely

$$L = \cup_{b\in B} \cup_{(p,q)\in P(M)\&\alpha(p,q,q,\dots)=b} f^{-1}(p)[f^{-1}(q)]^\omega.$$

Hence L itself is regular.

Conversely, suppose L is a regular ω-language. Let $f : A^* \to M$ be a morphism saturating L. Consider the disjoint union $U = M \cup I(M) \cup \{0\}$, where 0 is a new symbol. On U we define a multiplication like as

$$x.y = \begin{cases} [sp, q] \text{ if } x = s \in M, \text{ and } y = [p,q] \in I(M), \\ x \text{ if } y = 1, \\ y \text{ if } x = 1, \\ 0, \text{ otherwise,} \end{cases}$$

where 1 is the unit in M. It is easy to check that with such a multiplication U becomes a monoid. Next, for any $\mathbf{x} \in U^\omega$ we put

$$\alpha(\mathbf{x}) = \begin{cases} [p,q] \text{ if } \mathbf{x} \in M^\omega, \text{ where } (p,q) \in P(M) \text{ is any couple compatible with } \mathbf{x}, \\ 0, \text{ otherwise.} \end{cases}$$

It is easy to see that α is an infinite product on U. Now, f may be considered as a morphism from A^* into the finite monoid with infinite product U. By Lemma 4.2 and the definition of f_α we have

$$\begin{aligned} L &= \cup_{(p,q)\in J} {f_\alpha}^{-1}[p,q] \\ &= \cup_{(p,q)\in J} \{w = a_1 a_2 \cdots \in A^\omega \mid \alpha(f(a_1), f(a_2), \dots) = [p,q]\} \\ &= \cup_{(p,q)\in J} {f_\alpha}^{-1}([p,q]), \end{aligned}$$

where $J = \{(p,q) \in P(M) \mid f^{-1}[p,q] \cap L \neq \emptyset\}$. By putting $B = \{[p,q] \mid (p,q) \in J\}$, it follows that

$$L = \cup_{[p,q]\in B} {f_\alpha}^{-1}([p,q]) = {f_\alpha}^{-1}(B).$$

Thus, L is ω-recognizable. ■

The following fact can be proved in a similar way as in the second part of the proof of the above theorem.

Corollary 4.5. *Let L be an ω-language over a finite alphabet A. Let M be a finite monoid having an infinite product $\alpha : M^\alpha \to M$, and $h : A^* \to M$ a monoid morphism. If L is ω-recognized by h then there exists a subset B of M such that L can be represented in the form*

$$L = \cup_{(e,f)\in J} h^{-1}[e,f]$$

with

$$J = \{(e,f) \in P(M) : \alpha(e,f,f,\dots) \in B\}.$$

The above results show that, in some sense, MWIPs are as powerful as syntactic monoids of regular ω-languages (ω-syntactic monoids, for short) in recognizing languages as well as in generating M-varieties. To make clear relative positions between these two classes of monoids we need some notions and results in [4].

Given a finite monoid M. A subset J of $M \times M$ is said to be *closed* under $\simeq$ if, for any $(p,q),(p',q') \in M \times M$,

$$(p,q) \simeq (p',q') \& (p,q) \in J \;\Rightarrow\; (p',q') \in J.$$

With every subset J of $M \times M$ we associate a congruence $\approx_J$ on M defined as:

$$m \approx_J m' \text{ iff } \forall\, p,q,r \in M \begin{cases} (pmq,r) \in J \Leftrightarrow (pm'q,r) \in J, \\ (r,pmq) \in J \Leftrightarrow (r,pm'q) \in J. \end{cases}$$

We denote by $\phi : M \times M \to P(M)$ the application mapping every (p,q) in $M \times M$ into (pq^k, q^k) in $P(M)$, where k is a positive integer such that q^k is an idempotent.

The smallest finite monoid recognizing a regular ω-language L is called syntactic monoid of L. We call ω-syntactic any monoid which is syntactic monoid of some regular ω-language. Let M be a finite monoid. A subset I of $P(M)$ is called *ω-rigid* if I is closed under $\simeq$ and $\approx_{\phi^{-1}(I)}$ is an identity relation.

Lemma 4.6. [4] *A finite monoid M is ω-syntactic iff there exits a subset I of $P(M)$ which is ω-rigid.*

Theorem 4.7. *The class L_1 of all finite MWIPs and the class L_2 of all ω-syntactic monoids are different.*

Proof. Let $U_1 = 0,1$ be the two element multiplicative monoid with 1 as unit element and 0 as zero element. As known (see [4, Example 2.6]) U_1 is in L_2. We now show that U_1 is in L_1 too. Evidently $P(U_1) = \{(0,1),(1,1)\}$, and $I(U_1) = \{[0,1],[1,1]\}$. Also, $\{0\}$ is a left ideal of U_1. It is easy to check that the mapping $h : I(U_1) \to \{0\}$, defined by $h[0,1] = h[1,1] = 0$, is a surjection satisfying (1) in Theorem 2.6. Hence U_1 is a MWIP. Thus $L_1 \cap L_2 \neq \emptyset$.

Consider the cyclic monoid M generated by a with 2 as its index and 3 as its period. Then $M\{1 = a^0, a, a^2, a^3, a^4\}$, $a^j a^3 = a^j$ for all $j \geq 3$, and $e = a^3$ is an idempotent. It is easy to check that

$$P(M) = \{(1,1)\} \cup \{(a^k,1) \mid 1 \le k \le 4\} \cup \{(a^j,e) \mid j \ge 3\},$$
$$I(M) = \{[1,1]\} \cup \{[a^k,1] \mid 1 \le k \le 4\} \cup \{[e,e]\},$$

where each equivalence class in $P(M)$ (i.e. each element of $I(M)$) consists of only one couple. Put $I = \{(a^2,1),(a^4,1)\}$. Evidently I is closed under $\simeq$. It is easy to check that $\phi^{-1}(I) = I$. An easy computation shows that any two different elements u and v of M are not $\approx_I$. Indeed, we have, for example, $(a^3.a.1,1) = (a^4,1) \in I$ whereas $(a^3.a^3.1,1) = (a^3,1) \notin I$ which imply $a \not\approx_I a^3$. This means that $\approx_I$ is an identity relation. Thus, I is an ω-regid set, and therefore, by Lemma 4.3 M is an ω-syntactic monoid. Suppose M has an infinite product. By Theorem 2.6, there exist a left ideal of M and a surjection $h : I(M) \to I$ satisfying (1). It follows that, for any $j \ge 0$, $a^j h[e,e] = h[a^j e,e] = h[e,e]$, which means that $h[e,e]$ is a right zero of M. But such an element does not exist. Thus M has no infinite product, i.e. $M \in L_2 - L_1$.

Let $S = \{p,q,r\}$ be the semigroup of right zeros defined as $xp = p, xq = q, xr = r$ for all $x \in S$. Let M be the monoid obtained by adding to S a unit, $M = S \cup \{1\}$. As known in [4] (Example 2.6) $M \notin L_2$. Since $\{p\}$ is a left ideal of M, again Theorem 2.6 allows us to construct an infinite product on M. Thus $M \in L_1 - L_2$. ∎

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