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Survey

Singular Difference Equations: An Overview*

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Dedicated to Professor Hoang Tuy on the occasion of his 80th-birthday

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Abstract. This paper gives a brief survey of the development to the theory of singular difference equations (SDEs) and singular stochastic difference equations (SSDEs). It focuses on our recent works on index notions of SDEs, the solvability of initial-value problems (IVPs) and multipoint boundary-value problems (MPBVPs), the stability and robust stability of solutions of SDEs.

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1. Introduction

In recent years, there has been a great interest in SDEs (also referred to as descriptor systems, implicit difference equations) because of their appearance in many practical areas, such as the Leontiev dynamic model of multisector economy, the Leslie population growth model, singular discrete optimal control problems and so forth (see [12 - 14]). On the other hand, SDEs occur in a natural way of using discretization techniques for solving differential-algebraic equations

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(DAEs) and partial differential-algebraic equations, which have already attracted much attention of researchers (cf. [12, 13, 25, 28]).

However, until now only linear SDEs with constant coefficients have been studied thoroughly (see [12–14] and references therein). Clearly, many results for linear constant coefficient SDEs cannot be directly generalized to linear time varying SDEs.

Bondarenko and his colleagues in [9 - 11] considered a special class of implicit nonautonomous difference equations $T_n x_{n+1} + x_n = f_n$, where T_n are degenerated matrices and established the solvability of IVPs and periodic boundary-value problems (BVPs) for this special class of SDEs.

The index notion for linear SDEs with varying coefficients $A_n x_{n+1} + B_n x_n = q_n$ was introduced in [6, 16, 31] and the solvability of IVPs as well as MPBVPs are studied in [1 - 5, 30, 31]. Later on, the index notion has been extended to nonlinear cases $f_n(x_{n+1}, x_n) = 0$ [5]. Further, the notion of quasi-index and strangeness index for linear SDEs have been introduced in [7, 27, 30, 31].

There is a close relation between linear SDEs and linear DAEs, namely, the explicit Euler method applied to a linear index-1 DAE leads to a linear index-1 SDE (see [1, 3]). Moreover the unique solutions of the discretized IVP/BVP converge to the solutions of the corresponding continuous problems. Further, it is proved that every linear index-1 SDE can be reduced to the so-called Kronecker normal form and a periodic linear index-1 SDE can be transformed to an SDE with constant coefficients. The Floquet theory, first established for regular ODEs, and later for difference equations and recently for DAEs [29], has been developed for index-1 SDEs in [6]. This theory can be used for investigating the stability of nonlinear SDEs and periodically switched singular discrete systems. The Lyapunov function method has been studied for singular quasilinear difference equations in [4]. A formula for the stability radius of a linear index-1 SDE with constant coefficients, when all the coefficients of the system including the leading one, are perturbed has been established in [20].

In [15] an IVP for the SSDE $A(\xi_n)X(n+1) = B(\xi_n)X(n) + q_n,\ X(0) = x_0 \in \mathbb{R}^m,\ n \in \mathbb{N}$, where $\{\xi_n : n \in \mathbb{N}\}$ is an i.i.d. sequence with values in a Polish space has been studied. The concept of index-1 is introduced for characterizing the set of initial values, for which the SSDE has explicit solutions. It is assumed that the above SSDE is of Furstenberg-Kifer type. Furstenberg-Kifer decompositions for the corresponding homogeneous SSDE are proved. The existence of bounded solutions to nonhomogeneous SSDE when the random variables q_n satisfy certain moment conditions has been established.

An outline of the remainder of the paper is as follows: In Sec. 2 we introduce the tractability index notion for linear SDEs. Its relationship with the index notion for linear DAEs is revealed. Further, the quasi-index 1 notion for linear SDEs and the tractability index concept for nonlinear SDEs are established. Then, the solvability and unique solvability for IVPs and MPBVPs involving linear index-1 SDEs are studied. Sec. 3 deals with qualitative study of SDEs, ranging from the Floquet theory, the Lyapunov function methods to the robust stability of linear SDEs. Finally in Sec. 4 we describe the development of the theory of SSDEs. The Lyapunov exponents of solutions of SSDEs are introduced and the multiplicative ergodic theorem as well as the Furstenberg-Kifer decomposition for linear SSDEs are established.

2. Index-1 Tractable Singular Difference Equations

2.1. The Tractability Index of a Singular Difference Equation

We begin this subsection by considering a linear SDE

$$A_n x_{n+1} + B_n x_n = q_n, \quad n \geq 0, \tag{2.1}$$

where the data $A_n, B_n \in \mathbb{R}^{m\times m}$, $q_n \in \mathbb{R}^m$ are given, and the matrices A_n are singular for all $n \geq 0$.

Together with (2.1) we consider a linear DAE

$$A(t)x' + B(t)x = q(t), \quad t \in J := [t_0, T], \tag{2.2}$$

where $A, B \in C(J, \mathbb{R}^{m\times m})$ are given continuous matrix functions, $q \in C(J, \mathbb{R}^m)$ is a given vector function, and $A(t)$ is a singular matrix for every $t \in J$.

It is well known that the singularity of $A(t)$ may cause some serious trouble in handling Eq. (2.2), such as the instability of numerical methods or the inconsistence of initial values, and so forth.

Many authors have introduced various notions of index of a given DAE, such as the global index, the tractability index, the differentiation index, the perturbation index and the strangeness index. These index notions may be considered as measures of the singularity of DAEs.

To define the tractability index of DAE (2.2), the following subspaces are introduced [25]:

$$\mathcal{N}(t) = \ker A(t), \quad \mathcal{S}(t) = \{z \in \mathbb{R}^m : B(t)z \in \operatorname{im} A(t)\},$$

where $\mathcal{N}(t)$ is supposed to be smooth in t. Obviously, the subspace $\mathcal{S}(t)$ contains all the solutions of the corresponding homogeneous equation

$$A(t)x' + B(t)x = 0.$$

It can be proved that the smoothness of $\mathcal{N}(t)$ and the existence of a C^1- projection onto $\mathcal{N}(t)$ are equivalent [32]. Therefore, in what follows we will use C^1- projection $Q(t)$ satisfying conditions

$$Q \in C^1(J, \mathbb{R}^{m\times m}), \quad Q(t)^2 = Q(t), \quad \operatorname{im} Q(t) = \mathcal{N}(t), \quad \forall\, t \in J.$$

Further, let $P := I - Q$, where I denotes the $m \times m$ identity matrix.

Definition 2.1. [25] *The* DAE (2.2) *is called index*-1 *tractable on* J *if*

$$\mathcal{N}(t) \oplus \mathcal{S}(t) = \mathbb{R}^m, \quad \forall\, t \in J. \tag{2.3}$$

The index-1 tractability implies the regularity of the matrix $G(t) = A(t) + B(t)Q(t)$ for all $t \in J$. Moreover, the matrix $P_{\mathrm{can}}(t) = I - Q(t)G^{-1}(t)B(t)$

represents the canonical projection onto $\mathcal{S}(t)$ along $\mathcal{N}(t)$. Besides, the subspace $\mathcal{S}(t) = \operatorname{im} P_{\mathrm{can}}(t)$ is filled in by solutions of the corresponding homogeneous equation. From the smoothness of $\mathcal{N}(t) = \ker A(t)$ it also implies that $\operatorname{rank} A(t) = \mathrm{const}, t \in J$.

Coming back to the linear SDE (2.1), we have introduced the following definition [6, 16, 31].

Definition 2.2. *The linear* SDE (2.1) *is called index*-1 *tractable if the following conditions hold*

(i) $\operatorname{rank} A_n = r \quad (n \geq 0)$,
(ii) $\mathcal{S}_n \cap \mathcal{N}_{n-1} = \{0\} \quad (n \geq 1)$,
where $\mathcal{N}_{n-1} = \ker A_{n-1}$ *and* $\mathcal{S}_n = \{z \in \mathbb{R}^m : B_n z \in \operatorname{im} A_n\}$.

In what follows we always assume that $\dim \mathcal{S}_0 = r$ and let $A_{-1} \in \mathbb{R}^{m\times m}$ be any fixed matrix satisfying the relation $\mathbb{R}^m = \mathcal{S}_0 \oplus \ker A_{-1}$. Thus, the condition (ii) in Definition 2.2 holds for all $n \geq 0$.

Since $\operatorname{rank} A_n = \operatorname{rank} A_{n-1}$, there exists an invertible operator T_n, whose restriction onto $\mathcal{N}_n$ is an isomorphism between $\mathcal{N}_n$ and $\mathcal{N}_{n-1}$. Such an operator can be constructed as follows. Let $Q_n \in \mathbb{R}^{m\times m}$ be any projection onto $\mathcal{N}_n$. There is a nonsingular matrix $V_n \in \mathbb{R}^{m\times m}$, such that $Q_n = V_n \widetilde{Q} V_n^{-1}$, where $\widetilde{Q} = \operatorname{diag}(O_r, I_{m-r})$ and O_r, I_{m-r} stand for $r \times r$ zero and $(m-r)\times(m-r)$ identity matrices, respectively. Denote again by T_n the matrix induced by the operator T_n. Clearly $T_n = V_{n-1}V_n^{-1}$. Further, $T_n Q_n = V_{n-1}\widetilde{Q}V_n^{-1} =: Q_{n-1,n}$. The latter is called a connecting operator associated with projections Q_n and Q_{n-1} (see [5, 6]). The following lemma plays an important role in the theory of linear index-1 tractable SDEs (cf. [1-6, 16, 31]).

Lemma 2.3. *The following assertions are equivalent:*

(i) $\mathcal{S}_n \cap \mathcal{N}_{n-1} = \{0\}$;
(ii) *the matrix* $G_n = A_n + B_n T_n Q_n$ *is nonsingular;*
(iii) $\mathbb{R}^m = \mathcal{S}_n \oplus \mathcal{N}_{n-1}$.

From Lemma 2.3 it follows that the index-1 notion of a linear SDE as well as the nonsingularity of the matrix G_n do not depend on the choice of projection Q_n and transformation T_n. Thus, it is sufficient to restrict ourselves to orthogonal projections as it was done in [1, 2, 31]. However the use of orthogonal projections seem to be inconvenient for a further extension of the index notion to nonlinear SDE and to SSDE. A device for overcoming this difficulty was proposed in [3-6, 16], where instead of orthogonal projections, an arbitrary projection onto $\mathcal{N}_n$ and a discribed above transformation T_n were considered. Although the index-1 notions of linear DAEs and linear SDEs are similar, it is not difficult to reveal their differences. Firstly, in the discrete case besides projections onto $\mathcal{N}_n$, we need to consider the so-called connecting operators $Q_{n-1,n}$, or operators $T_n : \mathbb{R}^m \to \mathbb{R}^m$, whose restrictions on $\mathcal{N}_n$ are isomorphisms between $\mathcal{N}_n$ and $\mathcal{N}_{n-1}$. Secondly, the index-1 of the pencil $\{A(t), B(t)\}$ for a linear index-1 tractable DAE (2.2) always equals 1, while the pencil $\{A_n, B_n\}$ of a linear index-1 tractable

SDE (2.1) is not necessarily of index-1. An example of such SDEs can be found in [6, 30]. These differences require new techniques for dealing with linear SDEs.

On the other hand, there exists a close relationship between two index-1 tractability notions, namely an explicit discretization method applied to linear index-1 DAE must lead to a linear index-1 SDE. More precisely the following result has been established in [1].

Theorem 2.4. *Suppose that the linear index-*1 *tractable* DAE (2.2) *is discretized by the explicit Euler method*

$$A_n \frac{x_{n+1} - x_n}{\tau} + B_n x_n = q_n, \quad n = 0, \ldots, N-1,$$

or equivalently,

$$A_n x_{n+1} + (\tau B_n - A_n) x_n = \tau q_n, \quad n = 0, \ldots, N-1. \tag{2.4}$$

Then for a sufficiently small stepsize τ*, Eq.* (2.4) *is a linear index-*1 *tractable* SDE.

From the vast literature on DAEs, it seems that the rank-constancy of leading coefficients $A(t)$ is a minimal requirement for the successfull treatment of linear DAEs. It is interesting to remark that a class of linear SDEs, whose leading coefficients have varying ranks, were studied in [7, 31].

Consider Eq. (2.1) and assume that $r_n = \operatorname{rank} A_n$, where $0 < r_n < m$ for all $n \geq 0$. Let $A_n = U_n \Sigma_n V_n^T$ be a singular value decomposition (SVD) of A_n, where U_n (resp. V_n) are orthogonal matrices, whose columms are left (resp. right) singular vectors of A_n respectively, and $\Sigma_n = \operatorname{diag}(\sigma_1^{(n)}, \sigma_2^{(n)}, \ldots, \sigma_{r_n}^{(n)}, 0, \ldots, 0)$ are diagonal matrices with singular values of A_n, $\sigma_1^{(n)} \geq \sigma_2^{(n)} \geq \cdots \geq \sigma_{r_n}^{(n)} > 0$ on their diagonals.

Let $Q^{(n)} = \operatorname{diag}(O_{r_n}, I_{m-r_n})$ and $P^{(n)} = I - Q^{(n)}$. The following definition was introduced in [7, 31].

Definition 2.5. *The linear* SDE (2.1) *is called quasi-index-*1 *tractable, if for all* $n \geq 0$ *there hold two conditions*

(i) $0 < r_1 \leq \cdots \leq r_n \leq \cdots \leq r < m$.
(ii) *There exists a connecting operator* $Q_{n,n-1} = V_n Q^{(n)} V_{n-1}^T$ *such that* $Q_{n,n-1}(\mathcal{S}_n \cap \mathcal{N}_{n-1}) = \{0\}$.
Here A_{-1} *is an arbitrary fixed matrix satisfying the relation* $\mathcal{S}_0 \cap \mathcal{N}_{-1} = \{0\}$.

It can be proved that the quasi-index-1 tractability of a linear SDE is invariant under scaling and linear transformations by nonsingular matrices. Obviously, a linear quasi-index-1 tractable SDE with constant-rank leading coefficients is index-1 tractable.

The following result is needed in the next subsection.

Lemma 2.6. *Assume that the linear* SDE (2.1) *is quasi-index-*1 *tractable. Then the matrices* $G_n = A_n + B_n V_{n-1} Q^{(n)} V_n^T$ *are nonsigular for all* $n \geq 0$.

In the rest of this subsection, we shall present an index-1 tractable concept for nonlinear SDEs. Firstly, we recall the index-1 notion of nonlinear DAEs. Given a nonlinear DAE

$$f\big(x'(t), x(t), t\big) = 0, \tag{2.5}$$

where $f : \mathbb{R}^m \times \mathcal{D} \times J \to \mathbb{R}^m$ is continuous with continuous partial derivatives $f'_y, f'_x : \mathbb{R}^m \times \mathcal{D} \times J \to \mathrm{L}(\mathbb{R}^m)$. Here $\mathcal{D} \subseteq \mathbb{R}^m$ is an open subset and $J = [t_0, T]$ is an interval. The nullspace of the leading Jacobian $f'_y(y, x, t)$ is assumed to be independent of y and x, that is

$$\ker \frac{\partial f}{\partial y}(y, x, t) = \mathcal{N}(t), \quad (y, x, t) \in \mathbb{R}^m \times \mathcal{D} \times J.$$

Moreover, let $\mathcal{N}(t)$ vary smoothly in t. Here we denote by Q any C^1- projection onto $\mathcal{N}(t)$, i.e.,

$$Q \in C^1\big(J, \mathrm{L}(\mathbb{R}^m)\big), \quad Q(t)^2 = Q(t), \quad \operatorname{im} Q(t) = \mathcal{N}(t), \quad t \in J.$$

Besides the nullspace $\mathcal{N}(t)$ we consider the subspace

$$\mathcal{S}(y, x, t) = \big\{\xi \in \mathbb{R}^m : \frac{\partial f}{\partial x}(y, x, t)\xi \in \ \operatorname{im} \frac{\partial f}{\partial y}(y, x, t)\big\}$$

and, moreover, the matrix

$$G(y, x, t) = \frac{\partial f}{\partial y}(y, x, t) + \frac{\partial f}{\partial x}(y, x, t)Q(t).$$

Definition 2.7. [25, 32] *The nonlinear* DAE (2.5) *is called index-*1 *tractable on the open set* $\mathcal{G} \subseteq \mathbb{R}^m \times \mathcal{D} \times J$ *if the relation*

$$\mathcal{S}(y, x, t) \oplus \mathcal{N}(t) = \mathbb{R}^m \tag{2.6}$$

holds for all $(y, x, t) \in \mathcal{G}$.

It is well known that the condition (2.6) is equivalent to the nonsingularity of the matrix $G(y, x, t)$, and to the index-1 property of the pencil $\big\{f'_y(y, x, t), f'_x(y, x, t)\big\}$.

Together with Eq. (2.5), we now consider a nonlinear SDE given by

$$f_n(x_{n+1}, x_n) = 0, \quad (n \geq 0), \tag{2.7}$$

where $f_n : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ are given continuously differentiable functions. The nullspaces of the leading Jacobians $\dfrac{\partial f_n}{\partial y}(y, x)$ are assumed to be independent of y and x, i.e.,

$$\ker \frac{\partial f_n}{\partial y}(y, x) = \mathcal{N}_n \quad \forall\ y, x \in \mathbb{R}^m,\ \forall\ n \geq 0.$$

The definition of index-1 tractability for linear SDEs can be extended to a class of nonlinear SDEs as follows [3-6].

Definition 2.8. *The nonlinear* SDE (2.7) *is said to be of index-*1 *if*

(i) *the functions* f_n *are continuously differentiable; moreover, the subspaces* $\mathcal{N}_n = \ker \dfrac{\partial f_n}{\partial y}(y,x)$ *are independent of* $y, x \in \mathbb{R}^m$ *and have the same dimension, i.e.,* $\dim \mathcal{N}_n = m - r$ *for some integer* r *between* 1 *and* $m-1$, *for all* $n \geq 0$,

(ii) $\mathcal{S}_n(y,x) \cap \mathcal{N}_{n-1} = \{0\}$, $\forall\ n \geq 1$, *where* $\mathcal{S}_n(y,x) = \Big\{\xi \in \mathbb{R}^m : \dfrac{\partial f_n}{\partial x}(y,x)\xi \in \operatorname{im} \dfrac{\partial f_n}{\partial y}(y,x)\Big\}$.

For definiteness, we put $\mathcal{N}_{-1} = \mathcal{N}_0$. Similarly, as in the linear case, we denote by Q_n any fixed projection onto subspaces $\mathcal{N}_n$ and let $T_n \in \mathrm{GL}(\mathbb{R}^m)$ be such an invertible operator, that $T_n|_{\mathcal{N}_n}$ is an isomorphism between $\mathcal{N}_n$ and $\mathcal{N}_{n-1}$ for all $n \geq 0$. It can be verified that the condition (ii) in Definition 2.8 is equivalent to the nonsingularity of the matrices $G_n(y,x) = \dfrac{\partial f_n}{\partial y}(y,x) + \dfrac{\partial f_n}{\partial x}(y,x)T_nQ_n$ for all $n \geq 0$. As we see below, this useful remark plays an important role for solving an intial value problems involving nonlinear index-1 SDE.

The unique solvability of nonlinear SDEs obtained via discretization by applying explicit schemes to nonlinear DAEs have been studied in [3]. In particular, the compatibility between index notions for nonlinear index-1 DAEs and nonlinear index-1 SDEs has been established.

Let us discretize Eq. (2.5) by the explicit Euler scheme, namely

$$f\Big(\frac{x_{n+1} - x_n}{\tau}, x_n, t_n\Big) = 0, \quad n = 0, \ldots, N-1, \tag{2.8}$$

where $t_n = t_0 + n\tau$, $\tau = (T - t_0)/N$, $n = 0, \ldots, N$. A nonlinear version of Theorem 2.4 is stated as follows [3].

Theorem 2.9. *Assume that the nonlinear* DAE (2.5) *is of index*-1 *and the matrices* $G^{-1}(y,x,t)$ *and* $f'_x(y,x,t)$ *are uniformly bounded. Then for a sufficiently small stepsize* τ, *the discretized equation* (2.8) *is also of index*-1.

We end this subsection by noting that recently, a notion of the strangeness index for linear SDEs has been introduced in [27]. A relationship between the tractability index and the strangeness index of linear SDEs has been studied.

2.2. Initial Value Problems for Singular Difference Equations

We begin with a lemma that plays an important role in decoupling linear SDEs.

Lemma 2.10. [16, 31] *Suppose that the linear* SDE (2.1) *is of index*-1 *tractable and* Q_n *are arbitrary projections onto* $\mathcal{N}_n$, $n \geq 0$. *Then, the following relations hold:*

(i) $P_n = G_n^{-1}A_n$, *where* $P_n := I - Q_n$; (2.9)

(ii) $P_nG_n^{-1}B_n = P_nG_n^{-1}B_nP_{n-1}$; $\quad Q_nG_n^{-1}B_n = Q_nG_n^{-1}B_nP_{n-1} + T_n^{-1}Q_{n-1}$; (2.10)

(iii) $\widetilde{Q}_{n-1} := T_nQ_nG_n^{-1}B_n$ *is the canonical projection onto* $\mathcal{N}_{n-1}$ *along* $\mathcal{S}_n$. (2.11)

We now describe the decomposition technique for linear index-1 tractable SDEs. Multiplying both sides of Eq. (2.1) by $P_nG_n^{-1}$ and $Q_nG_n^{-1}$ respectively and applying the relation (2.9), we can decouple this equation into system

$$P_nx_{n+1} + P_nG_n^{-1}B_nx_n = P_nG_n^{-1}q_n, \quad n \geq n_0, \tag{2.12}$$
$$Q_nG_n^{-1}B_nx_n = Q_nG_n^{-1}q_n, \quad n \geq n_0, \tag{2.13}$$

where n_0 is a given nonnegative integer.

From (2.10) it yields that Eqs. (2.12)–(2.13) are equivalent to

$$P_nx_{n+1} + P_nG_n^{-1}B_nP_{n-1}x_n = P_nG_n^{-1}q_n, \quad n \geq n_0,$$
$$T_nQ_nG_n^{-1}B_nP_{n-1}x_n + Q_{n-1}x_n = T_nQ_nG_n^{-1}q_n, \quad n \geq n_0.$$

Putting $x_n = P_{n-1}x_n + Q_{n-1}x_n =: u_n + v_n$ $(n \geq n_0)$, we can rewrite the above equations as

$$u_{n+1} + P_nG_n^{-1}B_nu_n = P_nG_n^{-1}q_n, \quad n \geq n_0, \tag{2.14}$$
$$v_n = -T_nQ_nG_n^{-1}B_nu_n + T_nQ_nG_n^{-1}q_n, \quad n \geq n_0.$$

Hence,

$$x_n = u_n + v_n = (I - T_nQ_nG_n^{-1}B_n)u_n + T_nQ_nG_n^{-1}q_n, \quad \forall\ n \geq n_0, \tag{2.15}$$

where u_n solves the inherent regular ordinary difference equation (2.14).

Thus, if Eq. (2.1) is index-1 tractable, then, for given $u_{n_0} = P_{n_0-1}x_{n_0} \in \operatorname{im} P_{n_0-1}$, where n_0 is a fixed nonnegative integer, we can compute u_n and x_n $(n \geq n_0)$ by (2.14) and (2.15), respectively. As in the DAEs case, we need only to initialize the P_{n_0-1}-component of x_{n_0}, i.e.,

$$P_{n_0-1}\big(x_{n_0} - x^{(0)}\big) = 0. \tag{2.16}$$

As mentioned in [4], the initial condition (2.16) is equivalent to the condition

$$A_{n_0-1}\big(x_{n_0} - x^{(0)}\big) = 0 \tag{2.17}$$

which is independent of the choice of the projection P_{n_0-1}. Further, from (2.13) it follows that the solution of (2.1) with initial condition $x_{n_0} = x^{(0)}$ does exist if

$$Q_{n_0}G_{n_0}^{-1}B_{n_0}x^{(0)} = Q_{n_0}G_{n_0}^{-1}q_{n_0}. \tag{2.18}$$

Denote

$$\Delta_n := \{\xi \in \mathbb{R}^m : Q_nG_n^{-1}B_n\xi = Q_nG_n^{-1}q_n\}$$

and

$$\Omega_n := \{\xi \in \mathbb{R}^m : q_n - B_n\xi \in \ \operatorname{im} A_n\}.$$

From Lemma 3.8 ([4, Lemma 2.3]), it follows that $\Delta_n = \Omega_n$. In particular, Δ_n does not depend on the choice of projection Q_n and transformation T_n. Observing that if $\{x_n\}$ $(n \geq n_0)$ is a certain solution of the IVP (2.1) and (2.16), then, by virtue of Eq. (2.13), it follows that $x_n \in \Delta_n$ for all $n \geq n_0$. Conversely, for a given $n_0 \geq 0$ and for each $x^{(0)} \in \Delta_{n_0}$ satisfying (2.18), there exists a solution of (2.1) passing through $x^{(0)}$. Indeed, let $x_n\big(n_0; x^{(0)}\big)$, $n \geq n_0$ be a solution of (2.1) satisfying the initial condition (2.16). Formula (2.15) gives

$$\begin{aligned} x_{n_0}\big(n_0; x^{(0)}\big) &= P_{n_0-1}x^{(0)} - T_{n_0}\big(Q_{n_0}G_{n_0}^{-1}B_{n_0}P_{n_0-1}x^{(0)} - Q_{n_0}G_{n_0}^{-1}q_{n_0}\big) \\ &= P_{n_0-1}x^{(0)} + T_{n_0}Q_{n_0}G_{n_0}^{-1}B_{n_0}Q_{n_0-1}x^{(0)} - T_{n_0}\big(Q_{n_0}G_{n_0}^{-1}B_{n_0}x^{(0)} - Q_{n_0}G_{n_0}^{-1}q_{n_0}\big). \end{aligned}$$

Using Condition (iii) of Lemma 2.10 and the fact that $\widetilde{Q}_{n_0-1}Q_{n_0-1} = Q_{n_0-1}$, and recalling that $x^{(0)}$ satisfies Eq. (2.18) we come to the desired relation

$$x_{n_0}\big(n_0; x^{(0)}\big) = P_{n_0-1}x^{(0)} + Q_{n_0-1}x^{(0)} = x^{(0)}.$$

Now, we are interested in the IVP (2.1) and (2.16). Clearly, the formula (2.15) can be rewritten as follows

$$\begin{aligned} x_n = \widetilde{P}_{n-1}\Big\{&(-1)^{n-n_0}\Big(\prod_{i=1}^{n-n_0} P_{n-i}G_{n-i}^{-1}B_{n-i}\Big)P_{n_0-1}x_{n_0} + P_{n-1}G_{n-1}^{-1}q_{n-1} \\ &+ \sum_{k=n_0}^{n-2}(-1)^{n-k-1}\Big(\prod_{i=1}^{n-k-1} P_{n-i}G_{n-i}^{-1}B_{n-i}\Big)P_kG_k^{-1}q_k\Big\} \\ &+ T_nQ_nG_n^{-1}q_n, \quad \forall\, n \geq n_0 \end{aligned}$$

with $\widetilde{P}_{n-1} = I - \widetilde{Q}_{n-1}$. Combining the above formula with the initial condition (2.16) and using Eq. (2.10) as well as the fact that $\widetilde{P}_{n-1}P_{n-1} = \widetilde{P}_{n-1}$, we get the solution of the IVP (2.1) and (2.16) in the form

$$x_n = \widetilde{P}_{n-1}\big(M_{n_0-1}^{(n-1)}x^{(0)} + G_{n-1}^{-1}q_{n-1} + \sum_{k=n_0}^{n-2} M_k^{(n-1)}G_k^{-1}q_k\big) + T_nQ_nG_n^{-1}q_n \quad (2.19)$$

for all $n \geq n_0$, where

$$M_k^{(n-1)} := (-1)^{n-k-1}\prod_{i=1}^{n-k-1} G_{n-i}^{-1}B_{n-i} \ (k = n_0-1, \ldots, n-2)$$

for all $n > n_0$, $M_{n_0-1}^{(n_0-1)} := I$ and $x^{(0)} \in \mathbb{R}^m$ is a given vector.

Next, consider a linear homogeneous equation (2.1). The formula (2.15) yields that the IVP for the homogeneous equation

$$A_nx_{n+1} = B_nx_n, \quad n \geq 0$$

with the initial condition

$$P_{n_0-1}x_{n_0} = 0$$

has only trivial solution $x_n = 0$ for all $n \geq n_0$. Consequently, if Eq. (2.1) is index-1 tractable, then the IVP (2.1) and (2.16) always has a unique solution given by the formula (2.19). Moreover, the solution formula (2.19) is independent of the choice of projections Q_n and linear transformations T_n. However, the latter fact can be established in another way, by using the following lemma.

Lemma 2.11. [16,31] *Suppose that the linear* SDE (2.1) *is index*-1 *tractable. Then the matrices* $\widetilde{P}_n G_n^{-1}$ $(n \geq 0)$ *are independent of the choice of* Q_n *and* T_n.

Note that in the homogeneous case, the solution space Ω_n coincides with the subspace $\mathcal{S}_n$.

Summing up, we obtain the following theorem.

Theorem 2.12. [2, 31] *Let the linear* SDE (2.1) *be index*-1 *tractable. Then the following statements hold.*

(i) *The* IVP (2.1) *and* (2.16) *has a unique solution given by the formula* (2.19). *In addition, the solution formula is independent of the choice of projections* Q_n *and linear transformations* T_n.

(ii) *If* $\{x_n\}$ $(n \geq n_0)$ *solves the* IVP (2.1) *and* (2.16), *then* $x_n \in \Omega_n$, $\forall n \geq n_0$. *Conversely, through each* $x^{(0)} \in \Omega_{n_0}$ *there passes exactly one solution of* (2.1) *at* n_0, *namely, the* IVP (2.1) *with the initial condition* $x_{n_0} = x^{(0)}$ *possesses a unique solution.*

We now consider the IVP (2.1) and (2.17) for quasi-index-1 SDEs. Recall that $A_n = U_n \Sigma_n V_n^T$ is an SVD of A_n, where U_n, $V_n \in \mathbb{R}^{m\times m}$ are orthogonal matrices and $\Sigma_n = \operatorname{diag}(\sigma_1^{(n)}, \sigma_2^{(n)}, \ldots, \sigma_{r_n}^{(n)}, 0, \ldots, 0)$ with singular values $\sigma_1^{(n)} \geq \sigma_2^{(n)} \geq \cdots \geq \sigma_{r_n}^{(n)} > 0$, here $r_n = \operatorname{rank} A_n$. As discussed before, we can transform the problem (2.1) and (2.17) into a simpler form

$$\Sigma_n \overline{x}_{n+1} + \overline{B}_n \overline{x}_n = \overline{q}_n, \quad n \geq n_0, \tag{2.20}$$

$$\Sigma_{n_0-1}\big(\overline{x}_{n_0-1} - \overline{x}^{(0)}\big) = 0, \tag{2.21}$$

where $\overline{x}^{(0)} = V_{n_0-1}^T x^{(0)}$. It is worth of noting that Eq. (2.20) is also of quasi-index-1. By Lemma 2.6, the matrices $\overline{G}_n = \Sigma_n + \overline{B}_n Q^{(n)}$ are nonsingular, where $Q^{(n)} = \operatorname{diag}(I_{r_n}, O_{m-r_n})$. Acting as in the decoupling process for a linear index-1 tractable SDE and noting that

$$P^{(n)} = \overline{G}_n^{-1}\Sigma_n, \quad Q^{(n)} = \overline{G}_n^{-1}\overline{B}_n Q^{(n)}, \quad P^{(n)} = P^{(n)}P^{(n+1)},$$

where $P^{(n)} = I - Q^{(n)}$, we have proved the following theorem on the solvability of the IVP for linear quasi-index-1 SDEs.

Theorem 2.13. [31] *Let the linear* SDE (2.1) *be of quasi-index*-1. *Then the* IVP (2.20) *and* (2.21) *is solvable for any right-hand side* $\overline{q}_n$ *and there holds a solution formula:*

$$\begin{aligned} &\overline{x}_n = \big(I - Q^{(n)}\overline{G}_n^{-1}\overline{B}_n\big)\overline{u}_n + Q^{(n)}\overline{G}_n^{-1}\overline{q}_n, \\ &\overline{u}_{i+1} + \overline{G}_i^{-1}\overline{B}_i\overline{u}_i = \overline{G}_i^{-1}\overline{q}_i + Q^{(i)}\xi_i, \quad i = n_0, \ldots, n-1, \quad \forall\, n \geq n_0, \end{aligned} \tag{2.22}$$

where $\overline{u}_{n_0} = P^{(n_0-1)}\overline{x}^{(0)}$ *and* $\xi_i \in \mathbb{R}^m$ $(i = n_0, \ldots, n-1)$ *are arbitrary vectors.*

Notice that under the asumptions of Theorem 2.13, the IVP (2.20) and (2.21) always possesses a solution $\overline{x}_n$ depending on $n-n_0$ arbitrary vectors ξ_i $(i=n_0,\dots,n-1)$. Consequently, the IVP (2.1) and (2.16) always has a solution x_n depending on $n-n_0$ arbitrary vectors ξ_i $(i=n_0,\dots,n-1)$ as well as on the choice of SVDs of A_n. In particular, when $\ker\Sigma_n$ are constant, then $\mathrm{rank}A_n\equiv r$, $P^{(n)}=\widetilde{P}$, $Q^{(n)}=\widetilde{Q}$ and the linear SDE (2.1) is index-1 tractable. In this case, it is obvious that

$$\big(I-Q^{(n)}\overline{G}_n^{-1}\overline{B}_n\big)Q^{(n-1)}=\big(I-\widetilde{Q}\overline{G}_n^{-1}\overline{B}_n\big)\widetilde{Q}=O.$$

This leads to the formula (2.19) again.

Next, we shall study the solvability of IVP for nonlinear index-1 SDEs (2.7). As in the linear case, the initial condition (2.16) should be considered. In the remainder of this subsection, for the sake of simplicity the norm of $\mathbb{R}^m$ is assumed to be Euclidean. For nonlinear index-1 DAEs, under the asumptions that the matrix $G(y,x,t)$ has a bounded inverse on $\mathcal{G}$ and $\|f'_x(y,x,t)\|$ is bounded on $\mathcal{G}$, then it is not difficult to show the global existence and uniqueness of the solution of (2.5) with initial condition $P(t_0)\big(x(t_0)-x^{(0)}\big)=0$ (cf. [25, Theorem 15]).The following theorem can be considered as a discrete version of the corresponding result of [25].

Theorem 2.14. [5] *Let the nonlinear* SDE (2.7) *be of index*-1. *Moreover, suppose that*

$$\|G_n^{-1}(y,x)\|\le\alpha_n\|y\|+\beta_n\|x\|+\gamma_n\quad\forall\ y,x\in\mathbb{R}^m, n\ge n_0,$$

where α_n, $\beta_n\ge 0$, $\gamma_n>0$ *are constants and* n_0 *is a given nonnegative integer. Then the* IVP (2.7), (2.16) *has a unique solution.*

Now we deal with nonlinear SDEs of the special form

$$g_n(x_{n+1},x_n)+h_n(x_{n+1},x_n)=0,\quad n\ge 0, \tag{2.23}$$

where $g_n,h_n:\mathbb{R}^m\times\mathbb{R}^m\to\mathbb{R}^m$ are given continuously differentiable functions. Suppose that the SDE $g_n(x_{n+1},x_n)=0$ is of index-1 and $h_n(y,x)$ is small. In the next theorem, without loss of generality, we will use orthogonal projections onto $\mathcal{N}_n$, that is, $Q_n=V_n\widetilde{Q}V_n^T$ and $V_nV_n^T=V_n^TV_n=I$. In this case, $\|Q_n\|=\|P_n\|=\|V_n\|=1$. Moreover, we can choose $T_nQ_n=V_{n-1}\widetilde{Q}V_n^T$.

Theorem 2.15. [5] *Suppose that the following properties hold*

(i) $g_n(y,x)$ *is continuously differentiable; moreover*

$$\ker\frac{\partial g_n}{\partial y}(y,x)=\mathcal{N}_n,\quad \dim\mathcal{N}_n=m-r,\ \forall\,n\ge n_0,\ x,y\in\mathbb{R}^m;$$

(ii) $G_n(y,x)=\dfrac{\partial g_n}{\partial y}(y,x)+\dfrac{\partial g_n}{\partial x}(y,x)T_nQ_n$ $(n\ge n_0)$ *has uniformly bounded inverse, i.e.,* $\|G_n^{-1}(y,x)\|\le\gamma_n$ *for all* $n\ge n_0,\ y,x\in\mathbb{R}^m$;

(iii) $h_n(y,x)=h_n(P_ny,x)$ *for all* $n\ge n_0,y,x\in\mathbb{R}^m$;

(iv) $\|h_n(y,x)-h_n(\overline{y},\overline{x})\|\le L_n\big(\|y-\overline{y}\|^2+\|x-\overline{x}\|^2\big)^{1/2}$ *for all* $n\ge n_0,y,x,\overline{y},\overline{x}\in\mathbb{R}^m$.

Then the IVP (2.23) *and* (2.16) *has a unique solution, provided* $\gamma_n L_n < 1/\sqrt{2}$ *for all* $n \geq n_0$.

When the principal parts g_n are linear, i.e., $g_n(y,x) = A_n y + B_n x$, where $A_n, B_n \in \mathbb{R}^{m\times m}$ are given matrices, a necessary condition for the unique solvability of quasi-linear SDEs is given and an algorithm for finding an approximate solution of the IVP is proposed.

Corollary 2.16. [5, 7] *Suppose that the nonlinear* SDE (2.23) *satisfies the following conditions:*

(i) $g_n(y,x) = A_n y + B_n x$, *where* $A_n, B_n \in \mathbb{R}^{m\times m}$ *are a given matrices, further, the linear* SDE
$$A_n x_{n+1} + B_n x_n \; = 0 \quad (n \geq n_0)$$
is of index-1 *tractable,*

(ii) $h_n(y,x)$ *is continuously differentiable, moreover*
$$\ker A_n \subseteq \ker \frac{\partial h_n}{\partial y}(y,x), \quad \forall\, n \geq n_0,\; y,x \in \mathbb{R}^m,$$

(iii) *the nonlinear function* $h_n(y,x)$ *satisfies the inequality*
$$\|h_n(y,x) - h_n(\overline{y},\overline{x})\| \leq L_n\big(\|y-\overline{y}\|^2 + \|x-\overline{x}\|^2\big)^{1/2},\; \forall\, n \geq n_0,\; y,x,\overline{y},\overline{x} \in \mathbb{R}^m.$$

Then, the IVP (2.23) *and* (2.16) *is uniquely solvable, provided* $L_n\|G_n^{-1}\| < 1/\sqrt{2}$.

Theorem 2.17. [7] *Let the assumptions* (i)–(ii) *of Corollary* 2.16 *be satisfied. Moreover, suppose that the following conditions are satisfied:*

(i) $h_n(y,x)$ *satisfies the inequality*
$$\|h_n(y,x) - h_n(\overline{y},\overline{x})\| \leq \alpha_n\|y-\overline{y}\| + \beta_n\|x-\overline{x}\|$$
with nonnegative constants α_n *and* β_n *for all* $n \geq n_0, y, x, \overline{y}, \overline{x} \in \mathbb{R}^m$;

(ii) $I - T_nQ_nG_n^{-1}B_n$ *is uniformly bounded, i.e.,* $\|I - T_nQ_nG_n^{-1}B_n\| \leq C_1$;

(iii) *the norm of the matrix* $P_nG_n^{-1}B_n$ *is uniformly bounded by a constant less than* 1, *i.e.,* $\|P_nG_n^{-1}B_n\| \leq \delta_0 < 1$;

(iv) *the coefficients* α_n β_n *are small enough, such that*
$$\alpha_n\|P_nG_n^{-1}\| + \beta_n\|T_nQ_nG_n^{-1}\| \leq \omega < 1,$$
$$(1-\omega)^{-1}(\delta_0\alpha_n + C_1\beta_n)\|P_nG_n^{-1}\| \leq \delta_1 < 1 - \delta_0,$$
and
$$(1-\omega)^{-1}(\delta_0\alpha_n + C_1\beta_n)\|T_nQ_nG_n^{-1}\| \leq C_2.$$

Then within a given tolerance $\epsilon > 0$, *we can always find an approximate solution* $\{\overline{x}_n\}$ $(n \geq n_0)$ *via iterations, such that* $\|\overline{x}_n - x_n\| \leq \epsilon$ *for all* $n \geq n_0$, *where* $\{x_n\}$ $(n \geq n_0)$ *is an exact solution of the IVP* (2.23) *and* (2.16).

A similar result for quasi-linear quasi-index-1 SDE (2.23) has also been presented in [7].

Theorem 2.18. *Assume that the nonlinear* SDE (2.23) *satisfies the following conditions:*

(i) $g_n(y,x) = A_n y + B_n x$, where $A_n, B_n \in \mathbb{R}^{m\times m}$ *are given matrices; moreover,* $A_n x_{n+1} + B_n x_n = 0$, $(n \geq n_0)$ *is a linear quasi-index*-1 SDE;

(ii) *the nonlinear function* $h_n(y,x)$ *satisfies the condition* (ii) *of Corollary* 2.16 *and the following growth condition:*

$$\|h_n(y,x)\| \leq a_n\|y\|^{\nu_n} + b_n\|x\|^{\mu_n} + c_n, \quad \forall\, n \geq n_0, y, x \in \mathbb{R}^m$$

with nonnegative constants $a_n, b_n, c_n, \nu_n, \mu_n$*, where* $\theta_n = \max\{\nu_n, \mu_n\} \leq 1$*. Additionally, we suppose that if* $\theta_n = 1$ *then* $(a_n + b_n)\|G_n^{-1}\| \leq 1$*.*

Then the IVP (2.23) *and* (2.16) *possesses a solution.*

Finally, we turn to the convergence problem of the explicit Euler method for linear and nonlinear index-1 tractable DAEs. As mentioned above, Theorems 2.4 and 2.9 ensure that if the linear DAE (2.2) and nonlinear DAE (2.5) are index-1 tractable, then the corresponding discretized equations (2.4) and (2.8) are also index-1 tractable. For finding a solutions of Eqs. (2.2) and (2.5), satisfying the initial condition

$$P(t_0)\big(x(t_0) - x^{(0)}\big) = 0, \tag{2.24}$$

we use the explicit Euler method, i.e., we seek for a solutions of Eqs. (2.4) and (2.8) respectively, satisfying condition

$$P_0\big(x_0 - x^{(0)}\big) = 0. \tag{2.25}$$

Theorem 2.19. [1] *Assume that the linear* DAE (2.2) *is index*-1 *tractable. Then the explicit Euler method applied to the* IVP (2.2) *and* (2.24) *does converge, i.e.,*

$$x_n \to x(t_n) \text{ as } \tau \to 0,$$

where $x(t_n)$ *is the value of the solution of the* IVP (2.2), (2.24) *at* t_n*, and* x_n *is the solution of the* IVP (2.4) *and* (2.25).

Theorem 2.20. [3] *Under the assumptions of Theorem* 2.9, *the explicit Euler method applied to the* IVP (2.5) *and* (2.24) *is convergent.*

2.3. Multipoint Boundary-Value Problems for Singular Difference Equations

In this subsection, we are interested in MPBVPs for linear index-1 tractable SDEs (2.1). For the sake of simplicity, we shall restrict our consideration to the following two-point boundary value problem (TPBVP):

$$A_n x_{n+1} + B_n x_n = q_n, \quad n = 0, \ldots, N-1, \tag{2.26}$$

$$C_0 x_0 + C_N x_N = \gamma, \tag{2.27}$$

where A_n, B_n $(0 \leq n \leq N-1)$, C_0, $C_N \in \mathbb{R}^{m\times m}$ and q_n $(n = 0, \ldots, N-1)$, $\gamma \in \mathbb{R}^m$ are given matrices and vectors respectively. Further, assume that (2.26) is a linear index-1 tractable SDE. Let $A_n = U_n\Sigma_n V_n^T$ be an SVD of A_n, $Q_n = V_n\widetilde{Q}V_n^T$ be an orthogonal projection onto $\mathcal{N}_n$ and $G_n = A_n + B_n V_{n-1}\widetilde{Q}V_n^T$.

Clearly, the TPBVP (2.26), (2.27) represents a system of $m(N+1)$ linear equations, hence we can use well known facts from linear algebra to verify the solvability as well as to construct a solution formula for the system (2.26) and (2.27). However, when N becomes large, while m remains small, this linear system is a large-scale one, hence, the problem of verifying its solvability as well as finding its solution becomes an uneasy task. In order to overcome these difficulties, instead of considering the linear system in $\mathbb{R}^{m(N+1)}$, we shall treat it in $\mathbb{R}^m$.

Firstly, recall two notations which have already been introduced in the above subsection, namely

$$\widetilde{P}_{n-1} := I - \widetilde{Q}_{n-1} = I - V_{n-1}\widetilde{Q}V_n^T G_n^{-1} B_n$$

and

$$M_k^{(n-1)} := (-1)^{n-k-1} \prod_{i=1}^{n-k-1} G_{n-i}^{-1} B_{n-i} \ (k = -1, \dots, n-2) \text{ for all } n > 0.$$

We denote $X_0 = \widetilde{P}_{-1}$, $X_n = \widetilde{P}_{n-1} M_{-1}^{(n-1)}$ $(n = 1, \dots, N-1)$, $X_N = P_{N-1} M_{-1}^{(N-1)}$ and define the shooting matrix $D = C_0 X_0 + C_N X_N$. In what follows, we shall deal with the $(m \times 2m)$ column matrix $(D|C_N Q_{N-1})$ and the $(2m \times 2m)$ column matrix $R = \mathrm{diag}(P_{-1}, Q_{N-1})$. The regularity condition for the TPBVP (2.26) and (2.27) has been proposed in [31].

Definition 2.21. *The TPBVP* (2.26) *and* (2.27) *is called regular if*

$$\ker(D|C_N Q_{N-1}) = \ker R. \tag{2.28}$$

Clearly, the matrices D, Q_{N-1} and R depend on the orthogonal matrices V_n $(0 \le n \le N-1)$, hence, on SVDs of the matrices A_n $(n = 0, \dots, N-1)$. However, the regularity condition is independent of the choice of SVDs of A_n.

Lemma 2.22. [31] *The regularity condition* (2.28) *for the* TPBVP (2.26) *and* (2.27) *does not depend on the chosen* SVDs *of* A_n $(n = 0, \dots, N-1)$.

Moreover, the dimensions of $\ker R$ and $\ker(D|C_N Q_{N-1})$ do not depend on the choice of SVDs of A_n, neither.

Lemma 2.23. [2] *The dimensions of the subspaces* $\ker R$ *and* $\ker(D|C_N Q_{N-1})$ *are independent of the choice of* SVDs *of* A_n $(n = 0, \dots, N-1)$. *Moreover,* $\dim\big(\ker R\big) = m$ *and* $\dim\big(\ker(D|C_N Q_{N-1})\big) := p \ge m$.

On the other hand, the following inclusion always holds

$$\ker R \subseteq \ker(D|C_N Q_{N-1}). \tag{2.29}$$

Combining (2.29) with Lemmas 2.22 and 2.23 we come to the following.

Corollary 2.24. [2] *The* TPBVP (2.26) *and* (2.27) *is regular if and only if*

$$\dim\big(\ker(D|C_N Q_{N-1})\big) = m$$

and its regularity does not depend on the choice of SVDs *of* A_n $(n = 0, \dots, N-1)$.

For a further discussion we put

$$z_0 := V_{-1}\widetilde{Q}V_0^T G_0^{-1} q_0,$$

$$z_n := \widetilde{P}_{n-1}\big(\sum_{k=0}^{n-2} M_k^{(n-1)} G_k^{-1} q_k + G_{n-1}^{-1} q_{n-1}\big) + V_{n-1}\widetilde{Q}V_n^T G_n^{-1} q_n,\ n = 1, ..., N-1,$$

$$z_N := P_{N-1}\big(\sum_{k=0}^{N-2} M_k^{(N-1)} G_k^{-1} q_k + G_{N-1}^{-1} q_{N-1}\big) \text{ and } \gamma^* := \gamma - C_0 z_0 - C_N z_N.$$

The following relationship between the regularity and the solvability of TPBVPs has been established in [31].

Theorem 2.25. *Suppose that the linear* SDE (2.26) *is index-*1 *tractable. Then the regularity* (2.28) *of the* TPBVP (2.26) *and* (2.27) *is a necessary and sufficient condition for its unique solvability. Moreover, there holds the following solution formula*

$$\begin{aligned} x_n &= X_n x^{(0)} + z_n, \quad n = 0, \dots, N-1, \\ x_N &= X_N x^{(0)} + z_N + Q_{N-1}\xi, \end{aligned} \tag{2.30}$$

where $(x^{(0)T}, \xi^T)^T = (D|C_N Q_{N-1})^+ \gamma^*$ *and* $(D|C_N Q_{N-1})^+$ *denotes the generalized inverse in Moore-Penrose's sense of* $(D|C_N Q_{N-1})$.

It is worth of mentioning that the vectors $x^{(0)}$ and ξ in (2.30) are undetermined, while $X_n x^{(0)}$ $(n = 0, \dots, N)$ and $Q_{N-1}\xi$ are uniquely determined.

Next, we are interested in the irregular case, namely, when $p > m$. Denote by $\{w_i^0\}_{i=0}^m$ a basis of $\ker R$. By virtue of the relation (2.29) we can extend $\{w_i^1\}_{i=1}^m$ to a basis $\{w_i^0\}_{i=1}^p$ of $\ker(D|C_N Q_{N-1})$. For every $i = m+1, \dots, p$, let $u_i^0 \in \mathbb{R}^m$ and $v_i^0 \in \mathbb{R}^m$ be the first and the second groups of the components of w_i^0, i.e., $w_i^0 = (u_i^{0T}, v_i^{0T})^T$. We introduce the following column matrices

$$\Phi_n := \big(X_n u_{m+1}^0,\ \dots,\ X_n u_p^0\big) \in \mathbb{R}^{m\times(p-m)}\ (n = 0, \dots, N-1)$$

and

$$\Phi_N := \big(X_N u_{m+1}^0 + Q_{N-1} v_{m+1}^0,\ \dots,\ X_N u_p^0 + Q_{N-1} v_p^0\big) \in \mathbb{R}^{m\times(p-m)}.$$

Additionally, let us consider a linear operator $\mathcal{L}$ acting in $\mathbb{R}^{m(N+1)}$, defined by

$$\mathcal{L}x := \big((A_0 x_1 + B_0 x_0)^T,\ \dots,\ (A_{N-1} x_N + B_{N-1} x_{N-1})^T,\ (C_0 x_0 + C_N x_N)^T\big)^T.$$

Then solving the TPBVP (2.26) and (2.27) is equivalent to finding a solution $x \in \mathbb{R}^{m(N+1)}$ of $\mathcal{L}x = q$, where $q = (q_0^T,\ \dots,\ q_{N-1}^T,\ \gamma^T)^T \in \mathbb{R}^{m(N+1)}$. We have the following useful property.

Lemma 2.26. [2] $\ker \mathcal{L} = \{\big((\Phi_0 a)^T, \dots, (\Phi_N a)^T\big)^T : a \in \mathbb{R}^{p-m}\}$.

Lemma 2.26 yields that the subspace $\ker\mathcal{L}$ can be represented by SVDs of A_n $(n = 0, \dots, N-1)$. Furthermore, if the problem (2.26) and (2.27) is solvable, then its solution has the form $x = x^* + \overline{x}$, where $\overline{x} \in \ker \mathcal{L}$ and x^* is a certain particular solution. Clearly, the TPBVP (2.26) and (2.27) may have no solution. The following lemma gives a necessary and sufficient condition for its solvability.

Lemma 2.27. [2] *A necessary and sufficient condition for the solvability of* TPBVP (2.26) *and* (2.27) *is the following orthogonality condition*

$$w^T\gamma^* = 0, \quad \forall\ w \in \ker(D|C_N Q_{N-1})^T.$$

Putting $q := \dim\big(\ker(D|C_N Q_{N-1})^T\big)$ and letting W be a row matrix whose rows are vectors w_i $(i = 1, \dots, q)$ of the basis of $\ker(D|C_N Q_{N-1})^T$. Lemmas 2.26 and 2.27 lead to the existence theorem in the irregular case.

Theorem 2.28 [2] *Assume that the linear* SDE (2.26) *is index*-1 *tractable and the* TPBVP (2.26) *and* (2.27) *is irregular, namely, $p > m$. Then the* TPBVP *is solvable if and only if*

$$W\gamma^* = 0.$$

Moreover, its general solution can be represented as:

$$\begin{aligned} x_n &= X_n x^{(0)} + z_n + \Phi_n a, \quad n = 0, \dots, N-1, \\ x_N &= X_N x^{(0)} + z_N + Q_{N-1}\xi + \Phi_N a, \end{aligned} \tag{2.31}$$

where $a \in \mathbb{R}^{p-m}$ is an arbitrary vector, $(x^{(0)T}, \xi^T)^T = (D|C_N Q_{N-1})^+\gamma^*$.

Clearly, when $p = m$ the formula (2.31) turns to (2.30). Combining Theorem 2.25 and Theorem 2.28 for both regular and irregular cases, we arrive at the Fredholm alternative for a special large-scale system of the linear equations (2.26) and (2.27) as follows.

Corollary 2.29. [2] *Suppose that the linear* SDE (2.26) *is index*-1 *tractable. Then,*

(i) *Either $p = m$ and the* TPBVP (2.26) *and* (2.27) *is uniquely solvable for any data q_n $(n = 0, \dots, N-1)$ and γ;*

(ii) *Or $p > m$ and the* TPBVP (2.26) *and* (2.27) *is solvable if and only if*

$$W\gamma^* = 0.$$

Moreover, there hold the formulae (2.30) *or* (2.31) *for the cases* (i) *or* (ii), *respectively.*

Finally, we study the relationship between the TPBVP (2.2) with the boundary condition

$$C_0 x(t_0) + C_T x(T) = \gamma, \tag{2.32}$$

where the data $C_0, C_T \in \mathbb{R}^{m\times m}$ and $\gamma \in \mathbb{R}^m$ are given, and the corresponding discretized problem for (2.2) and (2.32) by applying the explicit Euler scheme

$$A_n x_{n+1} + (\tau B_n - A_n)x_n = \tau q_n, \quad n = 0, \dots, N-1, \tag{2.33}$$

$$C_0 x_0 + C_T x_N = \gamma. \tag{2.34}$$

Firstly, we recall (cf. [25]), that the TPBVP (2.2) and (2.32) is uniquely solvable for any $q \in C(J, \mathbb{R}^m)$ and $\gamma \in \mathrm{im}(C_0, C_T)$ if and only if the shooting matrix $D = C_0 X(t_0) + C_T X(T)$, where $X(t)$ is the fundamental solution matrix satisfying

$$A(t)X'(t) + B(t)X(t) = 0, \;\; t \in J,$$
$$P(t_0)(X(t_0) - I) = 0,$$

has the properties

$$\ker D = \ker A(t_0), \quad \mathrm{im} D = \mathrm{im}(C_0, C_T). \tag{2.35}$$

By virtue of Theorem 2.4, it follows that Eq. (2.33) is index-1 tractable provided the stepsize τ is small enough. For the TPBVP (2.33) and (2.34) we need some notations

$$B_n(\tau) = \tau B_n - A_n, \;\; G_n(\tau) = A_n + B_n(\tau)V_{n-1}\widetilde{Q}V_n^T,$$

$$\widetilde{P}_{n-1}(\tau) = I - V_{n-1}\widetilde{Q}V_n^T G_n^{-1}(\tau)B_n(\tau),$$

$$M_k^{(n-1)}(\tau) := (-1)^{n-k-1} \prod_{i=1}^{n-k-1} G_{n-i}^{-1}(\tau)B_{n-i}(\tau) \; (k = -1, \dots, n-2) \text{ for all } n > 0.$$

Further, define the shooting matrix $D(\tau) = C_0 X_0(\tau) + C_T X_N(\tau)$, where $X_0(\tau) = \widetilde{P}_{-1}(\tau)$, $X_n(\tau) = \widetilde{P}_{n-1}(\tau)M_{-1}^{(n-1)}(\tau)$ $(n = 1, \dots, N-1)$, $X_N(\tau) = P_{N-1}M_{-1}^{(N-1)}(\tau)$. Applying Corrollary 2.24 and Theorem 2.25, we come to the following necessary and sufficient condition for the unique solvability of the TPBVP (2.33) and (2.34)

$$\dim\big(\ker\left(D(\tau)|C_T Q_{N-1}\right)\big) = m. \tag{2.36}$$

Unfortunately, the regular conditions (2.35) and (2.36) for continuous and discretized problems are not compatible, i.e., there are examples showing that the unique solvability of the continuous problem (2.2) and (2.32) does not necessarily imply the unique solvability of the discretized problem (2.33) and (2.34) (cf. [1, 30]).

The difference between regular conditions for the continuous problem and the discretized one can be explained as follows. Assume that (2.2) and (2.32) are regular; then its solution is a solution of the IVP (2.2) and (2.24), where $x^{(0)}$ satisfies a linear system

$$Dx^{(0)} = \beta$$

with $\beta := \gamma - C_0 Q(t_0)G^{-1}(t_0)q(t_0) - C_T Q(T)G^{-1}(T)q(T)$. In this case, for every initial vector $x^{(0)}$ the problem (2.2) and (2.24) has a unique solution $x(t, x^{(0)})$. Further, this solution solves the TPBVP (2.2) and (2.32) if and only if $x^{(0)}$ satisfies the above linear system. On the other hand, the TPBVP (2.33) and (2.34) has a solution $\{x_n\}_{n=0}^N$. Here, the first N values $\{x_n\}_{n=0}^{N-1}$ is a solution of the IVP (2.33) and (2.25) and the last value x_N has the form

$$x_N = X_N(\tau)x^{(0)} + Q_{N-1}\xi + z_N(\tau),$$

where two vectors $x^{(0)}$ and ξ satisfy the relation

$$D(\tau)x^{(0)} + C_N Q_{N-1}\xi = \gamma(\tau),$$

and

$$z_N(\tau) := \tau P_{N-1}\sum_{k=0}^{N-2} M_k^{(N-1)}(\tau)q_k,$$
$$\gamma(\tau) := \gamma - \tau C_0 V_{-1}\widetilde{Q}V_0^T G_0^{-1}(\tau)q_0 - C_T z_N(\tau).$$

Thus, for each vector $x^{(0)}$ the relations (2.33) and (2.25) only determine uniquely the first N values $\{x_n(x^{(0)})\}_{n=0}^{N-1}$ of the solution $\{x_n(x^{(0)})\}_{n=0}^{N}$. The last value $x_N(x^{(0)})$ is decomposed into the uniquely determined term $P_{N-1}x_N(x^{(0)})$ and an undetermined one $Q_{N-1}x_N(x^{(0)})$.

For revealing a connection between the TPBVPs (2.2), (2.32) and (2.33), (2.34), we introduce an augmented TPBVP by adding to (2.33) the following equation

$$A_N x_{N+1} + B_N(\tau)x_N = \tau q_N.$$

In other words, the augmented problem has the form

$$A_n x_{n+1} + B_n(\tau)x_n = \tau q_n, \quad n = 0,\dots,N, \tag{2.37}$$
$$C_0 x_0 + C_T x_N = \gamma. \tag{2.38}$$

Clearly, if $\{x_n\}_{n=0}^{N+1}$ is a solution of the augmented TPBVP (2.37) and (2.38), then its first $N+1$ values $\{x_n\}_{n=0}^{N}$ form a solution of the original TPBVP (2.33) and (2.34). Further, we need the following definition.

Definition 2.30. [1] *The augmented TPBVP* (2.37) *and* (2.38) *is said to be uniquely solvable with respect to the first $N+1$ components if for any $\{q_n\}_{n=0}^{N}$ and $\gamma \in \operatorname{im}(C_0, C_T)$ it possesses a solution $\{x_n\}_{n=0}^{N+1}$ with uniquely determined first $N+1$ values $\{x_n\}_{n=0}^{N}$, i.e., if $\{y_n\}_{n=0}^{N+1}$ is another solution of* (2.37) *and* (2.38) *then $x_n = y_n$ for all $n = 0,\dots,N$.*

Theorem 2.31. [1] *The augmented problem* (2.37) *and* (2.38) *is uniquely solvable with respect to the first $N+1$ components if and only if the shooting matrix $D(\tau)$ satisfies the following relations*

$$\ker D(\tau) = \ker A(t_0), \quad \operatorname{im} D(\tau) = \operatorname{im}(C_0, C_T).$$

The desired relationship between continuous and discretized TPBVP is revealed in the following theorem [1].

Theorem 2.32. *Suppose that the continuous* TPBVP (2.2) *and* (2.32) *is uniquely solvable for any $q \in C(J, \mathbb{R}^m)$ and $\gamma \in \operatorname{im}(C_0, C_T)$. Then*

(i) *The corresponding discretized augmented problem* (2.37) *and* (2.38) *is also uniquely solvable with respect to the first $N+1$ components, provided τ is sufficiently small.*

(ii) *The discretization method is convergent, i.e.,*

$$\|x_n - x(t_n)\| \to 0 \text{ as } \tau \to 0,$$

where $\{x_n\}_{n=0}^N$ *are the first* $N+1$ *values of the solution of the augmented problem and* $x(t)$ *is a unique solution of the continuous problem.*

3. Qualitative Study of Singular Difference Equations

3.1. Floquet Theory for Linear Index-1 Singular Difference Equations

We begin this section with the Floquet theory for linear index-1 SDEs presented in [6]. Although many results in the discrete case are similar to those already known in the DAE case [29], it requires some special devices to obtain the desired results.

Lemma 3.1. *Suppose* Eq. (2.1) *is of index*-1. *Let* $Q_{n-1} = V_{n-1}\tilde{Q}V_{n-1}^{-1}$ *be an arbitrary projection onto* $\ker A_{n-1}$ $(n \geq 1)$. *Then*

(i) $\tilde{Q}_{n-1} := Q_{n-1,n}G_n^{-1}B_n$ *is the canonical projection onto* $\ker A_{n-1}$ *along* S_n;

(ii) $\tilde{Q}_{n-1} = \tilde{V}_{n-1}\tilde{Q}\tilde{V}_{n-1}^{-1}$, *where* $\tilde{V}_{n-1} = (s_n^1, \dots, s_n^r, h_{n-1}^{r+1}, \dots, h_{n-1}^m)$ *is a matrix, whose columns form certain bases of* S_n *and* $\ker A_{n-1}$, *respectively, i.e.,* $S_n = \operatorname{span}(\{s_n^i\}_{i=1}^r)$ *and* $\ker A_{n-1} = (\{h_{n-1}^j\}_{j=r+1}^m)$.

The canonical projections play an important role in the reduction of an SDE to the canonical form. Now consider a linear system, obtained from (2.1) via scaling and transforming variables

$$\overline{A}_n\overline{x}_{n+1} + \overline{B}_n\overline{x}_n = \overline{q}_n,$$

where $\overline{A}_n = E_nA_nF_n$; $\overline{B}_n = E_nB_nF_{n-1}$; $\overline{q}_n = E_nq_n$ and the matrices E_n, F_n are nonsingular. Here E_n are scaling matrices, while the transformations of variables are defined by $x_n = F_{n-1}\overline{x}_n$. Since $\overline{\mathcal{S}}_n \cap \ker\overline{A}_{n-1} = F_{n-1}^{-1}(\mathcal{S}_n \cap \ker A_{n-1})$, the index-1 property of linear SDEs is invariant under scaling and linear transformations. The following theorem is similar to that of linear index-1 DAEs.

Theorem 3.2. *Every linear index*-1 SDE *can be reduced to the Kronecker normal form*

$$\operatorname{diag}(I_r, O_{m-r})\overline{x}_{n+1} + \operatorname{diag}(W_n, I_{m-r})\overline{x}_n = \overline{q}_n.$$

Definition 3.3. *System* (2.1) *is called periodic of period* $N \in \mathbb{N}$ *if*

$$A_{n+N} = A_n, \quad B_{n+N} = B_n, \quad \text{and} \quad q_{n+N} = q_n \quad \forall n \geq 0.$$

For an N*-periodic singular difference system we define* $A_{-1} := A_{N-1}$.

Defintion 3.4. *The matrix* $X_n \in \mathbb{R}^{m\times m}$, *satisfying the IVP*

$$A_nX_{n+1} + B_nX_n = 0; \quad P_{-1}(X_0 - I) = 0,$$

where $P_{-1} = P_{N-1}$ *is a projection onto* $\mathcal{S}_{N-1}$ *along* $\ker A_{-1} = \ker A_{N-1}$, *will be called the fundamental matrix of Eq.* (2.1).

Theorem 3.5. *There exist an N-periodic nonsingular matrix F_n and a constant matrix $R \in \mathbb{C}^{r\times r}$ such that the fundamental matrix of the linear index*-1 *periodic* SDE (2.1) *with nonsingular matrices B_n can be represented as*

$$X_n = F_{n-1}\mathrm{diag}\,(R^n, O_{m-r})F_{-1}^{-1} \ \ (n \ge 1).$$

The following theorem, frequently referred to as the Lyapunov's theorem, has numerous applications in the stability theory.

Theorem 3.6. *Every index*-1 *periodic* SDE (2.1) *with nonsingular matrices B_n can be reduced to the Kronecker normal form with constant coefficients as follows*

$$\mathrm{diag}\,(I_r, O_{m-r})\widetilde{x}_{n+1} + \mathrm{diag}(-R, I_{m-r})\widetilde{x}_n = \widetilde{q}_n.$$

Before presenting some applications of the Floquet theory we will introduce some stability concepts and the Lyapunov function method [4].

3.2. Stability and Robust Stability of Singular Difference Equations

Consider a system

$$A_n x_{n+1} + B_n x_n = f_n(x_n) \ \ (n \ge 0). \tag{3.1}$$

In what follows we suppose that the corresponding linear homogeneous equation (2.1) is of index-1. Let us associate the SDE (3.1) with the initial condition

$$P_{n_0-1}x_{n_0} = P_{n_0-1}\gamma, \ \ n_0 \ge 0, \tag{3.2}$$

where γ is an arbitrary vector in $\mathbb{R}^m$ and n_0 is a fixed nonnegative integer. We always assume that $\dim S_0 = r$ and let $A_{-1} \in \mathbb{R}^{m\times m}$ be a fixed chosen matrix, such that $\mathbb{R}^m = S_0 \oplus \ker A_{-1}$, where as above, $\mathcal{S}_n = \{\xi \in \mathbb{R}^m : B_n\xi \in \mathrm{im}A_n\}$, $n \ge 0$. The unique solvability of the IVP (3.1)-(3.2) is guaranteed by the following theorem.

Theorem 3.7. *Let $f_n(x)$ be a Lipschitz continuous function with a sufficient small Lipschitz coefficient*, i.e.,

$$\|f_n(x) - f_n(\widetilde{x})\| \le L_n\|x - \widetilde{x}\|, \quad \forall\ x, \widetilde{x} \in \mathbb{R}^m,$$

where

$$\omega_n := L_n\|Q_{n-1,n}G_n^{-1}\| < 1, \quad \forall\ n \ge 0.$$

Then the IVP (3.1), (3.2) *has a unique solution.*

Thus, without loss of generality we can assume that $f_n(0) = 0$ for all $n \ge 0$. Then Eq. (3.1) always possesses a trivial solution $x_n \equiv 0$ $(n \ge 0)$. Set

$$\Delta_n := \big\{x \in \mathbb{R}^m : Q_{n-1}x = Q_{n-1,n}G_n^{-1}\big(f_n(x) - B_nP_{n-1}x\big)\big\}.$$

If $\{x_n\}$ is any solution of the IVP (3.1), (3.2), then obviously, $x_n \in \Delta_n$ $(n \ge n_0)$. Conversely, for each $\alpha \in \Delta_n$, there exists a solution of (3.1) passing α. The following lemma shows that the set Δ_n does not depend on the choice of projections.

Lemma 3.8. *The following hold*

(i) $\Delta_n = \Omega_n := \{x \in \mathbb{R}^m : f_n(x) - B_n x \in \mathrm{im} A_n\}$ $(n \geq 0)$.

(ii) $\Omega_n \cap \ker A_{n-1} = \{0\}$.

We observe that the initial condition (3.2) is equivalent to the condition

$$A_{n_0-1}x_{n_0} = A_{n_0-1}\gamma \quad (n_0 \geq 1), \tag{3.3}$$

which is independent of the choice of projections. Indeed, acting on both sides of (3.3) by $G^{-1}_{n_0-1}$ and using the equality $G^{-1}_{n_0-1}A_{n_0-1} = P_{n_0-1}$ we get (3.2). Conversely, multiplying on both sides of (3.2) by A_{n_0-1} and noting that $A_{n_0-1}P_{n_0-1} = A_{n_0-1}$ we obtain (3.3). For the sake of convenience, we choose a matrix $B_{-1} \in \mathbb{R}^{m\times m}$ such that the matrix pencil $\{A_{-1}, B_{-1}\}$ is of index-1. Then the matrix $G_{-1} = A_{-1} + B_{-1}Q_{-1}$ is nonsingular. Moreover, $A_{-1} = A_{-1}P_{-1}$ and $G^{-1}_{-1}A_{-1} = P_{-1}$. Thus both initial conditions (3.2) and (3.3) are equivalent for all $n_0 \geq 0$. The unique solution of the IVP (3.1), (3.2) or (3.1), (3.3) will be denoted by $x_n(n_0;\gamma)$. We shall restrict ourselves to the canonical projection onto $\ker A_{n-1}$, i.e., the projection from $\mathbb{R}^{m\times m}$ into $\ker A_{n-1}$ along $\mathcal{S}_n$ and will denote it again by Q_{n-1}. Then $P_{n-1} := I - Q_{n-1}$ is the canonical projection from $\mathbb{R}^{m\times m}$ into $\mathcal{S}_n$ along $\ker A_{n-1}$. Thanks to the decomposition $\mathbb{R}^m = \mathcal{S}_n \oplus \ker A_{n-1}$ the canonical projections are determined uniquely from the data A_n, B_n and A_{n-1}. Note that if $\widetilde{Q}_{n-1}$ is an any projection onto $\ker A_{n-1}$ $(n \geq 1)$ and $\widetilde{Q}_{n-1,n}$ is the associate connecting operator, then the canonical projection Q_{n-1} can be computed as $Q_{n-1} = \widetilde{Q}_{n-1,n}\widetilde{G}^{-1}_n B_n$, where $\widetilde{G}_n := A_n + B_n\widetilde{Q}_{n-1,n}$. Let $\mathbb{R}_+$ and $\mathbb{Z}_+$ be the set of nonnegative real numbers and nonnegative integers, respectively.

Definition 3.9. *The trivial solution of* (3.1) *is said to be*

(i) *A-stable (P-stable) if for each $\epsilon > 0$ and any $n_0 \geq 0$ there exists a $\delta = \delta(\epsilon, n_0) \in (0, \epsilon]$ such that the inequality $\|A_{n_0-1}\gamma\| < \delta$ $(\|P_{n_0-1}\gamma\| < \delta)$ implies $\|x_n(n_0;\gamma)\| < \epsilon$ for all $n \geq n_0$.*

(ii) *A-uniformly* (*P-uniformly*) *stable if it is A-stable* (*P-stable*) *and the number δ mentioned in part* (i) *of this definition does not depend on n_0.*

(iii) *A-asymptotically* (*P-asymptotically*) *stable if for any $n_0 \geq 0$ there exists a $\delta_0 = \delta_0(n_0) > 0$ such that the inequality $\|A_{n_0-1}\gamma\| < \delta_0$ $(\|P_{n_0-1}\gamma\| < \delta_0)$ implies $\|x_n(n_0;\gamma)\| \to 0$ $(n \to 0)$.*

From the relation $G^{-1}_n A_n = P_n$ and $A_n P_n = A_n$, it is easy to show that the notions A-stability and P-stability are equivalent. The same conclusion is true for the A-asymptotical stability and P-asymptotical stability. Further, if the matrices A_n are uniformly bounded, then A-uniform stability implies P-uniform stability. Conversely, if G^{-1}_n have uniformly bounded inverses, then A-uniform stability follows from P-uniform stability. Thus, in what follows we can drop the prefices A and P when talking about the stability or asymptotical stability.

Denote by $\mathcal{K}$ a class of all continuous and strictly increasing functions ψ from $[0,\infty)$ into itself, such that $\psi(0) = 0$. In what follows we study the stability of SDEs via Lyapunov functions $V : \mathbb{Z}_+ \times \mathbb{R}^m \to \mathbb{R}_+$ and denote $\Delta V(n,\xi_n) := V(n+1,\xi_{n+1}) - V(n,\xi_n)$ for all $n \geq 0$, $\xi_n \in \mathbb{R}^m$.

Lemma 3.10. *The trivial solution of* (3.1) *is A-uniformly* (*P-uniformly*) *stable if and only if there exists a function* $\psi \in \mathcal{K}$, *such that for any solution* x_n *of* (3.1) *and for any nonnegative integer* n_0, *there holds the inequality*

$$\|x_n\| \le \psi\big(\|A_{n_0-1}x_{n_0}\|\big)\ \forall\ n \ge n_0\ \big(\|x_n\| \le \psi\big(\|P_{n_0-1}x_{n_0}\|\big)\ \forall\ n \ge n_0\big).$$

Theorem 3.11. *The existence of the Lyapunov function* $V : \mathbb{Z}_+ \times \mathbb{R}^m \to \mathbb{R}_+$ *being continuous in the second variable at* 0 *and functions* $\psi_n \in \mathcal{K}$, *such that*

(i) $V(n,0) = 0, \quad n \ge 0;$

(ii) $\|y\| \le V(n, P_{n-1}y) \le \psi_n(\|P_{n-1}y\|), \quad \forall\ y \in \Delta_n, \quad n \ge 0;$

(iii) $\Delta V(n, P_{n-1}y_n) \le 0$ *for any solution* y_n *of the system* (3.1)

is a necessary and sufficient condition for the stability of the trivial solution of (3.1).

Theorem 3.12. *Assume that there exist a function* $\psi \in \mathcal{K}$ *and a Lyapunov function* $V : \mathbb{Z}_+ \times \mathbb{R}^m \to \mathbb{R}_+$, *which is continuous with respect to the second variable at* $\gamma = 0$, *such that*

(i) $V(n,0) = 0, \quad \forall\ n \ge 0;$

(ii) $\psi(\|x\|) \le V(n, A_{n-1}x), \quad \forall\ x \in \Omega_n, \quad n \ge 0;$

(iii) $\Delta V(n, A_{n-1}x_n) \le 0$ *for any solution* x_n *of the* SDE (3.1).

Then the trivial solution of the SDE (3.1) *is stable.*

Theorem 3.13. *The trivial solution of the* SDE (3.1) *is P-uniformly stable if and only if there exist two functions* $a, b \in \mathcal{K}$ *and a Lyapunov function* $V : \mathbb{Z}_+ \times \mathbb{R}^m \to \mathbb{R}_+$, *such that*

(i) $a(\|x\|) \le V(n, P_{n-1}x) \le b(\|P_{n-1}x\|), \ \forall x \in \Delta_n, \ n \ge 0.$

(ii) $\Delta V(n, P_{n-1}x_n) \le 0$ *for any solution* x_n *of the* SDE (3.1).

Theorem 3.14. *Suppose that there exist two functions* $a, b \in \mathcal{K}$ *and a Lyapunov function* $V : \mathbb{Z}_+ \times \mathbb{R}^m \to \mathbb{R}_+$, *such that*

(i) $a(\|x\|) \le V(n, A_{n-1}x) \le b(\|A_{n-1}x\|), \ \forall x \in \Omega_n, \ n \ge 0.$

(ii) $\Delta V(n, A_{n-1}x_n) \le 0$ *for any solution* x_n *of the system* (3.1).

Then the trivial solution of (3.1) *is A-uniformly stable. Moreover, if the condition* (ii) *is replaced by the following*

(iii) $\Delta V(n, A_{n-1}x_n) \le -c(\|A_{n-1}x_n\|)$ *for any solution* x_n *of* (3.1), *where c is a certain function of the class* $\mathcal{K}$,

then the trivial solution of the system (3.1) *is asymptotically stable.*

Further, we discribe an application of Lyapunov reduction theorem to study the stability of trivial solutions of a nonlinear periodic index-1 SDE

$$f_n(x_{n+1}, x_n) = 0\ (n \ge 0), \tag{3.4}$$

where $f_n : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ is a continuously differentiable function, $f_n(0,0) = 0$, $f_{n+N}(y,x) = f_n(y,x) \quad \forall\ n \ge 0, y, x \in \mathbb{R}^m$. Assume that (3.4) is of index-1 (see Definition 2.8), i.e.,

(i) $\ker \dfrac{\partial f_n}{\partial y}(y,x) = \mathcal{N}_n$, $\dim \mathcal{N}_n = m-r$, for some $1 \le r \le m-1$, $\forall\, n \ge 0$, $y, x \in \mathbb{R}^m$;

(ii) $\mathcal{S}_n(y,x) \cap \mathcal{N}_{n-1} = \{0\}$,
where $\mathcal{S}_n(y,x) = \big\{\xi \in \mathbb{R}^m : \dfrac{\partial f_n}{\partial x}(y,x)\xi \in \mathrm{im}\dfrac{\partial f_n}{\partial y}(y,x)\big\}$.

Let $A_n := \dfrac{\partial f_n}{\partial y}(0,0); B_n := \dfrac{\partial f_n}{\partial x}(0,0)$. We rewrite (3.4) as

$$A_n x_{n+1} + B_n x_n + h_n(x_{n+1}, x_n) = 0, \tag{3.5}$$

where $h_n(y,x) := f_n(y,x) - A_n y - B_n x$. Assuming that B_n are nonsingular matrices and using the periodic transformations discribed in the proof of Theorem 3.6 we can reduce (3.5) to a simpler system

$$\mathrm{diag}\,(I_r, O_{m-r})\overline{x}_{n+1} + \mathrm{diag}\,(-R, I_{m-r})\overline{x}_n + \overline{h}_n(\overline{x}_{n+1}, \overline{x}_n) = 0,$$

where the nonlinear part $\overline{h}_n$ satisfies conditions $\overline{h}_n(0,0) = 0$; $\dfrac{\partial \overline{h}_n}{\partial y}(0,0) = 0$ and $\dfrac{\partial \overline{h}_n}{\partial x}(0,0) = 0$. If all eigenvalues of R have modulus less than one, then the trivial solution of (3.4) is exponentially asymtotically stable, i.e., $\|x_n\| \le c\|\widetilde{P}_{-1}x_0\|e^{-\alpha n}$, for some positive constants α and c, where $\widetilde{P}_{-1} = \widetilde{P}_{N-1}$ -the canonical projection along $\ker A_{N-1}$, provided $\|\widetilde{P}_{-1}x_0\|$ is sufficiently small.

A stability analysis of a switched system consisting of linear singular time-invariant subsystems and a periodic switching rule by using Floquet theory can be carried out similarly as in [24].

The robust stability of linear dynamical systems has been studied for decades from different points of view by several authors, such as, Van Loan, Hinrichsen and Prichard, Martin and Byers, Qiu and Davison, etc... Problems of determining the stability radii of linear singular systems have attracted much attention of researchers (see [17 - 21, 23, 24]).

Consider a linear constant coefficient SDE

$$Ax_{n+1} = Bx_n, \;\; n = 0, 1, \ldots \tag{3.6}$$

where x_n, $n = 0, 1, \ldots$, are vectors in $\mathbb{K}^m$, $\mathbb{K} \in \{\mathbb{C}, \mathbb{R}\}$, A, B are constant matrices in $\mathbb{K}^{m\times m}$ and A is a singular matrix. Moreover, the pencil $\{A, B\}$ is supposed to be regular, i.e., there exists $\lambda \in \mathbb{C}$, such that $\det(\lambda A - B) \ne 0$. We determine the infimum $r_{\mathbb{K}}$ of the norm of disturbances under which the perturbed systems

$$(A + E\Delta_1 F_1)x_{n+1} = (B + E\Delta_2 F_2)x_n \tag{3.7}$$

are no longer asymptotically stable or regular. Here, Δ_i $(i = 1, 2)$ denote unknown disturbance matrices in $\mathbb{K}^{p\times q}$, E, F_1 and F_2 are matrices of appropriate sizes specifying the structure of perturbation. For definiteness, the Euclidean norm of vectors and the corresponding matrix norm will be used. Let $\|(\Delta_1, \Delta_2)\| := \|\Delta_1\| + \|\Delta_2\|$, then, by the definition

$$r_{\mathbb{K}} = \inf\{\|(\Delta_1, \Delta_2)\| : \Delta_1, \Delta_2 \in \mathbb{K}^{p\times q}, \text{ s.t. (3.7) is unstable or irregular}\}.$$

Theorem 3.15. *The complex stability radius $r_{\mathbb{C}}$ for the system* (3.6) *is given by*

$$r_{\mathbb{C}} = \Big\{ \max \Big[\sup_{|t|\geq 1} \|G(t)\|, \sup_{|t|\geq 1} \|H(t)\| \Big] \Big\}^{-1}, \tag{3.8}$$

with the transfer functions $G(t) = F_2(tA - B)^{-1}E$ and $H(t) = F_1 t(tA - B)^{-1}E$.

Formula (3.8) slightly generalizes the corresponding ones in [20]. Further, it has been proved in [20] that the stability radius with respect to perturbations in the leading matrix only, for the discrete system

$$A\frac{x_{n+1} - x_n}{h} = Bx_{n+1},$$

obtained by the implicit Euler method, monotone decreasingly tends to the stability radius of the DAE

$$Ax'(t) = Bx(t),$$

as the discretized step tends to zero.

4. Singular Stochastic Difference Equations

4.1. Solution of Linear Singular Difference Equations on $\mathbb{Z}$

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space satisfying the normal conditions (see [33]) and let $\theta : (\Omega, \mathcal{F}, \mathbf{P}) \to (\Omega, \mathcal{F}, \mathbf{P})$ be an invertible, $\mathbf{P}$-preserving transformation. For given two random variables valued in the space of $m \times m$-matrices $A(\cdot)$ and $B(\cdot)$, we consider the system

$$\begin{cases} A(\theta^n\omega)X_{n+1}(\omega) = B(\theta^n\omega)X_n(\omega), \ n \in \mathbb{Z}, \\ X_0 = x \in \mathbb{R}^m \text{ a.s.}, \end{cases} \tag{4.1}$$

where $\theta^n = \theta \circ \theta^{n-1}$ and $\mathbb{Z}$ is the set of all integers. System (4.1) is called a real noise linear SDE.

Assume that $\operatorname{rank} A(\omega) = r$ for $\mathbf{P}$-a.s. $\omega \in \Omega$, where r $(0 < r \leq m)$ is a nonrandom constant. Let T be a random variable with values in $\mathrm{GL}(\mathbb{R}^m)$, such that $T(\omega)\big|_{\ker A(\omega)}$ is an isomorphism between $\ker A(\omega)$ and $\ker A(\theta^{-1}\omega)$. Let $Q(\omega)$ be a measurable projection onto $\ker A(\omega)$. Denote

$$S(\omega) = \{z : B(\omega)z \in \operatorname{im} A(\omega)\}.$$

Definition 4.1. *The SSDE* (4.1) *is said to be index-*1 *tractable if*

$$S(\omega) \cap \ker A(\theta^{-1}\omega) = \{0\} \quad \text{for a.s. } \omega \in \Omega. \tag{4.2}$$

By virtue of Lemma 2.3, System (4.1) is index-1 tractable if and only if the matrices

$$G(\omega) := A(\omega) + B(\omega)T(\omega)Q(\omega)$$

are nonsingular with probability one.

Remark 1.

1. In fact, what we are doing below is true if the space Ω can be divided into θ-invariant subsets $\Omega_i, i = 1, 2, ..., q$ such that rank$A$ is constant on every Ω_i and (4.2) is satisfied for $\mathbf{P}$-almost sure ω.

2. In general, since $A(\omega)$ is degenerate, the dimension of the space of solutions varies in n. If the dimension of the space of solutions at step $n+1$ is greater than that at the step n, it may cause bifurcations or multi-valued functions. Up to now, there has been no comprehensive and systematic analysis of such systems. As we can see below, $S(\omega)$ is in fact the space of solutions. Therefore, with the index-1 tractability assumption (Condition (4.2)), dim$S(\omega)$ is constant. That implies the regularity of inherent ordinary difference equation.

For the sake of simplicity, we put

$$\begin{aligned}&Q_n(\omega) = Q(\theta^n\omega),\ P_n = I - Q_n,\ A_n(\omega) = A(\theta^n\omega),\ B_n(\omega) = B(\theta^n\omega),\\ &G_n(\omega) = G(\theta^n\omega),\ T_n(\omega) = T(\theta^n\omega).\end{aligned}$$

In what follows, if there is no confusion, we will omit ω in formulae, further, we always suppose that Equation (4.1) is index-1 tractable. Then it can be rewritten as

$$A_n X_{n+1} = B_n X_n,$$

i.e., we obtain (2.1) with $q_n = 0$. For $n \geq 0$, according to (2.10) the above equation is reduced to

$$\begin{cases} Y_{n+1} = P_n G_n^{-1} B_n Y_n, \\ Z_n = -T_n Q_n G_n^{-1} B_n Y_n, \end{cases} \quad n = 0, 1, \dots, \tag{4.3}$$

where $Y_n = P_{n-1} X_n$ and $Z_n = Q_{n-1} X_n$.

From (4.3) it follows that the solution with the initial condition $X_0 = x$ exists if

$$Q_0 G_0^{-1} B_0 x = 0$$

with probability one. By virtue of Lemma 3.8 we see that the random space $\mathcal{J}_{\mathrm{f}}(\omega) = \{\xi \in \mathbb{R}^m : (Q_0 G_0^{-1} B_0)(\omega)\xi = 0\}$ does not depend on the choice of the projection Q and the transformation T. Moreover, by Lemma 2.10

$$\widetilde{Q}_{n-1} := T_n Q_n G_n^{-1} B_n,$$

is the projection onto $\ker A_{n-1}$ along $S_n = \{\xi \in \mathbb{R}^m : B_n \xi \in \ \mathrm{im} A_n\}$. Hence, the matrices

$$\widetilde{G}_n = A_n + B_n T_n \widetilde{Q}_n$$

are also nonsingular with probability one.

Lemma 4.2. *The canonical projection $\widetilde{Q}_{n-1}(\omega)$ and the matrix $(\widetilde{P}_n G_n^{-1} B_n)(\omega)$ are independent of the choice of $Q(\omega)$ and $T(\omega)$, where $\widetilde{P}_n = I - \widetilde{Q}_n$.*

Using the canonical projections we have $\widetilde{Q}_{n-1}(\omega) X_n(\omega) = 0$ which implies $X_n(\omega) = \widetilde{P}_{n-1}(\omega) X_n(\omega)$ for a.s. $\omega \in \Omega$. Therefore, the forward equation of (4.1) for $n \geq 0$ with the initial condition $X_0 = x \in \mathcal{J}_{\mathrm{f}}$ is reduced to a classical difference equation

$$\begin{cases} X_{n+1} = \widetilde{P}_n \widetilde{G}_n^{-1} B_n \widetilde{P}_{n-1} X_n, \ n = 0, 1, \dots, \\ X_0 = x, \end{cases} \tag{4.4}$$

which gives

$$X_n(\omega) = \big(\prod_{i=1}^{n} \widetilde{P}_{n-i} \widetilde{G}_{n-i}^{-1} B_{n-i} \widetilde{P}_{n-i-1}\big)(\omega)x, \ \ n \geq 1, \ \ X_0(\omega) = x. \tag{4.5}$$

Summing up, we see that the IVP of (4.1) for $n \geq 0$ has a unique solution given by (4.5) provided $x \in \mathcal{J}_{\mathrm{f}}$.

Note that since $(\widetilde{P}_n \widetilde{G}_n^{-1} B_n)(\omega) = (\widetilde{P}_n G_n^{-1} B_n)(\omega)$, the equation (4.4) can be also rewritten as

$$\begin{cases} X_{n+1} = \widetilde{P}_n G_n^{-1} B_n \widetilde{P}_{n-1} X_n, \ n = 0, 1, \dots, \\ X_0 = \widetilde{P}_{-1} x, \end{cases}$$

where $x \in \mathbb{R}^m$ is a given vector.

In the case $n < 0$, Equation (4.1) turns into the SSDE

$$\begin{cases} B_n X_n = A_n X_{n+1}, \ n = -1, -2, \dots, \\ X_0 = x. \end{cases} \tag{4.6}$$

We suppose that $\operatorname{rank} B(\omega) = k$ for $\mathbf{P}$-a.s. $\omega \in \Omega$, where k $(0 < k \leq m)$ is a nonrandom constant. Let $\overline{T}$ be a random variable with values in $\mathrm{GL}(\mathbb{R}^m)$, such that $\overline{T}(\omega)\big|_{\ker B(\omega)}$ is an isomorphism between $\ker B(\omega)$ and $\ker B(\theta\omega)$. Let $\overline{Q}(\omega)$ be a measurable projection onto $\ker B(\omega)$. Put

$$\bar{G}(\omega) := B(\omega) + A(\omega)\bar{T}(\omega)\overline{Q}(\omega).$$

Suppose that $\overline{G}$ is nonsingular with probability one. This assumption implies the index-1 tractability of Eq. (4.6). Let

$$\bar{Q}_n(\omega) = \overline{Q}(\theta^n\omega), \ \bar{G}_n(\omega) = \overline{G}(\theta^n\omega), \ \bar{P}_n = I - \overline{Q}_n, \ \bar{T}_n(\omega) = \bar{T}(\theta^n\omega), \ n < 0.$$

For any $n < 0$, denote $Y_n = \overline{P}_n X_n, \ Z_n = \overline{Q}_n X_n$. Equation (4.6) leads to

$$\begin{cases} Y_n = \overline{P}_n \bar{G}_n^{-1} A_n Y_{n+1}, \\ Z_{n+1} = -\overline{T}_n \overline{Q}_n \overline{G}_n^{-1} A_n Y_{n+1}, \\ X_n = Y_n + Z_n, \ \ n = -1, -2, \dots \end{cases}$$

Denote $\widehat{Q}_n := \bar{T}_{n-1} \, \overline{Q}_{n-1} \, \overline{G}_{n-1}^{-1} A_{n-1}$ the projection onto $\ker B_n$ along $\overline{S}_{n-1} = \{\xi : A_{n-1}\xi \in \operatorname{im} B_{n-1}\}$. Let $\widehat{G}_n(\omega) = B_n(\omega) + A_n(\omega)\overline{T}_n \widehat{Q}_n$. By a similar argument as above we obtain

Lemma 4.3. *The canonical projection $\widehat{Q}_n$ and the matrix $\widehat{P}_n \bar{G}_n^{-1} A_n$ are independent of the choice of $\overline{Q}$ and $\overline{T}$, where $\widehat{P}_n = I - \widehat{Q}_n$.*

With the canonical projections $\widehat{Q}_n$ we have $\widehat{Q}_n X_n = 0$ for any $n < 0$. Therefore,

$$\begin{cases} X_n = \prod_{i=n}^{-1} \widehat{P}_i \widehat{G}_i^{-1} A_i \widehat{P}_{i+1} x, \quad n = -1, -2, ..., \\ X_0 = x \in \mathcal{J}_{\mathrm{b}}, \end{cases} \tag{4.7}$$

where $\mathcal{J}_{\mathrm{b}} = \{\xi \in \mathbb{R}^m : \widehat{Q}_0 \xi = 0\}$. Thus, the IVP of (4.1) for $n \leq 0$ has a unique solution given by (4.7) provided $x \in \mathcal{J}_{\mathrm{b}}$.

Remark 2.

1. Let us give some comments on the expressions of solutions (4.5) and (4.7). To obtain (4.5) for $n \geq 0$ we suppose the initial condition $X_0 = x \in \mathcal{J}_f$ (equivalently $X_0 = \widetilde{P}_{-1} x,\ x \in \mathbb{R}^m$) to be satisfied and to obtain (4.7) for $n \leq 0$ the equality $X_0 = \widehat{P}_0 x$ is required. Thus, there exists a unique solution of (4.1) with the initial condition $X_0 = x$ if and only if $x \in \mathcal{J}_{\mathrm{f}} \cap \mathcal{J}_{\mathrm{b}}$.

2. Unlike random ordinary difference equations, in general, the existence of solution of SSDEs implies that the initial condition must be a measurable selection of the corresponding $\omega \mapsto \mathcal{J}_{\mathrm{f}}(\omega) \cap \mathcal{J}_{\mathrm{b}}(\omega)$.

4.2. Dynamic Property

Since $\widehat{Q}_0$ is the projection onto $\ker B_0$,

$$\widetilde{Q}_{-1} \widehat{Q}_0 = T_0 Q_0 G_0^{-1} B_0 \widehat{Q}_0 = 0.$$

Similarly, $\widehat{Q}_0 \widetilde{Q}_{-1} = 0$. Hence, the projections $\widetilde{P}_{-1}$ and $\widehat{P}_0$ commute, i.e.,

$$\widetilde{P}_{-1} \widehat{P}_0 = \widehat{P}_0 \widetilde{P}_{-1}$$

with probability one.

Put

$$\Phi(n, \omega) = \begin{cases} \prod_{i=1}^{n} \big(\widetilde{P}_{n-i} G_{n-i}^{-1} B_{n-i}\big)(\omega) & \text{if } n > 0, \\ \big(\widetilde{P}_{-1} \widehat{P}_0\big)(\omega) & \text{if } n = 0, \\ \prod_{i=n}^{-1} \big(\widehat{P}_i \overline{G}_i^{-1} A_i\big)(\omega) & \text{if } n < 0. \end{cases} \tag{4.8}$$

We come to the so-called co-cycle property of linear SSDEs (cf. [8]).

Theorem 4.4. *For any $m,\ n \in \mathbb{Z}$ the following relation holds*

$$\Phi(n + m, \omega) = \Phi(n, \theta^m \omega) \cdot \Phi(m, \omega).$$

Denote by $X_n\big(z(\omega), \omega\big)$ the solution of (4.1) satisfying $X_0\big(z(\omega), \omega\big) = z(\omega)$. It is clear that

$$X_n\big(\bar{x}(\omega), \omega\big) = \Phi(n, \omega) x \text{ with } \bar{x}(\omega) = \big(\widetilde{P}_{-1} \widehat{P}_0\big)(\omega) x.$$

4.3. Lyapunov Exponents and Multiplicative Ergodic Theorem

Suppose that θ is an ergodic transformation on $(\Omega, \mathcal{F}, \mathbf{P})$ and the following condition is satisfied

Hypotheses 4.5.

$$\ln \|\widetilde{P}_0 \widetilde{G}_0^{-1} B_0\| \in L_1(\Omega, \mathcal{F}, \mathbf{P}) \text{ and } \ln \|\widehat{P}_0 \widehat{G}_0^{-1} A_0\| \in L_1(\Omega, \mathcal{F}, \mathbf{P}). \tag{4.9}$$

We note that these assumptions are independent of the choice of T, $\overline{T}$ and Q, $\overline{Q}$.

Since (Φ_n) is the product of ergodic stationary matrices $\widetilde{P}_n \widetilde{G}_n^{-1} B_n$ for $n > 0$ and $\widehat{P}_n \widehat{G}_n^{-1} A_n$ for $n < 0$, by [26] we get

i) Under the assumption (4.9), there exist the limits

$$\lim_{n\to\infty} \big(\Phi(n,\omega)^\top \Phi(n,\omega)\big)^{1/2n} =: \Delta(\omega), \quad \lim_{n\to -\infty} \big(\Phi(n,\omega)^\top \Phi(n,\omega)\big)^{1/2|n|} =: \overline{\Delta}(\omega).$$

ii) Let $0 < e^{\lambda_1} < e^{\lambda_2} < \cdots < e^{\lambda_\tau}$ be the different nonzero eigenvalues of Δ and $\lambda_0 = -\infty$. We denote $\mathcal{U}_0 = \ker \Delta(\omega)$, and $\mathcal{U}_i,\ i = 1, 2, \ldots, \tau$ the eigenspace with multipliers $d_i = \dim \mathcal{U}_i$ corresponding to the eigenvalue e^{λ_i}. Then, $\tau;\ d_i, \lambda_i, i = 1, 2, \ldots, \tau$ are nonrandom constants. Let $\mathcal{V}_k = \mathcal{U}_0 \oplus \mathcal{U}_1 \oplus \cdots \oplus \mathcal{U}_k, k = 0, 1, \ldots, \tau$ such that the sequence

$$\{0\} \subset \mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_\tau = \mathbb{R}^m$$

defines a filtration of $\mathbb{R}^m$. For each $x \in \mathbb{R}^m$ the Lyapunov exponent

$$\lambda(\omega, x) = \lim_{n\to\infty} \tfrac{1}{n} \ln \|\Phi(n,\omega)x\|$$

exists and

$$\lambda(\omega, x) = \lambda_k \Longleftrightarrow x \in (\mathcal{V}_k \setminus \mathcal{V}_{k-1})(\omega)$$

for $k = 1, 2, \ldots, \tau$. Further, $\lambda(\omega, x) = -\infty \Leftrightarrow x \in \mathcal{V}_0$ and the spaces $\mathcal{V}_i$ are invariant in the sense

$$\Phi(n,\omega)\mathcal{V}_i(\omega) \subset \mathcal{V}_i(\theta^n \omega)$$

for any $n \geq 0$, $i = 0, 1, \ldots, \tau$.

iii) A similar result can be formulated for the case $n \to -\infty$. That is, let $0 < e^{\overline{\lambda}_{\overline{\tau}}} < e^{\overline{\lambda}_{\overline{\tau}-1}} < \cdots < e^{\overline{\lambda}_1}$ be the different nonzero eigenvalues of $\overline{\Delta}$, $\overline{\lambda}_{\overline{\tau}+1} = -\infty$. Denote $\overline{\mathcal{U}}_{\overline{\tau}+1} = \ker \overline{\Delta}(\omega)$ and $\overline{\mathcal{U}}_{\overline{\tau}}, \ldots, \overline{\mathcal{U}}_1$ the corresponding eigenspace with multipliers $\overline{d}_i = \dim \overline{\mathcal{U}}_i,\ i = 1, 2, \ldots, \overline{\tau}$. Then $\overline{\tau};\ \overline{d}_i, \overline{\lambda}_i, i = 1, 2, \ldots, \overline{\tau}$ are nonrandom constants. Let $\overline{\mathcal{V}}_k = \overline{\mathcal{U}}_{\overline{\tau}+1} \oplus \overline{\mathcal{U}}_{\overline{\tau}} \oplus \cdots \oplus \overline{\mathcal{U}}_k, k = \overline{\tau}+1, \overline{\tau}, \ldots, 1$, such that the sequence

$$\{0\} \subset \overline{\mathcal{V}}_{\overline{\tau}+1} \subset \overline{\mathcal{V}}_{\overline{\tau}} \subset \cdots \subset \overline{\mathcal{V}}_1 = \mathbb{R}^m$$

defines another filtration of $\mathbb{R}^m$. For each $x \in \mathbb{R}^m$ the Lyapunov exponent

$$\overline{\lambda}(\omega, x) = \lim_{n\to -\infty} \frac{1}{|n|} \ln \|\Phi(n,\omega)x\|$$

exists and

$$\overline{\lambda}(\omega, x) = \overline{\lambda}_k \Longleftrightarrow x \in (\overline{\mathcal{V}}_k \setminus \overline{\mathcal{V}}_{k+1})(\omega)$$

for $k = 1, 2, \dots, \overline{\tau}$, and $\overline{\lambda}(\omega, x) = -\infty \Leftrightarrow x \in \overline{\mathcal{V}}_{\overline{\tau}+1}$. Moreover, the spaces $\overline{\mathcal{V}}_i$ are invariant in the sense

$$\Phi(n, \omega)\overline{\mathcal{V}}_i(\omega) \subset \overline{\mathcal{V}}_i(\theta^n\omega), \quad \forall\, n < 0$$

for $i = 1, 2, \dots, \overline{\tau} + 1$.

We arrive at the following MET for linear SSDEs [16].

Theorem 4.6. *Suppose that*

$$\ln\|\widetilde{P}_0\widetilde{G}_0^{-1}B_0\|,\ \ln\|\widehat{P}_0\widehat{G}_0^{-1}A_0\| \in L_1.$$

Then,

i) $\tau = \overline{\tau}$; $\lambda_i = -\overline{\lambda}_i$, $d_i = \overline{d}_i$, $i = 1, \dots, \tau$. *Further,* $\lambda_1, \lambda_2, \dots, \lambda_\tau$ *and* $d_1, d_2, \dots, d_\tau$ *are nonrandom numbers.*

(ii) *For any* $i = 1, 2, \dots, \tau$, *the set* $\mathcal{V}_i \cap \overline{\mathcal{V}}_i$ *is invariant.*

(iii) *There exist subspaces* $W_0 = \mathcal{V}_0, W_1, \dots, W_\tau$, *such that*

$$\mathbb{R}^m = \oplus_{i=0}^{\tau} W_i, \quad W_i \oplus \mathcal{V}_0 = \mathcal{V}_i \cap \overline{\mathcal{V}}_i,\ \ \dim W_i = d_i,\ i = 1, 2, \dots, \tau,$$

and for any $x \in W_i \setminus \{0\}$

$$\lim_{n\to\pm\infty} \frac{1}{n}\ln\|\Phi(n, \omega)x(\omega)\| = \lambda_i = \lim_{n\to\pm\infty} \frac{1}{n}\ln\|X_n(\overline{x}(\omega), \omega)\|$$

where $\overline{x} = \widetilde{P}_{-1}\widehat{P}_0 x$.

4.4. Furstenberg–Kifer Decomposition for Singular Stochastic Difference Equations

We turn to the Furstenberg-Kifer multiplicative ergodic theorem for the index-1 tractable SSDE (4.1). Let (ξ_n) be an i.i.d random sequence, defined on $(\Omega, \mathcal{F}, \mathbf{P})$, with values in a Polish space Y. Suppose that we are given two matrix-functions $A, B : Y \to \mathbb{R}^{m\times m}$. Denote $A(\omega) = A\big(\xi_0(\omega)\big)$, $B(\omega) = B\big(\xi_0(\omega)\big)$ and assume that $\operatorname{rank}A(y) = r$, $\operatorname{rank}B(y) = k$ for all $y \in Y$ and the functions $A(\cdot)$, $B(\cdot)$ are continuous on Y. Recall that

$$A(y) = U(y)\begin{pmatrix}\Sigma(y) & 0\\ 0 & 0\end{pmatrix}V(y)^T, \quad B(y) = \overline{U}(y)\begin{pmatrix}\overline{\Sigma}(y) & 0\\ 0 & 0\end{pmatrix}\overline{V}(y)^T,$$

where $U(y), \overline{U}(y), V(y), \overline{V}(y)$ are orthogonal matrix-functions, continuous in y and $\Sigma(y)$, $\overline{\Sigma}(y)$ are an $r\times r$ and $k\times k$ nonsingular matrices respectively. We recall some notations which have already been introduced in the above subsections, that is, $Q = \operatorname{diag}(O_r, I_{m-r})$, $\overline{Q} = \operatorname{diag}(O_k, I_{m-k})$ and

$$\begin{aligned}
G_n &= A(\xi_n) + B(\xi_n)V(\xi_{n-1})QV^T(\xi_n),\\
\widetilde{Q}_{n-1} &= V(\xi_{n-1})QV^T(\xi_n)G_n^{-1}B(\xi_n)\ (n \ge 0),\\
\overline{G}_n &= B(\xi_n) + A(\xi_n)\overline{V}(\xi_{n+1})\overline{Q}\,\overline{V}^T(\xi_n),\\
\widehat{Q}_n &= \overline{V}(\xi_n)\overline{Q}\,\overline{V}^T(\xi_{n-1})\overline{G}_{n-1}^{-1}A(\xi_{n-1})\ (n < 0).
\end{aligned}$$

Furthermore, we denote

$$H(\xi_{n+1}, \xi_n, \xi_{n-1}) = \widetilde{P}_n G_n^{-1} B(\xi_n),$$

where $\widetilde{P}_n = I - \widetilde{Q}_n$, then System (4.1) has a unique solution given by

$$X_n(x) = \prod_{i=1}^{n} H(\xi_{n-i+1}, \xi_{n-i}, \xi_{n-i-1})x, \quad n \geq 1,$$

provided $\widetilde{Q}_0 x = 0$ a.s. We remark that

$$H(\xi_{n+1}, \xi_n, \xi_{n-1}) = H(\xi_{n+1}, \xi_n, \xi_{n-1})\widetilde{P}_{n-1}$$

and the matrices $H(\xi_{n+1}, \xi_n, \xi_{n-1})$ are degenerate with $\text{rank} H(\xi_{n+1}, \xi_n, \xi_{n-1}) = r$ for all $n \geq 0$, so the hypothesis of Furtenberg–Kifer in [23] is not satisfied.

A similar result can be obtained for the case $n < 0$, namely, putting

$$\widehat{H}(\xi_{n+1}, \xi_n, \xi_{n-1}) = \widehat{P}_n \overline{G}_n^{-1} A(\xi_n),$$

where $\widehat{P}_n = I - \widehat{Q}_n$, then (4.1) has a unique solution

$$X_n(x) = \prod_{i=n}^{-1} \widehat{H}(\xi_{i+1}, \xi_i, \xi_{i-1})x, \quad n = -1, -2, \ldots$$

provided $\widehat{Q}_0 x = 0$ a.s.

It is worth of noting that System (4.1) possesses solutions only for those initial values x satisfying $\pi x = x$ a.s., where $\pi = \widetilde{P}_{-1}\widehat{P}_0$. Therefore, as is seen below, instead of a nonrandom filtration as is established in [22], here, we are able to find only a decomposition depending on the initial noise ξ_0. Since (ξ_n) is an equi-distributed sequence, we can write $\xi_n = \xi(\theta^n)$, with a random variable $\xi(\omega)$ defined on $(\Omega, \mathcal{F}, \mathbf{P})$. In addition, to simplify notations, we put $\eta_n = (\xi_{n+1}, \xi_n, \xi_{n-1})$, $\mathcal{Y} = Y \times Y \times Y$ and as the above mentioned, the solution of (4.1) is given by $X_n(x, \omega) = \Phi(n, \omega)x$, where $\Phi(n, \omega)$ is defined by the formula (4.8). Although the hypothesis of Furtenberg–Kifer in [23] for the index-1 tractable SSDE (4.1) is not satisfied, the Furstenberg–Kifer multiplicative ergodic theorem for System (4.1) has been established in [15].

Theorem 4.7. *There exist $\tau + 1$ numbers $-\infty = \lambda_0 < \lambda_1 < \cdots < \lambda_\tau$ and a family of filtrations consisting of linear subspaces of $\mathbb{R}^m$*

$$\{0\} \subset \mathcal{V}_0(y) \subset \mathcal{V}_1(y) \subset \cdots \subset \mathcal{V}_\tau(y) = \mathbb{R}^m, \ y \in \mathcal{Y}$$

such that the map $y \mapsto \mathcal{V}_i(y)$ is measurable and for any $0 < i \leq \tau$, if $x \in \mathcal{V}_i(\eta_0) \setminus \mathcal{V}_{i-1}(\eta_0)$ then

$$\lim_{n\to\infty} \frac{1}{n} \ln \|X_n(x, \omega)\| = \lambda_i,$$

$\mathbf{P}$- *a.s. $\omega \in \Omega$. Furthermore, the family of filtrations $\{\mathcal{V}_i(y)\}$ is invariant in the sense*

$$\Phi(n, \omega)\mathcal{V}_i(\eta_0) \subset \mathcal{V}_i(\eta_n)$$

for a.s. $\omega \in \Omega, \ n > 0$.

A similar result can be given for the case $n < 0$.

Theorem 4.8. *There exist* $\overline{\tau}+1$ *numbers* $-\infty = \overline{\lambda}_{\overline{\tau}+1} < \overline{\lambda}_{\overline{\tau}} < \cdots < \overline{\lambda}_1$ *and a family of filtrations consisting of linear subspaces of* $\mathbb{R}^m$

$$\{0\} \subset \overline{\mathcal{V}}_{\overline{\tau}+1}(y) \subset \overline{\mathcal{V}}_{\overline{\tau}}(y) \subset \ldots \subset \overline{\mathcal{V}}_1(y) = \mathbb{R}^m,\ y \in \mathcal{Y}$$

such that the map $y \mapsto \overline{\mathcal{V}}_i(y)$ *is measurable and if* $x \in \overline{\mathcal{V}}_{i+1}(\eta_0) \setminus \overline{\mathcal{V}}_i(\eta_0)$ *then*

$$\lim_{n\to-\infty} \frac{1}{|n|} \ln \|X_n(x,\omega)\| = \overline{\lambda}_i,$$

$\mathbf{P}$- *a.s.* $\omega \in \Omega$. *Moreover, the family of filtrations* $\{\overline{\mathcal{V}}_i(y)\}$ *is invariant in the sense*

$$\Phi(n,\omega)\overline{\mathcal{V}}_i(\eta_0) \subset \overline{\mathcal{V}}_i(\eta_n),$$

for a.s. $\omega \in \Omega,\ n < 0$.

It is shown that $\tau = \overline{\tau}$ and $\lambda_i = -\overline{\lambda}_i$ for any $1 \le i \le \tau$ (see [15] and references therein). Thus, putting

$$\mathcal{U}_0(y) = \ker\pi(y),\ \mathcal{U}_1(y) = \mathcal{V}_1 \cap \overline{\mathcal{V}}_1,\ \mathcal{U}_2 = \mathcal{U}_1 \oplus \mathcal{V}_2 \cap \overline{\mathcal{V}}_2, \ldots, \mathcal{U}_\tau = \mathcal{U}_{\tau-1} \oplus \mathcal{V}_\tau \cap \overline{\mathcal{V}}_\tau.$$

We have the following theorem.

Theorem 4.9. (Furstenberg–Kifer decomposition) *There exist* $\tau + 1$ *numbers* $-\infty = \lambda_0 < \lambda_1 < \cdots < \lambda_\tau$ *and a family of filtrations consisting of linear subspaces of* $\mathbb{R}^m$

$$\{0\} \subset \mathcal{U}_0(y) \subset \mathcal{U}_1(y) \subset \ldots \subset \mathcal{U}_\tau(y) = \mathbb{R}^m,\ y \in \mathcal{Y},$$

such that the map $y \mapsto \mathcal{U}_i(y)$ *is measurable and if* $x \in \mathcal{U}_i(\eta_0) \setminus \mathcal{U}_{i-1}(\eta_0)$ *then*

$$\lim_{n\to\pm\infty} \frac{1}{n} \ln \|X_n(x,\omega)\| = \lambda_i,$$

$\mathbf{P}$- *a.s.* $\omega \in \Omega$. *Furthermore, the family of filtrations* $\{\mathcal{U}_i(y)\}$ *is invariant in the sense*

$$\Phi(n,\omega)\mathcal{U}_i(\eta_0) \subset \mathcal{U}_i(\eta_n),$$

for a.s. $\omega \in \Omega,\ n \in \mathbb{Z}$.

Finally, an application of Furstenberg-Kifer decomposition for studying the existence of a bounded solution to the equation

$$A(\xi_n)X_{n+1} = B(\xi_n)X_n + q_n, \quad n \in \mathbb{Z} \tag{4.10}$$

has been given in [15] under the following hypothesis.

Hypotheses 4.10. *There exists a constant* C *such that*

$$\mathbf{P}\{\|H(\eta_1)\| \le C\} = 1, \quad \mathbf{P}\{\|\widehat{H}(\eta_1)\| \le C\} = 1.$$

Next, let L_α be the set of the sequences of random variables (q_n) such that

$$\sup_{n\in\mathbb{Z}} \mathbf{E}\|q_n\|^\alpha < \infty$$

and

$$L = \bigvee_{0<\alpha\leq\alpha_0} L_\alpha$$

for an $\alpha_0 > 0$. The unique solvability of Eq. (4.10) is guaranted by the following theorem.

Theorem 4.11. *Suppose that the spectrum of the system*

$$A(\xi_n)X_{n+1} = B(\xi_n)X_n$$

does not contain 0*, then for any* $(q_n) \in L$ *Equation* (4.10) *has a unique solution in* L.

Conclusion

Most of existing work on linear SDEs deals with constant coefficient systems. In our recent publications, time varying SDEs and SSDEs have been studied. This paper has summarized a number of results on deterministic and stochastic SDEs. Further studies on control problems for time varying SDEs and SSDEs may be of interest and should be performed.

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