*Vietnam Journal of Mathematics* **35**:4(2007) 541–561

Vietnam Journal of MATHEMATICS


# A Homogeneous Model for Mixed Complementarity Problems over Symmetric Cones*

**Yedong Lin**[1] **and Akiko Yoshise**[2]

[1] *AIMNEXT Inc. Minato-ku, Tokyo 105-0013, Japan*
[2] *Graduate School of Systems and Information Engineering, University of Tsukuba, Tsukuba, Ibaraki 305-8573, Japan*

Dedicated to Professor Hoang Tuy on the occasion of his 80th-birthday

Received October 5, 2007

**Abstract.** In this paper, we propose a homogeneous model for solving monotone mixed complementarity problems over symmetric cones, by extending the results in [11] for standard form of the problems. We show that the extended model inherits the following desirable features: (a) A path exists, is bounded and has a trivial starting point without any regularity assumption concerning the existence of feasible or strictly feasible solutions. (b) Any accumulation point of the path is a solution of the homogeneous model. (c) If the original problem is solvable, then every accumulation point of the path gives us a finite solution. (d) If the original problem is strongly infeasible, then, under the assumption of Lipschitz continuity, any accumulation point of the path gives us a finite certificate proving infeasibility. We also show that the homogeneous model is directly applicable to the primal-dual convex quadratic problems over symmetric cones.

2000 Mathematics Subject Classification: 90C22, 90C25, 90C33, 65K05, 46N10.

*Keywords:* Complementarity problem, nonlinear optimization, optimality condition, symmetric cone, Euclidean Jordan algebra, homogeneous algorithm, interior point method, detecting infeasibility.

---

*This work was supported in part by Grant-in-Aid for Scientific Research ((c)18560052) of the Ministry of Education, Culture, Sports, Science and Technology of Japan. An earlier of the paper was released as Discussion Paper Series 1130, Department of Social Systems and Management, University of Tsukuba, Tsukuba, Ibaraki 305-8573, Japan.

## 1. Introduction

Let $(V, \circ)$ be a Euclidian Jordan algebra with an identity element $e$. We denote by $K$ the symmetric cone of $V$ which is a self-dual closed convex cone such that for any two elements $x \in \mathrm{int}K$ and $y \in \mathrm{int}K$, there exists an invertible map $\Gamma : V \to V$ satisfying $\Gamma(K) = K$ and $\Gamma(x) = y$. It is known that a cone in $V$ is symmetric if and only if it is the cone of squares of $V$ given by $K = \{x \circ x : \ x \in V\}$.

The (nonlinear) complementarity problem (CP) over the symmetric cone $K$ is given by

$$\begin{array}{lll} \text{(CP)} & \text{Find} & (x,y,z) \in K \times K \times \Re^m \\ & \text{Subject to} & F(x,y,z) = 0, \ x \circ y = 0, \end{array} \tag{1}$$

where $F : K \times K \times \Re^m \to V \times \Re^m$ is continuous. The class of CPs covers a wide range of optimization problems such as primal-dual linear, quadratic, semidefinite, and second-order cone programs. Recently, an interior point map and associated trajectories have been studied in the paper [11] based on the results in [1, 4, 5,7]. The paper has also provided a *homogeneous model* for a special class of CPs of the form

$$\begin{array}{lll} \text{(SCP)} & \text{Find} & (x,y) \in K \times K \\ & \text{Subject to} & y - \psi(x) = 0, \ x \circ y = 0, \end{array} \tag{2}$$

where $\psi : K \to V$ is continuous. Choose an appropriate inner product $\langle \cdot, \cdot \rangle$ on $V \times V$, and suppose that the function $\psi$ is monotone on $K$, i.e., $\psi$ satisfies

$$\langle \psi(x) - \psi(x'), x - x' \rangle \geq 0 \ \text{ for all } x, x' \in K.$$

Then the homogenous model has the following remarkable features (cf. [11, Theorems 5.4 and 5.5]):

(a) A path exists, is bounded and has a trivial starting point without any regularity assumption concerning the existence of feasible or strictly feasible solutions.

(b) Any accumulation point of the path is a solution of the homogeneous model.

(c) If the original problem is solvable, then every accumulation point of the path gives us a finite certificate proving infeasibility.

(d) If the original problem is strongly infeasible, then under the assumption of Lipschitz continuity, every accumulation point of the path gives us a finite certificate proving infeasibility.

In this paper, we extend the above results for a wider class of CPs, which are so called *mixed* complementarity problem (cf. [3]) and given by

$$\begin{array}{lll} \text{(MiCP)} & \text{Find} & (x,y,z) \in K \times K \times \Re^m \\ & \text{Subject to} & F(x,y,z) := \begin{pmatrix} y - \psi_1(x,z) \\ \psi_2(x,z) \end{pmatrix}, \ x \circ y = 0, \end{array} \tag{3}$$

where $\psi := (\psi_1, \psi_2) : K \times \Re^m \to V \times \Re^m$.

The paper is organized as follows.

In Sec. 2, we summarize some basic properties of the MiCP, most of which have appeared in [11]. We propose a homogeneous model for the MiCP in Sec. 3, and show that the proposed model maintains the properties (a)–(d) above under certain assumptions on $F$. These assumptions are rather mild but slightly theoretical. In order to make them more tangible, we consider the following assumption in Sec. 4:

**Assumption 1.1.**

(i) *$\psi$ is affine. i.e., there exist linear operators $A$ and $B$, and $d \in V \times \Re^m$, such that $\psi(x, z) = Ax + Bz + d$.*

(ii) *$\psi$ is monotone on $K \times \Re^m$, i.e., for all $(x, z), (x', z') \in K \times \Re^m$,*
$$\langle \psi_1(x, z) - \psi_1(x', z'), x - x' \rangle + [\psi_2(x, z) - \psi_2(x', z')]^T (z - z') \geq 0.$$

(iii) *The rank of the linear operator $B$ is $m$.*

We show that if the function $\psi$ satisfies the above assumption then the properties (a)-(d) also hold. It should be noted that our results are new even for the (classical) mixed CPs where $K$ is the $n$-dimensional nonnegative orthant.

In Sec. 5, we apply our results to some well-known optimization problems. We conclude that our homogeneous model is directly applicable to the optimal condition of the following quadratic conic optimization problem:

$$\text{(QO)} \qquad \begin{array}{ll} \text{Minimize} & \frac{1}{2} z^T Q z + c^T z \\ \text{Subject to} & Az - b \in -K, \end{array} \tag{4}$$

where $A : \Re^m \to V$ is a linear operator of rank $m$, $Q \in \Re^{m \times m}$ is a symmetric positive semidefinite matrix, $b \in V$, and $c \in \Re^m$.

Note that the functions appearing in the paper are not necessarily defined on the boundary of the set $K \times K \times \Re^m$. By this reason, we introduce the definitions of *asymptotic feasibility* and *asymptotic solvability*:

**Definition 1.2.**

(i) *The CP is asymptotically feasible if and only if there exists a bounded sequence $\{x^{(k)}, y^{(k)}, z^{(k)}\} \subseteq \mathrm{int}K \times \mathrm{int}K \times \Re^m$ such that*
$$\lim_{k \to \infty} F(x^{(k)}, y^{(k)}, z^{(k)}) = 0.$$

(ii) *The CP is asymptotically solvable if and only if there exists a bounded sequence $\{x^{(k)}, y^{(k)}, z^{(k)}\} \subseteq \mathrm{int}K \times \mathrm{int}K \times \Re^m$ such that*
$$\lim_{k \to \infty} F(x^{(k)}, y^{(k)}, z^{(k)}) = 0 \quad \text{and} \quad \lim_{k \to \infty} x^{(k)} \circ y^{(k)} = 0.$$

(iii) *The CP is infeasible if and only if there is no feasible point $(x, y, z) \in K \times K \times \Re^m$ satisfying $F(x, y, z) = 0$.*

(iv) *The CP is strongly infeasible if and only if there is no sequence $\{x^{(k)}, y^{(k)}, z^{(k)}\} \subseteq \mathrm{int}K \times \mathrm{int}K \times \Re^m$ such that $\lim_{k \to \infty} F(x^{(k)}, y^{(k)}, z^{(k)}) = 0$.*

## 2. Preliminaries

Let $(V, \circ)$ be a Euclidean Jordan algebra with the identity element $e$, where $(x, y) \mapsto x \circ y : V \times V \to V$ is a bilinear map satisfying

(i) $x \circ y = y \circ x$,
(ii) $x \circ (y \circ x^2) = (x \circ y) \circ x^2$ where $x^2 = x \circ x$,
(iii) $x \circ e = e \circ x = x$,

for all $x, y, z \in V$. Note that the Jordan algebra $(V, \circ)$ is called Euclidean if there exists a symmetric, positive definite quadratic form $\mathcal{Q}$ on $V$ which is also associative, i.e., $\mathcal{Q}(x \circ y, z) = \mathcal{Q}(x, y \circ z)$ for all $x, y, z \in V$. Since $(x, y) \mapsto x \circ y$ is a bilinear map, for each $x \in V$, the linear transformation $L(x)$ is defined by $L(x)y = x \circ y$. For $x \in V$, the *degree* of $x$ is the smallest integer $d$ such that the set $\{e, x, x^2, \ldots, x^d\}$ is linearly independent. The *rank* $r$ of $V$ is the maximum of the degree of $x$ over all $x \in V$. For any element $x$ in $V$ of rank $r$, we can define the *characteristic polynomial* of $x$ of the form

$$p_x(\lambda) := \lambda^r - a_1(x)\lambda^{r-1} + \cdots + (-1)^r a_r(x)$$

(cf. Sec. 2 of [9]). We call the roots $\lambda_1, \cdots, \lambda_r$ of $p_x(\lambda)$ the *eigenvalues* of $x$ and define

$$\mathrm{tr}(x) := \sum_{i=1}^{r} \lambda_i = a_1(x), \quad \det\,(x) := \Pi_{i=1}^{r} \lambda_i = a_r(x). \tag{5}$$

Since $(x, y) \mapsto x \circ y$ is bilinear and $\mathrm{tr}(x \circ y)$ is a symmetric positive definite quadratic form which is associative, i.e., $\mathrm{tr}\,(x \circ (y \circ z)) = \mathrm{tr}\,((x \circ y) \circ z)$ for all $x, y, z \in V$, we define below the canonical inner product $\langle x, y \rangle$ of $x, y \in V$ and the canonical norm of $x \in V$, which we use throughout the paper:

$$\langle x, y \rangle := \mathrm{tr}(x \circ y), \quad \|x\| := \sqrt{\mathrm{tr}(x \circ x)}. \tag{6}$$

Note that $\|e\| = \sqrt{r}$. The property $\mathrm{tr}\,(x \circ (y \circ z)) = \mathrm{tr}\,((x \circ y) \circ z)$ implies that for each $x \in V$, $L(x)$ is self-adjoint with respect to $\langle \cdot, \cdot \rangle$, i.e., $\langle L(x)y, z \rangle = \langle y, L(x)z \rangle$ holds for all $y, z \in V$. We use the notation $L(u) \succ 0$ to mean that $L(u)$ is positively definite.

The set of squares $K := \{x^2 : \ x \in V\}$ is the symmetric cone of $V$, which is self-dual (i.e., $K = K^* := \{y : \ \langle x, y \rangle \geq 0 \ \text{ for all } x \in K\}$). In the next proposition, we give some properties of the symmetric cone $K$ for further discussion. For the proofs of the results, see Theorem III.1.2 and Corollary I.1.6 of [4], Lemma 2.6 and Proposition 2.7 of [11], etc.

**Proposition 2.1.** *Let $K$ be the symmetric cone of $V$.*

(i) *If $y \in intK$ and $\eta > 0$, then the set $\{x \in K : \langle x, y \rangle \leq \eta\}$ is compact.*
(ii) *If $x \in V$, then there exist real numbers $\lambda_1, \ldots, \lambda_r$ and a Jordan frame $c_1, \ldots, c_r$ such that $x = \sum_{j=1}^{r} \lambda_j c_j$. Here the numbers $\lambda_j$ (with their multiplicities) are uniquely determined by $x$ and $\lambda_j$'s are the eigenvalues (multiplicities included) of $x$.*
(iii) *If $x \in K$ and $y \in K$, then $\langle x, y \rangle = 0$ if and only if $x \circ y = 0$.*

(iv)
$$\mathrm{int}K \times \{\alpha e: \ \alpha \in \Re_{++}\} \subseteq \mathcal{U}, \quad \{\alpha e: \ \alpha \in \Re_{++}\} \times \mathrm{int}K \subseteq \mathcal{U},$$
$$K \times \{\alpha e: \ \alpha \in \Re_{+}\} \subseteq \mathrm{cl}(\mathcal{U}), \quad \{\alpha e: \ \alpha \in \Re_{+}\} \times K \subseteq \mathrm{cl}(\mathcal{U}),$$

*where*
$$\Re_+ := \{\alpha \in \Re: \ \alpha \geq 0\} \ \text{ and } \ \Re_{++} := \{\alpha \in \Re: \ \alpha > 0\}.$$

(v) *There exist $\omega_1 > 0$ and $\omega_2 > 0$ for which $0 < \omega_1 \leq \|c\| \leq \omega_2$ holds for any nonzero idempotent $c$ of $V$.*

Here we introduce the so-called *interior point map* $H : \mathrm{int}K \times \mathrm{int}K \times \Re^m \to V \times V \times \Re^m$ of the form

$$H := \begin{pmatrix} x \circ y \\ F(x,y,z) \end{pmatrix}. \tag{7}$$

Consider the following assumption on $F$:

**Assumption 2.2.**

(i) *$F$ is $(x,y)$-everywhere-monotone on its domain, i.e., there exist continuous functions $\phi$ from the domain of $F$ to the set $V \times \Re^m$ and $c : (V \times \Re^m) \times (V \times \Re^m) \to \Re$ such that for any $r \in V \times \Re^m$ and for any $(x,y,z)$ and $(x',y',z')$ in the domain of $F$, we have*

$$c(r,r) = 0$$

*and*

$$\begin{aligned}&\left[F(x,y,z) = r \text{ and } F(x',y',z') = r'\right] \\ &\Rightarrow \langle x - x', y - y'\rangle \geq \langle r - r', \phi(x,y,z) - \phi(x',y',z')\rangle_{V\times\Re^m} + c(r,r').\end{aligned}$$

*Here we define*
$$\langle (a,b),(a',b')\rangle_{V\times\Re^m} = \langle a, a'\rangle + b^T b'$$
*for any $(a,b),(a',b') \in V \times \Re^m$.*

(ii) *$F$ is $z$-bounded on its domain, i.e., for any sequence $\{(x^{(k)}, y^{(k)}, z^{(k)})\}$ in the domain of $F$, if $\{(x^{(k)}, y^{(k)})\}$ and $\{F(x^{(k)}, y^{(k)}, z^{(k)})\}$ are bounded then the sequence $\{z^{(k)}\}$ is also bounded.*

(iii) *$F(x,y,z)$ is $z$-injective on its domain, i.e., for any $(x,y,z)$ and $(x,y,z')$ lie in the domain of $F$, if $F(x,y,z) = F(x,y,z')$ then $z = z'$ holds.*

The following theorem has been proposed in [11] as Theorems 3.10 and 3.12. The theorem shows that the map $H$ is a *homeomorphism* under Assumption 2.2:

**Theorem 2.3.** *Suppose that a continuous map $F : \mathrm{int}K \times \mathrm{int}K \times \Re^m \to V \times \Re^m$ satisfies Assumption 2.2. Define the set*

$$\mathcal{U} := \{(x,y) \in \mathrm{int}K \times \mathrm{int}K : \ x \circ y \in \mathrm{int}K\}.$$

(i) *$H$ maps $\mathcal{U} \times \Re^m$ homeomorphically onto $\mathrm{int}K \times F(\mathcal{U} \times \Re^m)$, i.e., $H$ is bijective from $\mathcal{U} \times \Re^m$ onto $\mathrm{int}K \times F(\mathcal{U} \times \Re^m)$, and $H$ and $H^{-1}$ are continuous.*

(ii) *The set $F(\mathcal{U} \times \Re^m)$ is an open convex set.*

## 3. A Homogeneous Model for the MiCP

We define the homogeneous model (HMiCP) for the MiCP, which is a natural extension of the model in [11] for the CP with $K = \Re^n_+$, and of the one in [11] for the SCP (2):

$$\begin{array}{lll} \text{(HMiCP)} & \text{Find} & (x,\tau,y,\kappa,z) \in (K \times \Re_{++}) \times (K \times \Re_+) \times \Re^m \\ & \text{Subject to} & F_{\rm H}(x,\tau,y,\kappa,z) = 0, (x,\tau) \circ_{\rm H} (y,\kappa) = 0 \end{array} \tag{8}$$

where $F_{\rm H}$ and $(x,\tau) \circ_{\rm H} (y,\kappa)$ are given by

$$F_{\rm H}(x,\tau,y,\kappa,z) := \begin{pmatrix} y - \tau\psi_1(x/\tau, z/\tau) \\ \kappa + \langle \psi_1(x/\tau, z/\tau), x\rangle + \psi_2(x/\tau, z/\tau)^T z \\ \tau\psi_2(x/\tau, z/\tau) \end{pmatrix} \tag{9}$$

and

$$(x,\tau) \circ (y,\kappa) := \begin{pmatrix} x \circ y \\ \tau\kappa \end{pmatrix} \tag{10}$$

We also introduce the scalar product $\langle (x,\tau),(y,\kappa)\rangle_{\rm H}$ associated to the product above by

$$\langle (x,\tau),(y,\kappa)\rangle_{\rm H} := \langle x, y\rangle + \tau\kappa. \tag{11}$$

For ease of notation, we use the following symbols

$$V_{\rm H} := V \times \Re,\ K_{\rm H} := K \times \Re_+,\ x_{\rm H} := (x,\tau) \in V_{\rm H},\ y_{\rm H} := (y,\kappa) \in V_{\rm H} \tag{12}$$

and define the mapping $\psi_{\rm H} := (\psi_{{\rm H}_1}, \psi_{{\rm H}_2})$ by

$$\begin{array}{l} \psi_{{\rm H}_1}(x_{\rm H}, z) = \psi_{{\rm H}_1}(x,\tau,z) := \begin{pmatrix} \tau\psi_1(x/\tau, z/\tau) \\ -\langle \psi_1(x/\tau, z/\tau), x\rangle - \psi_2(x/\tau, z/\tau)^T z \end{pmatrix} \\ \psi_{{\rm H}_2}(x_{\rm H}, z) = \psi_{{\rm H}_2}(x,\tau,z) := \tau\psi_2(x/\tau, z/\tau) \end{array} \tag{13}$$

for every $(x_{\rm H}, z) = (x,\tau,z) \in (K \times \Re_{++}) \times \Re^m$. We can easily see that $\int K_{\rm H} = \int K \times \Re_{++}$ and

$$F_{\rm H}(x_{\rm H}, y_{\rm H}, z) = \begin{pmatrix} y_{\rm H} - \psi_{{\rm H}_1}(x_{\rm H}, z) \\ \psi_{{\rm H}_2}(x_{\rm H}, z) \end{pmatrix} \tag{14}$$

In addition,

$$K_{\rm H} = \{x_{\rm H}^2 = (x^2, \tau^2) : x_{\rm H} \in V_{\rm H}\}$$

holds, hence the closed convex cone $K_{\rm H}$ is the symmetric cone of $V_{\rm H}$.

Note that the function $F_{\rm H}$ is defined on the set $\mathrm{int}K_{\rm H} \times \mathrm{int}K_{\rm H} \times \Re^m$ (but not necessarily on its boundary). In what follows, we set the domain of $F_{\rm H}$ to be $\mathrm{int}K_{\rm H} \times \mathrm{int}K_{\rm H} \times \Re^m$.

Let us consider the map

$$H_{\mathrm{H}} := \begin{pmatrix} x_{\mathrm{H}} \circ_{\mathrm{H}} y_{\mathrm{H}} \\ F_{\mathrm{H}}(x_{\mathrm{H}}, y_{\mathrm{H}}, z) \end{pmatrix} \tag{15}$$

and choose an initial point $(x_{\mathrm{H}}^{(0)}, y_{\mathrm{H}}^{(0)}, z^{(0)})$ such that

$$(x_{\mathrm{H}}^{(0)}, y_{\mathrm{H}}^{(0)}, z^{(0)}) \in \mathrm{int}K_{\mathrm{H}} \times \mathrm{int}K_{\mathrm{H}} \times \Re^m \text{ and } x_{\mathrm{H}} \circ_{\mathrm{H}} y_{\mathrm{H}} \in \mathrm{int}K_{\mathrm{H}}.$$

For simplicity, we set

$$\begin{aligned}(x_{\mathrm{H}}^{(0)}, y_{\mathrm{H}}^{(0)}, z^{(0)}) &= (x^{(0)}, \tau_0, y^{(0)}, \kappa_0, z^{(0)}) \\ &= (e, 1, e, 1, 0) \in \mathrm{int}K_{\mathrm{H}} \times \mathrm{int}K_{\mathrm{H}} \times \Re^m. \end{aligned} \tag{16}$$

Define

$$h_{\mathrm{H}}^{(0)} := \begin{pmatrix} p_{\mathrm{H}}^{(0)} \\ f_{\mathrm{H}}^{(0)} \end{pmatrix} := \begin{pmatrix} (x_{\mathrm{H}}^{(0)}, y_{\mathrm{H}}^{(0)}) \\ F_{\mathrm{H}}(x_{\mathrm{H}}^{(0)}, y_{\mathrm{H}}^{(0)}, z^{(0)}) \end{pmatrix} = \begin{pmatrix} e_{\mathrm{H}} \\ \begin{pmatrix} y_{\mathrm{H}}^{(0)} - \psi_{\mathrm{H}_1}(x_{\mathrm{H}}^{(0)}, z^{(0)}) \\ \psi_{\mathrm{H}_2}(x_{\mathrm{H}}^{(0)}, z^{(0)}) \end{pmatrix} \end{pmatrix} \tag{17}$$

where $e_{\mathrm{H}} = (e, 1)$ is the identity element in $V_{\mathrm{H}}$ satisfying

$$\mathrm{Tr}(e_{\mathrm{H}}) = \mathrm{rank}(V_{\mathrm{H}}) = r + 1. \tag{18}$$

The next two theorems follow from the results described in Sec. 2. The proofs are analogous to the proofs of Theorems 5.4 and 5.5 in [11], and we relegate them to Appendices A and B, respectively. Theorem 3.1 below shows that we can find whether the MiCP is solvable, infeasible or in other cases, by observing any accumulation point of a bounded path, whose existence is guaranteed by Theorem 3.2.

**Theorem 3.1.** *Suppose that $F : K \times K \times \Re^m \to \Re^m$ satisfies Assumption* 2.2.

(i) *For every $(x_{\mathrm{H}}, z) \in intK_{\mathrm{H}} \times \Re^m$,*

$$\langle x_{\mathrm{H}}, \psi_{\mathrm{H}_1}(x_{\mathrm{H}}, z)\rangle_{\mathrm{H}} + z^T \psi_{\mathrm{H}_2}(x_{\mathrm{H}}, z) = 0.$$

(ii) *Every asymptotically feasible solution $(\hat{x}_{\mathrm{H}}, \hat{y}_{\mathrm{H}}, z)$ of (HCP) is an asymptotically complementary solution.*

(iii) *The HCP is asymptotically feasible.*

(iv) *The CP has a solution if and only if the HCP has an asymptotical solution $(x_{\mathrm{H}}^*, y_{\mathrm{H}}^*, z^*) = (x^*, \tau^*, y^*, \kappa^*, z^*)$ with $\tau^* > 0$.*
*In this case, $(x^*/\tau^*, y^*/\tau^*, z^*/\tau^*)$ is a solution of (CP).*

(v) *Suppose that $\psi$ satisfies the Lipschitz condition on $K$, i.e., there exists a constant $\gamma \geq 0$ such that*

$$\|\psi(x + h) - \psi(x)\| \leq \gamma\|h\| \quad \text{for any} \quad x \in K \text{ and } h \in V.$$

*If the CP is strongly infeasible then the HCP has an asymptotic solution $(x^*, \tau^*, y^*, \kappa^*, z^*)$ with $\kappa^* > 0$. Conversely, if the HCP has an asymptotic solution $(x^*, \tau^*, y^*, \kappa^*, z^*)$ with $\kappa^* > 0$ then the CP is infeasible. In the latter case, $(x^*/\kappa^*, y^*/\kappa^*)$ is a certificate to prove infeasibility of the CP. Here, the asymptotic feasibility and solvability of the problem are given in Definition* 1.2.

**Theorem 3.2.** *Suppose that $F_H : intK_{\rm H} \times intK_{\rm H} \times \Re^m \to V_{\rm H}$ defined by* (14) *satisfies Assumption* 2.2.

(i) *For any $t \in (0,1]$, there exists a point $(x_{\rm H}(t), y_{\rm H}(t), z(t)) \in intK_{\rm H} \times intK_{\rm H} \times \Re^m$ such that*

$$H_{\rm H}(x_{\rm H}(t), y_{\rm H}(t), z(t)) = th_{\rm H}^{(0)}.$$

(ii) *The set*

$$P := \{(x_{\rm H}(t), y_{\rm H}(t), z(t)) : H_{\rm H}(x_{\rm H}(t), y_{\rm H}(t), z(t)) = th_{\rm H}^{(0)}, t \in (0,1]\} \tag{19}$$

*forms a bounded path $\in intK_{\rm H} \times intK_{\rm H} \times \Re^m$. Any accumulation point $(x_{\rm H}(t), y_{\rm H}(t), z(t))$ is an asymptotically complementary solution to the HCP.*

(iii) *If the HCP has an asymptotically complementarity solution $(x^*_{\rm H}, y^*_{\rm H}, z^*) = (x^*, \tau^*, y^*, \kappa^*, z^*)$ with $\tau^* > 0$ $(\kappa^* > 0, respectively)$, then any accumulation point*

$$(x_{\rm H}(0), y_{\rm H}(0), z(0)) = (x(0), \tau(0), y(0), \kappa(0), z(0))$$

*of the bounded path $P$ satisfies $\tau(0) > 0$ $(\kappa(0) > 0, respectively)$.*

Note that we have to impose Assumption 2.2 on $F$ in Theorem 3.1 and on $F_{\rm H}$ in Theorem 3.2, respectively. In the next section, we will show that if the function $\psi$ of the MiCP satisfies Assumption 1.1 is sufficient, then both of $F$ and $F_{\rm H}$ satisfies Assumption 2.2 as well as $\psi$ is Lipschitzian on $K$ as required in (v) of Theorem 3.1.

## 4. A Sufficient Condition on the Function $\psi$

Monteiro and Pang [7] showed several sufficient conditions to ensure that the function $F$ satisfies Assumption 2.2, when $K$ is the cone of positive semidefinite matrices. The issue is more complicated in our analysis: Not only the function $F$ but also the homogeneous function $F_{\rm H}$ should satisfy Assumption 2.2 (see Theorems 3.1 and 3.2). In this section, we show that these requirements are satisfied under Assumption 2.2.

**Proposition 4.1.** *Suppose that the function $\psi = (\psi_1, \psi_2) : K \times \Re^m \to V \times \Re^m$ satisfies Assumption 1.1. Then the function $F : K \times K \times \Re^m \to V \times \Re^m$ satisfies Assumption 2.2, i.e., $F$ is $(x,y)$-everywhere-monotone, $z$-injective and $z$-bounded on $K \times K \times \Re^m$.*

*Proof.* Suppose that $\psi$ satisfies Assumption 1.1.

Define $\phi : K \times K \times \Re^m \to V \times \Re^m$ and $c : (V \times \Re^m) \times (V \times \Re^m) \to \Re$ by

$$\phi(x,y,z) := (x, -z), \;\; c := 0.$$

Let $r := (a,b) = F(x,y,z)$ and $r' := (a', b') = F(x', y', z')$ where $(x,z), (x', z') \in K \times \Re^m$. Then

$$\psi_1(x,z) - \psi_1(x', z') = (y - y') - (a - a'), \quad \psi_2(x,z) - \psi_2(x', z') = b - b',$$

and the monontonicity of $\psi$ implies that

$$\begin{aligned}0 &\le \langle \psi_1(x,z)-\psi_1(x',z'), x-x'\rangle + [\psi_2(x,z)-\psi_2(x',z')]^T(z-z')\\ &= \langle (y-y')-(a-a'), x-x'\rangle + (b-b')^T(z-z')\\ &= \langle y-y', x-x'\rangle - \langle a-a', x-x'\rangle + (b-b')^T(z-z')\\ &= \langle y-y', x-x'\rangle - \langle r-r', \phi(x,y,z)-\phi(x',y',z')\rangle + c(r,r').\end{aligned}$$

Thus, the function $F$ is $(x,y)$-everywhere-monotone.

By (i) and (iii) of Assumption 1.1, we can easily see that $\psi$ is $z$-bounded and $z$-injective, and hence, from the definition (3), $F$ is $z$-bounded and $z$-injective. ■

**Proposition 4.2.** *Suppose that the function* $\psi = (\psi_1, \psi_2) : K \times \Re^m \to V \times \Re^m$ *satisfies Assumption* 1.1*.*
(i) $\psi_{\mathrm{H}}$ *is monotone on* $intK_{\mathrm{H}} \times \Re^m$*.*
(ii) $F_{\mathrm{H}}$ : $intK_{\mathrm{H}} \times intK_{\mathrm{H}} \times \Re^m$ *satisfies Assumption* 2.2*, i.e.,* $F_{\mathrm{H}}$ *is* $(x_{\mathrm{H}}, y_{\mathrm{H}})$*-everywhere-monotone,* $z$*-bounded and* $z$*-injective on* $intK_{\mathrm{H}} \times intK_{\mathrm{H}} \times \Re^m$*.*

*Proof.* (i) For every $(x_{\mathrm{H}}, z)$, $(x'_{\mathrm{H}}, z') \in \mathrm{int}K_{\mathrm{H}} \times \Re^m$, it follows from the definition (13) that

$$\begin{aligned}&\langle \psi_{\mathrm{H}_1}(x_{\mathrm{H}}, z) - \psi_{\mathrm{H}_1}(x'_{\mathrm{H}}, z'), x_{\mathrm{H}} - x'_{\mathrm{H}}\rangle_{\mathrm{H}} + [\psi_{\mathrm{H}_2}(x_{\mathrm{H}}, z) - \psi_{\mathrm{H}_2}(x'_{\mathrm{H}}, z')]^T(z-z')\\ &= \langle \tau\psi_1(x/\tau, z/\tau) - \tau'\psi_1(x'/\tau', z'/\tau'), x-x'\rangle + [\tau\psi_2(x/\tau, z/\tau)\\ &\quad -\tau'\psi_2(x'/\tau', z'/\tau')]^T(z-z') - (\tau-\tau')\left\{\langle \psi_1(x/\tau, z/\tau), x\rangle - \langle \psi_1(x'/\tau', z'/\tau'), x'\rangle\right\}\\ &\quad - (\tau-\tau')\left\{\psi_2(x/\tau, z/\tau)^T z - \psi_2(x'/\tau', z'/\tau')^T z'\right\}\end{aligned}$$

By rearranging the right-hand side, we have

$$\begin{aligned}&\langle \psi_{\mathrm{H}_1}(x_{\mathrm{H}}, z) - \psi_{\mathrm{H}_1}(x'_{\mathrm{H}}, z'), x_{\mathrm{H}} - x'_{\mathrm{H}}\rangle_{\mathrm{H}} + [\psi_{\mathrm{H}_2}(x_{\mathrm{H}}, z) - \psi_{\mathrm{H}_2}(x'_{\mathrm{H}}, z')]^T(z-z')\\ &= \tau\tau'\big\{\langle \psi_1(x/\tau, z/\tau) - \psi_1(x'/\tau', z'/\tau'), (x/\tau) - (x'/\tau')\rangle\\ &\qquad + [\psi_2(x/\tau, z/\tau) - \psi_2(x'/\tau', z'/\tau')]^T[(z/\tau) - (z'/\tau')]\big\}\\ &\ge 0,\end{aligned}$$

where the last inequality follows from the monotonicity of $\psi = (\psi_1, \psi_2)$. Thus the map $\psi_{\mathrm{H}} = (\psi_{\mathrm{H}_1}, \psi_{\mathrm{H}_2})$ is monotone on the set $\mathrm{int}K_{\mathrm{H}} \times \Re^m$.
(ii) The monotonicity of $F_{\mathrm{H}}$ follows from (i) above and an analogous discussion to the proof of Proposition 4.1.

We are going to show that $F_{\mathrm{H}}$ is $z$-bounded and $z$-injective. Note that we have already seen that $\psi$ is $z$-bounded and $z$-injective in Proposition 4.1. Suppose that

$$\left\{(x_{\mathrm{H}}^{(k)}, y_{\mathrm{H}}^{(k)})\right\} \subseteq \int K_{\mathrm{H}} \times \int K_{\mathrm{H}}$$

and

$$\{F_{\mathrm{H}}(x_{\mathrm{H}}^{(k)}, y_{\mathrm{H}}^{(k)}, z^{(k)})\} = \{y_{\mathrm{H}}^{(k)} - \psi_{\mathrm{H}}(x_{\mathrm{H}}^{(k)}, z^{(k)})\} \subseteq V \times \Re^m$$

are bounded. Then $\{\psi_{\mathrm{H}}(x_{\mathrm{H}}^{(k)}, z^{(k)})\}$ is also bounded. Since we assume that $\psi = (\psi_1, \psi_2)$ is affine, $\psi_1$ and $\psi_2$ are given by

$$\psi_1(x, z) = A_1 x + B_1 z + d_1, \quad \psi_2(x, z) = A_2 x + B_2 z + d_2,$$

for some linear operators $A_i, B_i$ $(i = 1, 2)$, $d_1 \in V$ and $d_2 \in \Re^m$. Therefore, for any $\tau_k > 0$, we have

$$\begin{aligned} \tau_k \|\psi_1(x^{(k)}/\tau_k, z^{(k)}/\tau_k)\| &= \|A_1 x^{(k)} + B_1 z^{(k)} + \tau_k d_1\| \\ &= \|A_1 x^{(k)} + B_1 z^{(k)} + d_1 - (1 - \tau_k) d_1\| \\ &\geq \|A_1 x^{(k)} + B_1 z^{(k)} + d_1\| - \|(1 - \tau_k) d_1\|, \end{aligned}$$

and by the definition (13) of $\psi_{\mathrm{H}_1}$,

$$\begin{aligned} \|\psi_1(x^{(k)}, z^{(k)})\| &= \|A_1 x^{(k)} + B_1 z^{(k)} + d_1\| \\ &\leq \tau_k \|\psi_1(x^{(k)}/\tau_k, z^{(k)}/\tau_k)\| + \|(1 - \tau_k) d_1\| \\ &= \|\psi_{\mathrm{H}_1}(x_{\mathrm{H}}^{(k)}, z^{(k)})\| + \|(1 - \tau_k) d_1\|. \end{aligned}$$

By the boundedness of $\{(x_{\mathrm{H}}^{(k)}, z^{(k)}) = (x^{(k)}, \tau_k, z^{(k)})\}$ and of $\{\psi_{\mathrm{H}}(x_{\mathrm{H}}^{(k)}, z^{(k)})\}$, we know that $\{\psi_1(x^{(k)}, z^{(k)})\}$ is bounded. The boundedness of $\{\psi_2(x^{(k)}, z^{(k)})\}$ can be obtained similarly. Since we have seen that $\psi$ is $z$-bounded, the above facts guarantee that $\{z^{(k)}\}$ is bounded, which implies the $z$-boundedness of $F_{\mathrm{H}}$.

Next, we show that $F_{\mathrm{H}}$ is $z$-injective. Suppose that $(x_{\mathrm{H}}, y_{\mathrm{H}}, z), (x_{\mathrm{H}}, y_{\mathrm{H}}, z') \in \mathrm{int} K_{\mathrm{H}} \times \mathrm{int} K_{\mathrm{H}} \times \Re^m$ satisfy $F_{\mathrm{H}}(x_{\mathrm{H}}, y_{\mathrm{H}}, z) = F_{\mathrm{H}}(x_{\mathrm{H}}, y_{\mathrm{H}}, z')$. Then, by the definition (9) of $F_{\mathrm{H}}$, we have

$$y - \tau\psi_1(x/\tau, z/\tau) = y - \tau\psi_1(x/\tau, z'/\tau), \quad \tau\psi_2(x/\tau, z/\tau) = \tau\psi_2(x/\tau, z/\tau).$$

Since $\psi$ is $z$-injective, the equivalence $z = z'$ follows. ∎

## 5. Convex Quadratic Optimization Problems over $K$

Consider the quadratic convex optimization problem QO given by (4). The QO is a special case of the convex optimization problem CO:

$$\begin{array}{lll} \text{(CO)} & \text{Minimize} & f(z) \\ & \text{Subject to} & g(z) \in -K, \end{array} \tag{20}$$

where $f : \Re^m \to \Re$ is continuously differentiable and convex, $g : \Re^m \to V$ is continuously differentiable and $K$-*convex*, i.e., for any $z, z' \in \Re^m$ and $\tau \in (0, 1)$,

$$\tau g(z) + (1 - \tau) g(z') - g(\tau z + (1 - \tau) z') \in K$$

holds. Rockafellar [8] discussed the optimality condition of the CO, and showed that under a suitable constraint qualification, there must exist $z \in \Re^m$ and $x \in K$ such that

$$g(z) \in -K, \quad \nabla_z L(x, z) = 0, \quad \langle x, g(z) \rangle = 0, \tag{21}$$

where $L : K \times \Re^m \to \Re$ is the Lagrangian function given by

$$L(x,z) = f(z) + \langle x, g(z)\rangle$$

(see also Shapiro [10] for semidefinite programming cases). Define

$$\psi(x,z) := \begin{pmatrix} \psi_1(x,z) \\ \psi_2(x,z) \end{pmatrix} := \begin{pmatrix} -g(z) \\ \nabla_z L(x,z) \end{pmatrix}. \tag{22}$$

Then, by (iii) of Proposition 2.1, the optimal condition (21) is equivalent to the MiCP with (22). The following proposition shows that the function $\psi$ is monotone whenever the CO is convex.

**Proposition 5.1.** *Suppose that $f$ is continuously differentiable and convex, and $g$ is continuously differentiable and $K$-convex on $\Re^m$. Then the function $\psi$ given by* (22) *is continuous and monotone on $K \times \Re^m$ in the sense of Assumption* 1.1*.*

*Proof.* Since the cone $K$ is self-dual (i.e., $K = K^* := \{y : \langle x, y\rangle \geq 0 \text{ for all } x \in K\}$), it is easy to see that the $K$-convexity of $g$ implies that $\langle x, g(z)\rangle$ is convex on $\Re^m$ for any fixed $x \in K$. Thus, for any $x \in K$, the function $L(x,\cdot)$ is continuously differentiable and convex on $\Re^m$. Let $x, x' \in K$. Then, we can see that

$$\begin{aligned} &L(x,z') - L(x,z) - (z'-z)^T \nabla_z L(x,z) \geq 0, \\ &L(x',z) - L(x',z') - (z-z')^T \nabla_z L(x',z') \geq 0 \end{aligned}$$

hold for any $z, z' \in \Re^m$. By adding these two inequalities and by the definition (22) of $\psi$, we have

$$\begin{aligned} 0 \leq &L(x,z') - L(x,z) + L(x',z) - L(x',z') + (z-z')^T[\nabla_z L(x,z) - \nabla_z L(x',z')] \\ =&\langle x - x', -g(x) + g(z')\rangle + (z-z')^T[\nabla_z L(x,z) - \nabla_z L(x',z')] \\ =&\langle \psi_1(x,z) - \psi_1(x',z'), x - x'\rangle + [\psi_2(x,z) - \psi_2(x',z')]^T(z-z'). \end{aligned}$$

Thus, $\psi$ is monotone in the sense of Assumption 1.1. ∎

The Lagrangian function $L : K \times \Re^m \to \Re$ for the QO is given by

$$L(x,z) = \frac{1}{2} z^T Q z + c^T z + \langle x, Az - b\rangle$$

and we see that

$$\psi(x,z) := \begin{pmatrix} \psi_1(x,z) \\ \psi_2(x,z) \end{pmatrix} := \begin{pmatrix} -Az + b \\ Qz + c + A^* x \end{pmatrix} \tag{23}$$

where $A^* : V \to \Re^m$ is the adjoint of $A$. Suppose that $\operatorname{rank}(A) = m$ and $Q$ is positive semidefinite. It is easy to see that the function $\psi$ satisfies (i) and (iii) of Assumption 1.1. In addition, we can see that $g(z) = Az - b$ is $K$-convex, and $f(z) = \frac{1}{2} z^T Q z$ is convex. Thus, the function $\psi$ defined above is monotone by Proposition 5.1. Since the above properties are invalid if we restrict the domain of $\psi$ to $K \times \Re^m$, we obtain the following corollary.

**Corollary 5.2.** *Suppose that the function $\psi$ is given by* (23). *If the rank of the linear operator $A$ is $m$ and the symmetric matrix $Q$ is positive semidefinite, then $\psi$ whose domain is restricted to $K \times \Re^m$ satisfies Assumption* 1.1.

It should be noted that the QO can be equivalently represented as

$$\begin{array}{ll} \text{Minimize} & \alpha \\ \text{Subject to} & \dfrac{1}{2}z^T Qz + c^T z \leq \alpha, \quad Az - b \in -K, \end{array}$$

and by using the second order cone $K_0$, it turns out to be

$$\text{(QO1)} \qquad \begin{array}{ll} \text{Minimize} & \alpha \\ \text{Subject to} & h(z, \alpha) \in -K_0 \quad Az - b \in -K, \end{array}$$

where $h : \Re^{m+1} \to \Re^{r+1}$ with $r = \text{rank}(Q)$ is an affine function. Since the QO1 is a linear optimization problem over a symmetric cone, it can be solved by using a primal-dual framework for linear conic optimization problems. However, we should add $r = \text{rank}(Q)$ new variables to the dual form, and the size of primal-dual problems may become considerably larger than the size of the original problem QO. A merit of our homogeneous model is that we can deal with the QO directly without adding a large number of variables.

Finally, we consider a more special case of the QO, i.e., the linear optimization problem (LO) over the symmetric cone $K$, which can be obtained by setting $Q = O$ in the QO:

$$\text{(LO)} \qquad \begin{array}{ll} \text{Minimize} & c^T z \\ \text{Subject to} & Az - b \in -K. \end{array}$$

It is well known that the (primal) LO has the following mutually exclusive cases:

- The problem is strictly feasible, i.e., there exists $\bar{z}$ such that $Az - b \in -\text{int}K$.
- The problem is feasible, i.e., there exists $\bar{z}$ such that $Az - b \in -K$, but not strictly feasible.
- The problem is infeasible, but *asymptotically* feasible, i.e. there exists an (unbounded) sequence $\{z^{(k)}\}$ such that $\lim_{k\to\infty} Az^{(k)} - b \in -K$.
- The problem is strongly infeasible, i.e., infeasible and not asymptotically feasible.

Note that the terminology *asymptotically* above is used differently from the way in Definition 1.2, where the asymptotically converging sequence $\{z^{(k)}\}$ should be bounded. It is also known that the dual problem has the corresponding four cases similarly, and all of the possible 16 cases of primal and dual pair of LOs have concrete examples.

Table 1 shows the results obtained by applying our homogeneous model to primal-dual pair of LOs (see Theorem 3.1). Each case shows the possible signs of the variables $\tau(0)$ and $\kappa(0)$ at any accumulation point of the path (19). The results shown in Table 1 are weaker than those obtained for the self-dual embedding model for the LOs proposed in [6], in terms of the discriminant ability to detect the primal infeasibility or the dual infeasibility. A merit of our homogeneous model is that, as we have seen in Theorems 3.1 and 3.2, it has applicability to optimality conditions of nonlinear optimization problems over symmetric cones, whenever the corresponding functions $F$ and $F_{\mathrm{H}}$ satisfy Assumption 2.2.

*Table 1.* Theorem 3 for the linear conic optimization

| | | Primal: feasible, strictly fesible | Primal: feasible, others | Primal: infeasible, asympt. feasible | Primal: infeasible, strongly infeasible |
|---|---|---|---|---|---|
| Dual: feasible | strictly fesible | $\tau > 0$, $\kappa = 0$ | $\tau >= 0$, $\kappa = 0$ | $\tau = 0$, $\kappa >= 0$ | $\tau = 0$, $\kappa > 0$ |
| | others | $\tau >= 0$, $\kappa = 0$ | $\tau >= 0$, $\kappa = 0$ | $\tau = 0$, $\kappa >= 0$ | $\tau = 0$, $\kappa > 0$ |
| Dual: infeasible | asympt. feasible | $\tau = 0$, $\kappa >= 0$ | $\tau = 0$, $\kappa >= 0$ | $\tau = 0$, $\kappa >= 0$ | $\tau = 0$, $\kappa > 0$ |
| | strongly infeasible | $\tau = 0$, $\kappa > 0$ | $\tau = 0$, $\kappa > 0$ | $\tau = 0$, $\kappa > 0$ | $\tau = 0$, $\kappa > 0$ |

## References


1. F. Alizadeh and S. H. Schmieta, Symmetric cones, potential reduction methods and word-by-word extensions, *Handbook of semidefinite programming, Theory, algorithms, and applications*, H. Wolkowicz, R. Saigal and L. Vandenberghe (eds.), Kluwer Academic Publishers, 2000.
2. E. Andersen and Y. Ye, On a homogeneous algorithm for the monotone complementarity problems, *Mathematical Programming* **84** (1999) 375–400.
3. F. Facchinei and J.-S. Pang, *Finite-Dimensional Variational Inequalities and Complementarity Problems*, Volume I, Springer Series in Operations Research, Springer-Verlag, 2003.
4. J. Faraut and A. Korányi, *Analysis on symmetric cones*, Oxford Science Publishers, 1994.
5. L. Faybusovich, Euclidean Jordan algebras and interior-point algorithms, *Positivity* **1** (1997) 331–357.
6. Z.-Q. Luo, J.F. Sturm, and S. Zhang, Conic convex programming and self-dual embedding, *Optimization Methods and Software* **14** (2000) 196–218.
7. R. D. C. Monteiro and J. -S. Pang, On two interior-point mappings for nonlinear semidefinite complementarity problems, *Mathematics of Operations Research* **23** (1998) 39–60.
8. R. T. Rockafellar, *Conjugate Duality and Optimization*, SIAM, 1974.
9. S. H. Schmieta and F. Alizadeh, Extension of primal-dual interior-point algorithm to symmetric cones, *Mathematical Programming* **96** (2003) 409–438.
10. A. Shapiro, First and second order analysis of nonlinear semidefinite programs,

*Mathematical Programming* **77** (1997) 301–320.

11. A. Yoshise, Interior point trajectories and a homogeneous model for nonlinear complementarity problems over symmetric cones, *SIAM Journal on Optimization* **17** (2006) 1129–1153.

## Appendix

### A. Proof of Theorem 3.1

(i) The equation follows from the definition of (13).

(ii) Suppose that $(\hat{x}_{\mathrm{H}}, \hat{y}_{\mathrm{H}}, \hat{z})$ is an asymptotically feasible solution (see Definition 1.2). Then there exists a bounded sequence $(x_{\mathrm{H}}^k, y_{\mathrm{H}}^k, z^{(k)}) \in \mathrm{int}K_{\mathrm{H}} \times \mathrm{int}K_{\mathrm{H}} \times \Re^m$ such that

$$\lim_{k\to\infty}(y_{\mathrm{H}}^k - \psi_{\mathrm{H}_1}(x_{\mathrm{H}}^k, z^{(k)})) = \lim_{k\to\infty}\psi_{\mathrm{H}_2}(x_{\mathrm{H}}^k, z^{(k)})) = 0.$$

The assertion (i) implies that

$$\begin{aligned}\langle x_{\mathrm{H}}^k, y_{\mathrm{H}}^k\rangle_{\mathrm{H}} &= \langle x_{\mathrm{H}}^k, y_{\mathrm{H}}^k\rangle_{\mathrm{H}} - \left\{\langle \psi_{\mathrm{H}_1}(x_{\mathrm{H}}^k, z^{(k)}), x_{\mathrm{H}}^k\rangle_{\mathrm{H}} + \psi_{\mathrm{H}_2}(x_{\mathrm{H}}^k, z^{(k)})^T z^{(k)}\right\} \\ &= \langle y_{\mathrm{H}}^{(k)} - \psi_{\mathrm{H}_1}(x_{\mathrm{H}}^k, z^{(k)}), x_{\mathrm{H}}^k\rangle_{\mathrm{H}} - \psi_{\mathrm{H}_2}(x_{\mathrm{H}}^k, z^{(k)})^T z^{(k)}\end{aligned}$$

holds for any $k \geq 0$. Thus, by the boundedness of $z^{(k)}$, we see that $\lim_{k\to\infty}\langle x_{\mathrm{H}}^k, y_{\mathrm{H}}^k\rangle_{\mathrm{H}} = 0$ and $(\hat{x}_{\mathrm{H}}, \hat{y}_{\mathrm{H}}, \hat{z})$ is an asymptotically complementary solution.

(iii) For any $k \geq 0$, define

$$\begin{aligned}&x^{(k)} := (1/2)^k e \in \mathrm{int}K, \tau_k := (1/2)^k \in \Re_{++}, \\ &y^{(k)} := (1/2)^k e \in \mathrm{int}K, \kappa_k := (1/2)^k \in \Re_{++}, \quad z^{(k)} := 0 \in \Re^m.\end{aligned}$$

It is easy to see that the bounded sequence

$$(x_{\mathrm{H}}^k, y_{\mathrm{H}}^k, z^{(k)}) = \{(x^{(k)}, \tau_k, y^{(k)}, \kappa_k, z^{(k)})\} \subseteq \mathrm{int}K_{\mathrm{H}} \times \Re^m$$

satisfies

$$\lim_{k\to\infty}\left(y_{\mathrm{H}}^k - \psi_{\mathrm{H}_1}(x_{\mathrm{H}}^k, z^{(k)})\right) = 0, \ \lim_{k\to\infty}\psi_{\mathrm{H}_2}(x_{\mathrm{H}}^{(k)}, z^{(k)}) = 0.$$

(iv) If $(x^*, \tau^*, y^*, \kappa^*, z^*) \in (K \times \Re_{++}) \times (K \times \Re_+) \times \Re^m$ is a solution of the HMiCP with $\tau^* > 0$ then

$$y^*/\tau^* - \psi_1(x^*/\tau^*, z^*/\tau^*) = 0, \ \tau^*\psi_2(x^*/\tau^*, z^*/\tau^*) = 0, \ x^* \circ y^* = 0$$

and $(x^*/\tau^*, y^*/\tau^*, z^*/\tau^*) \in K \times K \times \Re^m$ is a solution of the MiCP. Conversely, if $(\hat{x}, \hat{y}, \hat{z}) \in K \times K \times \Re^m$ is a solution of the MiCP, then $(\hat{x}, 1, \hat{y}, 0, \hat{z}) \in (K \times \Re_{++}) \times (K \times \Re_+) \times \Re^m$ is a solution of the MiHCP.

(v) By (ii) of Theorem 2.3, the set $F(\mathcal{U} \times \Re^m)$ is open and convex.

If the MiCP is strongly infeasible, then we must have $0 \notin \mathrm{cl}\big(F(\mathcal{U} \times \Re^m)\big)$. Since the set $\mathrm{cl}\big(F(\mathcal{U}\times\Re^m)\big)$ is a closed convex set, by the separating hyperplane theorem, there exists $a = (a_1, a_2) \in V \times \Re^m$ with $\|a\| = 1$ and $\xi \in \Re$ that

$$\langle a, b\rangle \geq \xi > 0 \quad \text{for all} \quad b = (b_1, b_2) \in \mathrm{cl}\big(F(\mathcal{U} \times \Re^m)\big). \tag{24}$$

Since $F$ is continuous on the set $\mathrm{cl}(\mathcal{U} \times \Re^m) \subseteq K \times K \times \Re^m$, we can see that $F\big(\mathrm{cl}(\mathcal{U} \times \Re^m)\big) \subseteq \mathrm{cl}\big(F(\mathcal{U} \times \Re^m)\big)$. Therefore (24) implies that

$$\begin{aligned}\langle a, F(x, y, z)\rangle &= \langle a_1, y - \psi_1(x, z)\rangle + a_2^T \psi_2(x, z)\\ &= \langle a_1, y\rangle - \langle a_1, \psi_1(x, z)\rangle + a_2^T \psi_2(x, z)\\ &\geq \xi > 0\end{aligned} \tag{25}$$

for all $(x, y, z) \in \mathrm{cl}(\mathcal{U} \times \Re^m)$. Note that (iv) of Proposition 2.1 ensures that the above relation (25) holds at $(x, y, z) = (0, \alpha\bar{y}, 0)$ for any $\bar{y} \in K$ and $\alpha > 0$. Thus, it must be true that $\langle a_1, \bar{y}\rangle \geq 0$ for all $\bar{y} \in K$, which implies that $a_1 \in K$. Similarly, since $(x, 0, z) \in \mathrm{cl}(\mathcal{U} \times \Re^m)$ for any $(x, z) \in K \times \Re^m$, it follows from (25) that

$$-\langle a_1, \psi_1(x, z)\rangle + a_2^T \psi_2(x, z) \geq \xi > 0 \tag{26}$$

for all $(x, z) \in K \times \Re^m$. Combining with the fact that $(a_1, -a_2) \in K \times \Re^m$, we see that

$$-\langle a_1, \psi_1(\beta a_1, -\beta a_2)\rangle + a_2^T \psi_2(\beta a_1, -\beta a_2) \geq \xi > 0 \text{ for all } \beta \geq 0. \tag{27}$$

From the monotonicity of the map $\psi$ on the set $K \times \Re^m$, we see that

$$\begin{aligned}0 \leq&\langle \beta x - x, \psi_1(\beta x, \beta z) - \psi_1(x, z)\rangle + [\psi_2(\beta x, \beta z) - \psi_2(x, z)]^T (\beta z - z)\\ =& (\beta - 1)\langle x, \psi_1(\beta x, \beta z) - \psi_1(x, z)\rangle + (\beta - 1)[\psi_2(\beta x, \beta z) - \psi_2(x, z)]^T z\\ =& (\beta - 1)\left\{\langle x, \psi_1(\beta x, \beta z) - \psi_1(x, z)\rangle + [\psi_2(\beta x, \beta z) - \psi_2(x, z)]^T z\right\}\end{aligned}$$

for all $(x, z) \in K \times \Re^m$ and $\beta \geq 0$. Thus, for all $\beta \geq 1$, we should have

$$\langle x, \psi_1(\beta x, \beta z) - \psi_1(x, z)\rangle + [\psi_2(\beta x, \beta z) - \psi_2(x, z)]^T z \geq 0 \tag{28}$$

and hence,

$$\lim_{\beta\to\infty} \left\{\langle x, \psi_1(\beta x, \beta z)\rangle + \psi_2(\beta x, \beta z)^T z\right\} / \beta \geq 0. \tag{29}$$

Let $\{\beta_k\}$ be a subsequence such that $\beta_k \to +\infty$, and let

$$\frac{\psi_1(\beta_k a_1, -\beta_k a_2)}{\beta_k} = \sum_{i=1}^{r} \lambda_i^{(k)} c_i^{(k)}, \; (k = 1, 2, \ldots) \tag{30}$$

be a decomposition given by (ii) of Proposition 2.1. We also define

$$\begin{aligned}&\lambda_k := \min\{\lambda_i^{(k)}(i = 1, 2, ..., r)\}, \; j_k \in \; \mathrm{argmin}\{\lambda_i^{(k)}(i = 1, 2, ..., r)\},\\ &c^{(k)} := c_{j_k}^{(k)}.\end{aligned} \tag{31}$$

Note that $\{c^{(k)}\}$ is a sequence of primitive (i.e., nozero) idempotents of a Euclidean Jordan algebra $(V, \circ)$. Thus, by (v) of Proposition 2.1, there exist $\omega_1 > 0$ and $\omega_2 > 0$ such that

$$0 < \omega_1 \leq \|c^{(k)}\| \leq \omega_2 \text{ for all } k. \tag{32}$$

We first claim that $\liminf_{k\to\infty} \lambda_k \geq 0$. Suppose that $\liminf_{k\to\infty} \lambda_k < 0$. Then, by taking a subsequence if necessary, we may assume that there exists a $\delta > 0$ for which $\lambda_k \leq -\delta < 0$ for sufficiently large $k$'s. Define $x^{(k)} := a_1 + \epsilon c^{(k)}$ for $\epsilon > 0$. We can see that

$$\begin{aligned}
&\big\{\langle x^{(k)}, \psi_1(\beta_k x^{(k)}, -\beta_k a_2)\rangle - \psi_2(\beta_k x^{(k)}, -\beta_k a_2)^T a_2\big\}/\beta_k \\
&= \big\{\langle a_1 + \epsilon c^{(k)}, \psi_1(\beta_k x^{(k)}, -\beta_k a_2)\rangle - \psi_2(\beta_k x^{(k)}, -\beta_k a_2)^T a_2\big\}\ \beta_k \\
&= \big\{\langle a_1, \psi_1(\beta_k x^{(k)}, -\beta_k a_2)\rangle - \psi_2(\beta_k x^{(k)}, -\beta_k a_2)^T a_2\big\}/\beta_k \\
&\quad + \epsilon\langle c^{(k)}, \psi_1(\beta_k x^{(k)}, -\beta_k a_2)\rangle/\beta_k \\
&< \epsilon\langle c^{(k)}, \psi_1(\beta_k x^{(k)}, -\beta_k a_2)\rangle/\beta_k \qquad \text{(by (26))} \\
&= \epsilon\big\{\langle c^{(k)}, \psi_1(\beta_k x^{(k)}, -\beta_k a_2) - \psi_1(\beta_k a_1, -\beta_k a_2)\rangle + \langle c^{(k)}, \psi_1(\beta_k a_1, -\beta_k a_2)\rangle\big\}/\beta_k.
\end{aligned} \tag{33}$$

The definitions (30) and (31) and the boundedness (42) of $\{c^{(k)}\}$ ensure that

$$\langle c^{(k)}, \psi_1(\beta_k a_1, -\beta_k a_2)\rangle/\beta_k = \lambda_k \langle c^{(k)}, c^{(k)}\rangle \leq -\delta\omega_1^2 < 0 \tag{34}$$

holds for any $k \geq \bar{k}$. In addition, since we set $x^{(k)} = a_1 + \epsilon c^{(k)}$, by the Lipschitz continuity of $\psi_1$ and the boundedness of $\{c^{(k)}\}$, there exist $\bar{\gamma} > 0$ and $\bar{\epsilon} > 0$ independent of $k$, for which

$$\langle c^{(k)}, \psi_1(\beta_k x^{(k)}, -\beta_k a_2) - \psi_1(\beta_k a_1, -\beta_k a_2)\rangle/\beta_k \leq \bar{\gamma}\epsilon \leq \delta\omega_1^2/2 \tag{35}$$

holds for any $\epsilon \leq \bar{\epsilon}$. Thus, by (34) and (35),

$$\langle c^{(k)}, \psi_1(\beta_k x^{(k)}, -\beta_k a_2) - \psi_1(\beta_k a_1, -\beta_k a_2)\rangle/\beta_k + \langle c^{(k)}, \psi_1(\beta_k a_1, -\beta_k a_2)\rangle/\beta_k \leq -\delta\omega_1^2/2 < 0$$

holds for any $k \geq \bar{k}$ and $\epsilon \leq \bar{\epsilon}$. Therefore, by (33), we obtain

$$\big\{\langle x^{(k)}, \psi_1(\beta_k x^{(k)}, -\beta_k a_2)\rangle - \psi_2(\beta_k x^{(k)}, -\beta_k a_2)^T a_2\big\}/\beta_k \leq -\epsilon\delta\omega_1^2/2 < 0$$

for all such $k$'s and $\epsilon$'s. Since $x^{(k)} = a + \epsilon c^{(k)} \in K$, by fixing a suitably small $\epsilon \in (0, \bar{\epsilon}]$, the above inequality contradicts (29) and we must have

$$\liminf_{k\to\infty} \lambda_k = \liminf_{k\to\infty} \Big[\min\{\lambda_i^{(k)} (i = 1, 2, \ldots, r)\}\Big] \geq 0. \tag{36}$$

Next we claim that $\lim_{k\to\infty} \|\psi_2(\beta_k a_1, -\beta_k a_2)/\beta_k\| = 0$. Suppose that

$$\liminf_{k\to\infty} \|\psi_2(\beta_k a_1, -\beta_k a_2)/\beta_k\| \neq 0.$$

By taking a subsequence if necessary, there exist $0 \neq v \in \Re^m$ and $\omega > 0$ such that

$$v^T \psi_2(\beta_k a_1, -\beta_k a_2)/\beta_k < -\omega$$

for sufficiently large $k$'s. Define $z(\epsilon) = -a_2 + \epsilon v$ for $\epsilon > 0$. Since the $v$ is a constant vector, there exists $\bar{k}$ such that

$$v^T \psi_2(\beta_k a_1, -\beta_k a_2)/\beta_k < -\omega/2 < 0 \tag{37}$$

holds for any $k \geq \bar{k}$. In addition, since we set $z(\epsilon) = a_2 + \epsilon v$, by the Lipschitz continuity of $\psi_2$, there exist $\bar{\gamma} > 0$ and $\bar{\epsilon} > 0$ independent of $k$, for which

$$v^T [\psi_2(\beta_k a_1, -\beta_k z(\epsilon)) - \psi_2(\beta_k a_1, -\beta_k a_2)]/\beta_k \leq \bar{\gamma}\epsilon < \omega/2 \tag{38}$$

holds for any $k \geq \bar{k}$ and $\epsilon \leq \bar{\epsilon}$. Thus, by (37) and (38), we have

$$v^T [\psi_2(\beta_k a_1, -\beta_k z(\epsilon))] < 0 \tag{39}$$

for all such $k$'s and $\epsilon$'s. By (26), we can calculate that

$$\begin{aligned}
&\left\{\langle a_1, \psi_1(\beta_k a_1, \beta_k z(\epsilon))\rangle + \psi_2(\beta_k a_1, \beta_k z(\epsilon))^T z(\epsilon)\right\}/\beta_k \\
&= \left\{\langle a_1, \psi_1(\beta_k a_1, \beta_k z(\epsilon))\rangle - \psi_2(\beta_k a_1, -\beta_k z(\epsilon))^T a_2\right\}/\beta_k \\
&\quad + \epsilon\delta\psi_2(\beta_k a_1, -\beta_k z(\epsilon))^T v/\beta_k \\
&< \epsilon\delta\psi_2(\beta_k a_1, -\beta_k z(\epsilon))^T v/\beta_k
\end{aligned}$$

and hence, from (39),

$$\left\{\langle a_1, \psi_1(\beta_k a_1, \beta_k z(\epsilon))\rangle + \psi_2(\beta_k a_1, \beta_k z(\epsilon))^T z(\epsilon)\right\}/\beta_k < 0$$

for any $k \geq \bar{k}$ and $\epsilon \leq \bar{\epsilon}$. By fixing a suitably small $\epsilon > 0$, the above inequality contradicts (29). Thus, we must have $\lim_{k\to\infty} \|\psi_2(\beta_k a_1, -\beta_k a_2)/\beta_k\| = 0$, and hence,

$$\lim_{k\to\infty} \psi_2(\beta_k a_1, -\beta_k a_2)/\beta_k = 0 \in \Re^m. \tag{40}$$

We are going to show that $\{\psi(\beta_k a_1, -\beta_k a_2)/\beta_k\}$ is bounded. Since we have seen that $\lim_{k\to\infty} \psi_2(\beta_k a_1, -\beta_k a_2)/\beta_k = 0 \in \Re^m$, according to the definition in (30), it suffices to show that $\{\lambda_i^{(k)}\}$ is bounded for any $i = 1, 2, \ldots, r$.

Since $(\beta_k a_1, -\beta_k a_2)$ and $(e, 0)$ are in $K \times \Re^m$, by the monotonicity of $\psi$, we have

$$\begin{aligned}
0 &\leq \langle(\beta_k a_1, -\beta_k a_2) - (e, 0), \psi(\beta_k a_1, -\beta_k a_2) - \psi(e, 0)\rangle/\beta_k \\
&= \langle a_1, \psi_1(\beta_k a_1, -\beta_k a_2)\rangle - a_2^T \psi_2(\beta_k a_1, -\beta_k a_2) \\
&\quad - \langle a_1, \psi_1(e, 0)\rangle + a_2^T \psi_2(e, 0) - \langle e, \psi_1(\beta_k a_1, -\beta_k a_2)\rangle/\beta_k + \langle e, \psi_1(e, 0)\rangle/\beta_k \\
&\leq -\langle a_1, \psi_1(e, 0)\rangle + a_2^T \psi_2(e, 0) - \langle e, \psi_1(\beta_k a_1, -\beta_k a_2)\rangle/\beta_k + \langle e, \psi_1(e, 0)\rangle/\beta_k
\end{aligned}$$

where the last inequality follows from (26). This implies the existence of a constant $\sigma > 0$ such that

$$\langle e, \psi_1(\beta_k a_1, -\beta_k a_2)\rangle/\beta_k \le \sigma \tag{41}$$

holds for sufficiently large $k$'s. By the definition (30), the left-hand side of the above inequality is given by

$$\left\langle e, \frac{\psi_1(\beta_k a_1, -\beta_k a_2)}{\beta_k}\right\rangle = \left\langle e, \sum_{i=1}^{r} \lambda_i^{(k)} c_i^{(k)}\right\rangle = \sum_{i=1}^{r} \lambda_i^{(k)} \langle e, c_i^{(k)}\rangle.$$

Combining the above with (41), we have

$$\sum_{i=1}^{r} \lambda_i^{(k)} \langle e, c_i^{(k)}\rangle \le \sigma.$$

Here, (v) of Proposition 2.1 ensures the existence of $\omega_1 > 0$ and $\omega_2 > 0$ such that

$$0 < \omega_1 \le \langle e, c_i^{(k)}\rangle \le \omega_2 \text{ for all } i \text{ and } k. \tag{42}$$

Since we have shown the inequality (36), the set $\{\lambda_i^{(k)}\}$ must be bounded. Note that from (27) and (28), we have

$$\begin{aligned} -\xi &\ge \langle a_1, \psi_1(\beta_k a_1, -\beta_k a_2)\rangle - a_2^T \psi_2(\beta_k a_1, -\beta_k a_2) \\ &\ge \langle a_1, \psi_1(a_1, -a_2)\rangle - a_2^T \psi_2(a_1, -a_2). \end{aligned}$$

Thus the sequence $\{\langle a_1, \psi_1(\beta_k a_1, -\beta_k a_2)\rangle - a_2^T \psi_2(\beta_k a_1, -\beta_k a_2)\}$ is also bounded.

To summarize, by taking an appropriate subsequence and setting

$$\begin{aligned} &\hat{x}^{(k)} := a_1 + (1/\beta_k)e \in \mathrm{int}K, \quad \hat{\tau}_k := 1/\beta_k > 0, \\ &\hat{y}^{(k)} := \sum_{i=1}^{r} (\max\{\lambda_i^{(k)},\ 1/\beta_k\} c_i^{(k)}) \in \mathrm{int}K, \\ &\hat{\kappa}_k := -\langle a_1, \psi_1(\beta_k a_1, -\beta_k a_2)\rangle + a_2^T \psi_2(\beta_k a_1, -\beta_k a_2) \ge \xi > 0, \\ &\hat{z}^{(k)} := -a_2 + (1/\beta_k)v \in \Re^m, \end{aligned}$$

then, by (42), (36), (40) and the boundedness of $\{\lambda_i^{(k)}\}$ and $\{c_i^{(k)}\}$, we have

$$\lim_{k\to\infty} \sum_{i=1}^{r} (\max\{\lambda_i^{(k)},\ 1/\beta_k\} c_i^{(k)}) =: \psi_1^{\infty}(a) \in K$$

and conclude that $(x^*, \tau^*, y^*, \kappa^*, z^*) \in (K \times \Re_+) \times (K \times \Re_+) \times \Re^m$ given by

$$\begin{aligned} &x^* := a_1 \in K, \\ &\tau^* := \lim_{\beta_k \to \infty} \frac{1}{\beta_k} = 0, \\ &z^* := -a_2 \in \Re^m, \\ &y^* := \psi_1^{\infty}(a_1, -a_2) \in K, \\ &\kappa^* := \lim_{k\to\infty} \{-\langle a_1, \psi_1(\beta_k a_1, -\beta_k a_2)\rangle + a_2^T \psi_2(\beta_k a_1, -\beta_k a_2)\} \ge \xi > 0, \end{aligned}$$

is an asymptotical solution of the MiHCP with $\kappa^* > 0$.

Conversely, suppose that there exists a bounded sequence

$$\{(x^{(k)}, \tau_k, y^{(k)}, \kappa_k, z^{(k)})\} \subseteq (\mathrm{int}K \times \Re_{++}) \times (\mathrm{int}K \times \Re_{++}) \times \Re^m$$

such that

$$\begin{aligned} \lim_{k\to\infty} y^{(k)} &= \lim_{k\to\infty} \tau_k \psi_1(x^{(k)}/\tau_k, z^{(k)}/\tau_k) \in K, \\ \lim_{k\to\infty} \kappa_k &= \lim_{k\to\infty} \{-\langle \psi_1(x^{(k)}/\tau_k, z^{(k)}/\tau_k), x^{(k)}\rangle - \psi_2(x^{(k)}/\tau_k, z^{(k)}/\tau_k)^T z^{(k)}\} \\ &\geq \xi > 0, \\ 0 &= \lim_{k\to\infty} \tau_k \psi_2(x^{(k)}/\tau_k, z^{(k)}/\tau_k). \end{aligned}$$

Let us show that there is no feasible point $(x, y, z) \in K \times K \times \Re^m$ satisfying $y - \psi_1(x, z) = 0$ and $\psi_2(x, z) = 0$. Suppose that $(x, y, z) \in K \times K \times \Re^m$ satisfies $y - \psi_1(x, z) = 0$ and $\psi_2(x, z) = 0$, and define $x_H = (x, 1)$. Since $\psi_{\mathrm{H}}$ is monotone on $(K \times \Re_{++}) \times (K \times \Re_+) \times \Re^m$, by the definition (13) of $\psi_{\mathrm{H}}$ and by the assumptions $\psi_1(x, z) = y$ and $\psi_2(x, z) = 0$, we have

$$\begin{aligned} 0 &\leq \langle (x_{\mathrm{H}}^{(k)}, z^{(k)}) - (x_{\mathrm{H}}, z), \psi_{\mathrm{H}}(x_{\mathrm{H}}^{(k)}, z^{(k)}) - \psi_{\mathrm{H}}(x_{\mathrm{H}}, z)\rangle \\ &= \langle \psi_1(x^{(k)}/\tau_k, z^{(k)}/\tau_k), x^{(k)}\rangle + \psi_2(x^{(k)}/\tau_k, z^{(k)}/\tau_k)^T z^{(k)} - \langle x^{(k)}, y\rangle \\ &\quad - \langle \tau_k \psi_1(x^{(k)}/\tau_k, z^{(k)}/\tau_k), x\rangle - \tau_k \psi_2(x^{(k)}/\tau_k, z^{(k)}/\tau_k)^T z + \tau_k \langle x, y\rangle. \end{aligned} \tag{43}$$

Here, we see that $\langle x^{(k)}, y\rangle \geq 0$,

$$\lim_{k\to\infty} \langle x, y^{(k)}\rangle = \langle x, \lim_{k\to\infty} \tau_k \psi_1(x^{(k)}/\tau_k, z^{(k)}/\tau_k)\rangle \geq 0,$$

and $\lim_{k\to\infty} \tau_k = 0$ since $\lim_{k\to\infty} \kappa_k \geq \xi > 0$. Thus the relation (43) ensures that

$$\lim_{k\to\infty} \langle \psi_1(x^{(k)}/\tau_k, z^{(k)}/\tau_k), x^{(k)}\rangle + \psi_2(x^{(k)}/\tau_k, z^{(k)}/\tau_k)^T z^{(k)} \geq 0$$

which contradicts

$$\kappa_k = -\langle \psi_1(x^{(k)}/\tau_k, z^{(k)}/\tau_k), x^{(k)}\rangle - \psi_2(x^{(k)}/\tau_k, z^{(k)}/\tau_k)^T z^{(k)} \geq \xi > 0.$$

In addition, any limit of $x^{(k)}$ gives a separation hyperplane, i.e., a certificate proving infeasibility. ■

**Proof of Theorem 3.2**

(i) Since the map $F_{\mathrm{H}}$ satisfies Assumption 2.2, by (ii) of Theorem 2.3, the set $H_{\mathrm{H}}(\mathcal{U}_{\mathrm{H}} \times \Re^m)$ with

$$\mathcal{U}_{\mathrm{H}} := \{(x_{\mathrm{H}}, y_{\mathrm{H}}) \in \mathrm{int}K_{\mathrm{H}} \times \mathrm{int}K_{\mathrm{H}} : x_{\mathrm{H}} \circ_{\mathrm{H}} y_{\mathrm{H}} \in \mathrm{int}K_{\mathrm{H}}\}$$

is an open convex subset of $\mathrm{int}K_{\mathrm{H}} \times V_{\mathrm{H}} \times \Re^m$. Note that we have already seen that $0 \in \mathrm{cl}\,(H_{\mathrm{H}}(\mathcal{U}_{\mathrm{H}} \times \Re^m))$ in (ii) and (iii) of Theorem 3.1, and $h_{\mathrm{H}}^{(0)} \in H_{\mathrm{H}}(\mathcal{U}_{\mathrm{H}} \times \Re^m)$ by the definition (17). Since the set $H_{\mathrm{H}}(\mathcal{U}_{\mathrm{H}} \times \Re^m)$ is convex, the fact above implies that $th_{\mathrm{H}}^{(0)} \in H_{\mathrm{H}}(\mathcal{U}_{\mathrm{H}} \times \Re^m)$ for every $t \in (0,1]$. Combining this with the homeomorphism of the map $H_{\mathrm{H}}$ in (i) of Theorem 2.3, we obtain the assertion (i).

(ii) It follows from (i) of Theorem 2.3 that the map $H_{\mathrm{H}}$ is a homeomorphism and the set $P$ forms a path in $\mathrm{int}K_{\mathrm{H}} \times \mathrm{int}K_{\mathrm{H}} \times \Re^m$. Therefore, it suffices to show that the path $P$ is bounded. Let $(x_{\mathrm{H}}, y_{\mathrm{H}}, z) = (x, \tau, y, \kappa, z) \in P$. Then there exists a $t \in (0,1]$ for which $H_{\mathrm{H}}\,(x_{\mathrm{H}}, y_{\mathrm{H}}, z) = th_{\mathrm{H}}^{(0)}$ i.e.,

$$x_{\mathrm{H}} \circ_{\mathrm{H}} y_{\mathrm{H}} = te_{\mathrm{H}}, \quad y_{\mathrm{H}} - \psi_1(x_{\mathrm{H}}, z) = t\left(y_{\mathrm{H}}^{(0)} - \psi_{\mathrm{H}_1}(x_{\mathrm{H}}^{(0)}, z^{(0)})\right), \\ \psi_{\mathrm{H}_2}(x_{\mathrm{H}}, z) = t\psi_{\mathrm{H}_2}(x_{\mathrm{H}}^{(0)}, z^{(0)})$$

hold (see (17)). By analogous discussions as in the proof of Theorem 5.5 in [11], we can see that

$$\begin{aligned} \langle x_{\mathrm{H}}, y_{\mathrm{H}}^{(0)}\rangle_{\mathrm{H}} + \langle x_{\mathrm{H}}^{(0)}, y_{\mathrm{H}}\rangle_{\mathrm{H}} &\le t\langle x_{\mathrm{H}}^{(0)}, y_{\mathrm{H}}^{(0)}\rangle_{\mathrm{H}} + \langle x_{\mathrm{H}}^{(0)}, y_{\mathrm{H}}^{(0)}\rangle_{\mathrm{H}} \\ &= (1+t)\langle x_{\mathrm{H}}^{(0)}, y_{\mathrm{H}}^{(0)}\rangle_{\mathrm{H}} \\ &= (1+t)(r+1) \qquad \text{(by (17))} \\ &\le 2(r+1). \end{aligned}$$

Thus, the boundedness of the set $P$ follows from (i) of Proposition 2.1 and $z$-boundedness of $F_{\mathrm{H}}$.

(iii) Let $(x_{\mathrm{H}}^*, y_{\mathrm{H}}^*, z^*) = (x^*, \tau^*, y^*, \kappa^*, z^*)$ be an asymptotical solution to the HCP. Then there exists a bounded sequence

$$\{(x_{\mathrm{H}}^{(k)}, y_{\mathrm{H}}^{(k)}, z^{(k)})\} = \{(x^{(k)}, \tau_k, y^{(k)}, \kappa_k, z^{(k)})\} \subseteq \mathrm{int}K_{\mathrm{H}} \times \mathrm{int}K_{\mathrm{H}} \times \Re^m$$

such that

$$\begin{aligned} &\lim_{k\to\infty} (x_{\mathrm{H}}^{(k)}, y_{\mathrm{H}}^{(k)}, z^{(k)}) = (x_{\mathrm{H}}^*, y_{\mathrm{H}}^*, z^*), \\ &\lim_{k\to\infty} y_{\mathrm{H}}^{(k)} - \psi_{\mathrm{H}_1}(x_{\mathrm{H}}^{(k)}, z^{(k)}) = 0, \\ &\lim_{k\to\infty} \psi_{\mathrm{H}_2}(x_{\mathrm{H}}^{(k)}, z^{(k)}) = 0, \\ &\lim_{k\to\infty} x_{\mathrm{H}}^{(k)} \circ_{\mathrm{H}} y_{\mathrm{H}}^{(k)} = 0. \end{aligned}$$

Let $(x_{\mathrm{H}}(t), y_{\mathrm{H}}(t), z(t)) = (x(t), \tau(t), y(t), \kappa(t), z(t))$ be any point on the path $P$. Then

$$\begin{aligned} &x_{\mathrm{H}}(t) \circ_{\mathrm{H}} y_{\mathrm{H}}(t) = te_{\mathrm{H}}, \\ &y_{\mathrm{H}}(t) - \psi_{\mathrm{H}_1}(x_{\mathrm{H}}(t), z(t)) = t[y_{\mathrm{H}}^{(0)} - \psi_{\mathrm{H}_1}(x_{\mathrm{H}}^{(0)}, z^{(0)})], \\ &\psi_{\mathrm{H}_2}(x_{\mathrm{H}}(t), z(t)) = t\psi_{\mathrm{H}_2}(x_{\mathrm{H}}^{(0)}, z^{(0)}). \end{aligned} \tag{40}$$

By the boundedness of the set $P$ as we have seen in (ii) above, there exists an $\epsilon \in (0,1]$ such that

$$\|x_{\mathrm{H}}(t)\| \le 1/\epsilon, \ \|y_{\mathrm{H}}(t)\| \le 1/\epsilon, \ \|z(t)\| \le 1/\epsilon \tag{45}$$

holds for every $t \in (0,1]$. In addition, for each $t \in (0,1]$, there exists an index $k(t)$ such that for every $k \geq k(t)$, we have

$$\begin{aligned} &\|x_{\mathrm{H}}^{(k)} - x_{\mathrm{H}}^*\| \leq \epsilon,\ \|y_{\mathrm{H}}^{(k)} - y_{\mathrm{H}}^*\| \leq \epsilon,\ \|z^{(k)} - z^*\| \leq \epsilon, \\ &\|y_{\mathrm{H}}^{(k)} - \psi_{\mathrm{H}_1}(x_{\mathrm{H}}^{(k)}, z^{(k)})\| \leq t\epsilon,\ \|\psi_{\mathrm{H}_2}(x_{\mathrm{H}}^{(k)}, z^{(k)})\| \leq t\epsilon. \end{aligned} \tag{46}$$

Since $\psi_{\mathrm{H}}$ is monotone and (i) of Theorem 3.1 holds, by analogous calculations as in the proof of Theorem 5.5 in [11], we can see that

$$\begin{aligned} \langle x_{\mathrm{H}}(t), y_{\mathrm{H}}^{(k)}\rangle_{\mathrm{H}} + \langle y_{\mathrm{H}}(t), x_{\mathrm{H}}^{(k)}\rangle_{\mathrm{H}} \leq\ & \langle x_{\mathrm{H}}(t), y_{\mathrm{H}}^{(k)} - \psi_{\mathrm{H}_1}(x_{\mathrm{H}}^{(k)}, z^{(k)})\rangle_{\mathrm{H}} \\ & + \langle x_{\mathrm{H}}^{(k)}, y_{\mathrm{H}}(t) - \psi_{\mathrm{H}_1}(x_{\mathrm{H}}(t), z(t))\rangle_{\mathrm{H}} \\ & - \psi_{\mathrm{H}_2}(x_{\mathrm{H}}(t), z(t))^T z^{(k)} - \psi_{\mathrm{H}_2}(x_{\mathrm{H}}^{(k)}, z^{(k)})^T z(t) \end{aligned}$$

for every $t \in (0,1]$ and every $\kappa \geq \kappa(t)$. Therefore, it follows from (44), (45) and (46) that

$$\begin{aligned} &\langle x_{\mathrm{H}}(t), y_{\mathrm{H}}^{(k)}\rangle_{\mathrm{H}} + \langle y_{\mathrm{H}}(t), x_{\mathrm{H}}^{(k)}\rangle_{\mathrm{H}} \\ &\leq \langle x_{\mathrm{H}}(t), y_{\mathrm{H}}^{(k)} - \psi_{\mathrm{H}_1}(x_{\mathrm{H}}^{(k)}, z^{(k)})\rangle_{\mathrm{H}} + \langle x_{\mathrm{H}}^{(k)}, y_{\mathrm{H}}(t) - \psi_{\mathrm{H}_1}(x_{\mathrm{H}}(t), z(t))\rangle_{\mathrm{H}} \\ &\quad - \psi_{\mathrm{H}_2}(x_{\mathrm{H}}(t), z(t))^T z^{(k)} - \psi_{\mathrm{H}_2}(x_{\mathrm{H}}^{(k)}, z^{(k)})^T z(t) \\ &= \langle x_{\mathrm{H}}(t), y_{\mathrm{H}}^{(k)} - \psi_{\mathrm{H}_1}(x_{\mathrm{H}}^{(k)}, z^{(k)})\rangle_{\mathrm{H}} + \langle x_{\mathrm{H}}^{(k)}, t[y_{\mathrm{H}}^{(0)} - \psi_{\mathrm{H}_1}(x_{\mathrm{H}}^{(0)}, z^{(0)})]\rangle_{\mathrm{H}} \\ &\quad - [t\psi_{\mathrm{H}_2}(x_{\mathrm{H}}^{(0)}, z^{(0)})]^T z^{(k)} - \psi_{\mathrm{H}_2}(x_{\mathrm{H}}^{(k)}, z^{(k)})^T z(t) \\ &\leq \|x_{\mathrm{H}}(t)\|\|y_{\mathrm{H}}^{(k)} - \psi_{\mathrm{H}_1}(x_{\mathrm{H}}^{(k)}, z^{(k)})\| + t\|x_{\mathrm{H}}^{(k)}\|\|y_{\mathrm{H}}^{(0)} - \psi_{\mathrm{H}_1}(x_{\mathrm{H}}^{(0)}, z^{(0)})\| \\ &\quad + t\|\psi_{\mathrm{H}_2}(x_{\mathrm{H}}^{(0)}, z^{(0)})\|\|z^{(k)}\| + \|\psi_{\mathrm{H}_2}(x_{\mathrm{H}}^{(k)}, z^{(k)})\|\|z(t)\| \\ &\leq (1/\epsilon)(t\epsilon) + t(\|x_{\mathrm{H}}^*\| + \epsilon)\|y_{\mathrm{H}}^{(0)} - \psi_{\mathrm{H}_1}(x_{\mathrm{H}}^{(0)}, z^{(0)})\| \\ &\quad + t\|\psi_{\mathrm{H}_2}(x_{\mathrm{H}}^{(0)}, z^{(0)})\|(\|z^*\| + \epsilon) + (t\epsilon)(1/\epsilon) \\ &\leq t\delta, \end{aligned}$$

where $\delta := 2 + \|h_{\mathrm{H}}^{(0)}\|(\|x_{\mathrm{H}}^*\| + \|z^*\| + 2)$. Note that (44) implies

$$x_{\mathrm{H}}(t) = t y_{\mathrm{H}}(t)^{-1},\ y_{\mathrm{H}}(t) = t x_{\mathrm{H}}(t)^{-1}.$$

Combining the relations above, it must hold that for every $t \in (0,1]$ and $k \geq k(t)$

$$\begin{aligned} t\delta &\geq \langle x_{\mathrm{H}}(t), y_{\mathrm{H}}^{(k)}\rangle_{\mathrm{H}} + \langle y_{\mathrm{H}}(t), x_{\mathrm{H}}^{(k)}\rangle_{\mathrm{H}} \\ &= \langle t y_{\mathrm{H}}(t)^{-1}, y_{\mathrm{H}}^{(k)}\rangle_{\mathrm{H}} + \langle t x_{\mathrm{H}}(t)^{-1}, x_{\mathrm{H}}^{(k)}\rangle_{\mathrm{H}} \\ &= t\left\{\langle y(t)^{-1}, y^{(k)}\rangle + \frac{\kappa_k}{\kappa(t)} + \langle x(t)^{-1}, x^{(k)}\rangle + \frac{\tau_k}{\tau(t)}\right\} \end{aligned}$$

Since $\langle y(t)^{-1}, y^{(k)}\rangle > 0$ and $\langle x(t)^{-1}, x^{(k)}\rangle > 0$ ,we finally obtain that

$$\frac{\kappa_k}{\kappa(t)} < \delta,\ \frac{\tau_k}{\tau(t)} < \delta$$

for every $t \in (0,1]$ and $k \geq k(t)$. The assertion (iii) follows from the facts $\kappa_k \to \kappa^*$, $\tau_k \to \tau^*$ and $\delta > 0$. ∎