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Convex Conjugation and Duality for the Solutions of a Functional Equation

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Dedicated to Professor Hoang Tuy on the occasion of his 80th-birthday

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Abstract. We characterize the l.s.c. convex functions $f : (0, +\infty) \longrightarrow \overline{\mathbb{R}}$ that satisfy $f(x) = xf(1/x)$ in terms of their Fenchel conjugates. We also introduce a suitable duality theory for such functions and characterize the associated support sets.

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1. Introduction

In view of some applications in information theory, the following functional equation for extended real-valued functions $f : (0, +\infty) \longrightarrow \overline{\mathbb{R}}$ has been studied in [2]:

$$f(x) = xf\left(\frac{1}{x}\right).$$

This equation can be written as

$$f = f^{\diamond}, \tag{1}$$

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by using the transformation $f \longmapsto f^{\diamond}$ from the set of extended real-valued functions on $(0,+\infty)$ into itself defined by

$$f^{\diamond}(x) = xf\Big(\frac{1}{x}\Big).$$

In the above cited paper all the convex solutions to this functional equation were determined and a method for generating them was proposed.

The aim of this paper is twofold. On one hand, since Fenchel conjugation is an essential tool in convex analysis, it is natural to study the Fenchel conjugates of the convex solutions to (1). This is one of the objectives of this paper. Let us recall that the convex conjugate of $f : (0,+\infty) \longrightarrow \overline{\mathbb{R}}$ is the function $f^* : \mathbb{R} \longrightarrow \overline{\mathbb{R}}$ given by

$$f^*(x^*) = \sup_{x\in(0,+\infty)} \{x^*x - f(x)\};$$

similarly, for $g : \mathbb{R} \longrightarrow \overline{\mathbb{R}}$ we will consider its "conjugate" $g^* : (0,+\infty) \longrightarrow \overline{\mathbb{R}}$, defined by

$$g^*(x) = \sup_{x^*\in\mathbb{R}} \{xx^* - g(x^*)\}. \tag{2}$$

This "conjugate" function g^* considered here is nothing but the restriction to $(0,+\infty)$ of the usual Fenchel conjugate of g, which is defined on the whole of $\overline{\mathbb{R}}$ by the same expression (2). We will denote by Γ and $\Gamma_{(0,+\infty)}$ the sets of all l.s.c. proper convex functions $g : \mathbb{R} \longrightarrow \overline{\mathbb{R}}$ and $f : (0,+\infty) \longrightarrow \overline{\mathbb{R}}$, respectively (including the constant functions $\pm\infty$). It follows from classical results in convex analysis that $f^{**} = f$ if and only if $f \in \Gamma_{(0,+\infty)}$, and $g^{**} = g$ if and only if $g \in \Gamma$ and is either nondecreasing and nonconstant or one of the constant functions $\pm\infty$. For the basic notions and results in convex analysis that we will use, we refer to the standard books [1, 6]. Among these notions, let us recall that of the epigraph of a function $\varphi : S \longrightarrow \overline{\mathbb{R}}$ defined on an arbitrary set S :

$$epi\,\varphi = \{(s,\lambda) \in S \times \mathbb{R} : \varphi(s) \le \lambda\}.$$

A subset of $\mathbb{R}^2$ (of $(0,+\infty) \times \mathbb{R}$) will be called epigraphical if it is the epigraph of some function $f : (0,+\infty) \longrightarrow \overline{\mathbb{R}}$ (resp., $g : \mathbb{R} \longrightarrow \overline{\mathbb{R}}$). It can be easily seen that a closed convex set is epigraphical if and only if the vector $(0,1)$ is a recession direction, in which case it is the epigraph of a lower semicontinuous (l.s.c.) convex function.

We will characterize the convex solutions to (1) in terms of their Fenchel conjugates; it will be shown that such conjugate functions are the solutions to another functional equation involving the transformation

$$g \longmapsto g^h$$

that assigns to each function $g : \mathbb{R} \longrightarrow \overline{\mathbb{R}}$ the new function $g^h : \mathbb{R} \longrightarrow \overline{\mathbb{R}}$ defined by

$$g^h(r) = \sup\{s \in \mathbb{R} : g(s) \le r\}.$$

This transformation arises in the study of inversion of real-valued functions carried out in [5].

A second goal of this paper is to build a specific duality theory for convex solutions to (1) in the framework of generalized convex conjugation. We will do it by following the abstract scheme of [4], which we recall in Sec. 3. We refer to [3, 7, 8] for more details on abstract convexity and generalized convex duality theory.

The rest of the paper consists of three sections. In Sec. 2 we study the composition of the Fenchel conjugation operator with the mapping $f \longmapsto f^{\diamond}$ and characterize the convex solutions to (1) in terms of their conjugates. Section 3 provides a specific duality theory for such solutions within the framework of dualities between complete lattices. Finally, in Sec. 4 we consider the support sets associated with the dualities introduced in Sec. 3 and characterize them as special classes of convex sets.

2. Characterizations of Convex Solutions in Terms of Fenchel Conjugates

In this section we characterize the convex solutions to the functional equation (1) in terms of their Fenchel conjugates. Our first result deals with the composition of the Fenchel conjugation operator with the transformation $f \longmapsto f^{\diamond}$.

Proposition 1. *Let* $f : (0, +\infty) \longrightarrow \overline{\mathbb{R}}$. *Then*

$$f^{\diamond *}(x^*) = -f^{*h}(-x^*), \quad \forall x^* \in \mathbb{R}.$$

Proof. For every $x^* \in \mathbb{R}$, direct computation yields

$$\begin{aligned} f^{\diamond *}(x^*) &= \sup_{x \in (0,+\infty)} \{x^* x - f^{\diamond}(x)\} \\ &= \sup_{x \in (0,+\infty)} \left\{x^* x - x f\left(\frac{1}{x}\right)\right\} \\ &= \inf \left\{t \in \mathbb{R} : t \geq x^* x - x f\left(\frac{1}{x}\right), \ \forall x > 0\right\}. \end{aligned}$$

Making the change of variables $s = -t$ and $y = \dfrac{1}{x}$ in the last expression, we get

$$\begin{aligned} f^{\diamond *}(x^*) &= \inf \left\{ -s \in \mathbb{R} : -s \geq \frac{x^*}{y} - \frac{1}{y} f(y), \ \forall y > 0 \right\} \\ &= -\sup \{s \in \mathbb{R} : sy - f(y) \leq -x^*, \ \forall y > 0\} \\ &= -\sup \{s \in \mathbb{R} : f^*(s) \leq -x^*\} = -f^{*h}(-x^*). \end{aligned}$$

■

We next consider the composition of the same operators but in the reverse order.

Proposition 2. *Let* $g : \mathbb{R} \longrightarrow \overline{\mathbb{R}}$. *Then*

$$g^{*\diamond} = \left(-g^h(-\cdot)\right)^*.$$

Proof. For every $x \in (0, +\infty)$, one has

$$\begin{aligned}\left(-g^h(-\cdot)\right)^*(x) &= \sup_{x^* \in \mathbb{R}} \left\{xx^* + g^h(-x^*)\right\} \\ &= \sup_{x^* \in \mathbb{R}} \left\{xx^* + \sup\left\{s \in \mathbb{R} : g(s) \leq -x^*\right\}\right\} \\ &= \sup\left\{xx^* + s : x^* \in \mathbb{R},\ s \in \mathbb{R},\ g(s) \leq -x^*\right\} \\ &= \sup_{s \in \mathbb{R}} \left\{-xg(s) + s\right\} = x \sup_{s \in \mathbb{R}} \left\{\frac{s}{x} - g(s)\right\} \\ &= xg^*\left(\frac{1}{x}\right) = g^{*\diamond}(x).\end{aligned}$$

■

The preceding propositions yield the following immediate corollaries.

Corollary 1. *If $f : (0, +\infty) \longrightarrow \overline{\mathbb{R}}$ satisfies* (1) *then*

$$f^*(x^*) = -f^{*h}(-x^*), \quad \forall x^* \in \mathbb{R}. \tag{3}$$

Conversely, if $g : \mathbb{R} \longrightarrow \overline{\mathbb{R}}$ satisfies

$$g(x^*) = -g^h(-x^*), \quad \forall\ x^* \in \mathbb{R}, \tag{4}$$

then its conjugate g^* is a solution to the functional equation (1).

The first part of the preceding corollary is nothing but an analytic version of the observation made in [2, p. 1316] that $f \in \Gamma_{(0,+\infty)}$ satisfies (1) if and only if *epi* f^* is symmetric with respect to the second bisecting line in the plane (i.e., the line $\left\{(x_1^*, x_2^*) \in \mathbb{R}^2 : x_1^* + x_2^* = 0\right\}$).

Since for every $f \in \Gamma_{(0,+\infty)}$ one has $f^{**} = f$, from Corrollary 1 we easily obtain the following result.

Corollary 2. *Let $f \in \Gamma_{(0,+\infty)}$. Then f satisfies* (1) *if and only if* (3) *holds.*

In a similar way, since for every nondecreasing nonconstant function $g \in \Gamma$ one has $g^{**} = g$, Corrollary 1 also yields the next result.

Corollary 3. *Let $g \in \Gamma$ be nondecreasing and nonconstant. Then g satisfies* (4) *if and only if g^* is a solution to the functional equation* (1).

3. Specific Duality Theory

In this section we will develop a specific duality theory for convex solutions to the functional equation (1). Our approach will be based on the scheme introduced in [4] for dualities between complete lattices.

Let $G : (0, +\infty) \times \mathbb{R} \times \overline{\mathbb{R}} \longrightarrow \overline{\mathbb{R}}$ be the function defined by

$$G(x, x^*, a) = \max\left\{xx^* - a, \frac{x^* - a}{x}\right\}. \tag{5}$$

For every index set I one has

$$G\big(x, x^*, \inf_{i\in I} a_i\big) = \sup_{i\in I} G(x, x^*, a_i) \quad \big(x \in (0, +\infty), x \in \mathbb{R}, \{a_i\}_{i\in I} \subseteq \overline{\mathbb{R}}\big);$$

hence, by [4, Theorem 3.1], the mapping assigning to each function $f : (0, +\infty) \longrightarrow \overline{\mathbb{R}}$ its dual $f^{\Delta} : \mathbb{R} \longrightarrow \overline{\mathbb{R}}$, defined by

$$f^{\Delta}(x^*) = \sup_{x\in(0,+\infty)} G(x, x^*, f(x)), \tag{6}$$

is a duality, i.e., for every index set I and every family of functions $f_i : (0, +\infty) \longrightarrow \overline{\mathbb{R}}$ $(i \in I)$ it satisfies

$$\Big(\inf_{i\in I} f_i\Big)^{\Delta} = \sup_{i\in I} f_i^{\Delta}.$$

Direct computation yields

$$\begin{aligned} f^{\Delta}(x^*) &= \sup_{x\in(0,+\infty)} \max\Big\{xx^* - f(x), \frac{x^* - f(x)}{x}\Big\} \\ &= \max\Big\{\sup_{x\in(0,+\infty)} \{xx^* - f(x)\}, \sup_{x\in(0,+\infty)} \frac{x^* - f(x)}{x}\Big\} \\ &= \max\Big\{f^*(x^*), \sup_{x\in(0,+\infty)} \frac{x^* - f(x)}{x}\Big\}. \end{aligned}$$

By making the change of variables $y = 1/x$ in the last expression, we obtain

$$\begin{aligned} f^{\Delta}(x^*) &= \max\Big\{f^*(x^*), \sup_{y\in(0,+\infty)} \Big\{yx^* - yf\Big(\frac{1}{y}\Big)\Big\}\Big\} \\ &= \max\Big\{f^*(x^*), \sup_{y\in(0,+\infty)} \big\{yx^* - f^{\diamond}(y)\big\}\Big\} \\ &= \max\big\{f^*(x^*), f^{\diamond *}(x^*)\big\}. \end{aligned}$$

We thus conclude that

$$f^{\Delta} = \max\big\{f^*, f^{\diamond *}\big\}$$

or, equivalently,

$$f^{\Delta} = (\min\{f, f^{\diamond}\})^*. \tag{7}$$

Proposition 3. *For every $f : (0, +\infty) \longrightarrow \overline{\mathbb{R}}$, the dual function f^{Δ} belongs to Γ and is either nondecreasing, nonconstant and a solution to the functional equation* (4) *or one of the constant functions $\pm\infty$.*

Proof. Since, in view of (5), for each $(x, a) \in (0, +\infty) \times \overline{\mathbb{R}}$ the function $G(x, \cdot, a)$ is nondecreasing and solves the functional equation (3), it follows from (6) that f^{Δ} is nondecreasing and solves this functional equation too (as the transformation $f \longmapsto f^h$, when applied to nondecreasing functions, transforms suprema into infima). Moreover, if $f^{\Delta} \not\equiv -\infty$ then, by (6), for some $(x, a) \in (0, +\infty) \times \mathbb{R}$ the function $G(x, \cdot, a)$ is a minorant of f^{Δ}; since such a minorant is strictly increasing and $f \not\equiv +\infty$, we easily deduce that f^{Δ} is nonconstant. ∎

The dual operator Δ' is defined, according to [4], as the mapping assigning to each $g : \mathbb{R} \longrightarrow \overline{\mathbb{R}}$ the function $g^{\Delta'} : (0, +\infty) \longrightarrow \overline{\mathbb{R}}$ given by

$$g^{\Delta'} = \inf \left\{ f : (0, +\infty) \longrightarrow \overline{\mathbb{R}} : f^{\Delta} \leq g \right\}.$$

By [4, Theorem 3.5], one has

$$g^{\Delta'}(x) = \sup_{x^* \in \mathbb{R}} G'(x^*, x, g(x^*)), \quad \forall x \in (0, +\infty), \tag{8}$$

with $G' : \mathbb{R} \times (0, +\infty) \times \overline{\mathbb{R}} \longrightarrow \overline{\mathbb{R}}$ being the function defined by

$$G'(x^*, x, b) = \min \left\{ a \in \overline{\mathbb{R}} : G(x, x^*, a) \leq b \right\}.$$

Straightforward computations yield

$$G'(x^*, x, b) = \max \{x^* x - b, x^* - bx\}; \tag{9}$$

therefore

$$\begin{aligned} g^{\Delta'}(x) &= \sup_{x^* \in \mathbb{R}} \max \{x^* x - g(x^*), x^* - xg(x^*)\} \\ &= \max \left\{ \sup_{x^* \in \mathbb{R}} \{x^* x - g(x^*)\}, \sup_{x^* \in \mathbb{R}} \{x^* - xg(x^*)\} \right\} \\ &= \max \left\{ g^*(x), x \sup_{x^* \in \mathbb{R}} \left\{ \frac{x^*}{x} - g(x^*) \right\} \right\} \\ &= \max \left\{ g^*(x), xg^*\left(\frac{1}{x}\right) \right\} \\ &= \max \{g^*(x), g^{*\diamond}(x)\}. \end{aligned}$$

We have thus proved that

$$g^{\Delta'} = \max \{g^*, g^{*\diamond}\}; \tag{10}$$

hence, by Proposition 2,

$$g^{\Delta'} = \max \left\{ g^*, \left(-g^h(-\cdot)\right)^* \right\}$$

or, equivalently,

$$g^{\Delta'} = \left(\min \left\{ g, \left(-g^h(-\cdot)\right) \right\}\right)^*. \tag{11}$$

Proposition 4. *For every $g : \mathbb{R} \longrightarrow \overline{\mathbb{R}}$, the dual function $g^{\Delta'}$ belongs to $\Gamma_{(0,+\infty)}$ and solves the functional equation* (1).

Proof. By (8), since in view of (9) for every $(x^*, b) \in \mathbb{R} \times \overline{\mathbb{R}}$ the function $G'(x^*, \cdot, b)$ is continuous and solves the functional equation (1), $g^{\Delta'}$ belongs to $\Gamma_{(0,+\infty)}$ and solves this functional equation too. ∎

By [4, Theorem 3.6], for every $x \in (0, +\infty)$ one has

$$f^{\Delta\Delta'}(x) = \sup \left\{ G'(x^*, x, b) : G'(x^*, y, b) \leq f(y), \ \forall y \in (0, +\infty) \right\}. \tag{12}$$

Using this expression we will prove the next result.

Theorem 1. *For every $f:(0,+\infty)\longrightarrow\overline{\mathbb{R}}$, the second dual $f^{\Delta\Delta'}$ is the largest minorant of f in $\Gamma_{(0,+\infty)}$ that solves the functional equation* (1).

Proof. It immediately follows from (12) that $f^{\Delta\Delta'}$ is a minorant of f, and by Proposition 7 it belongs to $\Gamma_{(0,+\infty)}$ and solves the functional equation (1). It only remains to prove that it is the largest minorant of f with these properties. Let $h\in\Gamma_{(0,+\infty)}$ be a minorant of f such that $h=h^{\diamond}$. For every $x\in(0,+\infty)$, by Fenchel inequality we have

$$x^{*}-h^{*}(x^{*})\,x=x\Big(x^{*}\frac{1}{x}-h^{*}(x^{*})\Big)\leq xh\Big(\frac{1}{x}\Big)=h^{\diamond}(x)=h(x)\,,\ \ \forall x^{*}\in\mathbb{R};$$

hence

$$\begin{aligned}h(x)=h^{**}(x)&=\sup_{x^{*}\in\mathbb{R}}\{xx^{*}-h^{*}(x^{*})\}\\&\leq\sup_{x^{*}\in\mathbb{R}}G^{'}(x^{*},x,h^{*}(x^{*}))\\&=\sup_{x^{*}\in\mathbb{R}}\max\{xx^{*}-h^{*}(x^{*})\,,x^{*}-h^{*}(x^{*})\,x\}\\&\leq\sup_{x^{*}\in\mathbb{R}}\max\{xx^{*}-h^{*}(x^{*})\,,h(x)\}\\&=\max\Big\{\sup_{x^{*}\in\mathbb{R}}\{xx^{*}-h^{*}(x^{*})\}\,,h(x)\Big\}\\&\leq\max\{h(x)\,,h(x)\}=h(x)\,.\end{aligned}$$

We thus have $\sup_{x^{*}\in\mathbb{R}}G^{'}(x^{*},x,h^{*}(x^{*}))=h(x)\leq f(x)$; since x is arbitrary this shows that, for each $x^{*}\in\mathbb{R}$, $G^{'}(x^{*},\cdot,h^{*}(x^{*}))$ is a minorant of f. Therefore, in view of (12), $f^{\Delta\Delta'}(x)\geq\sup_{x^{*}\in\mathbb{R}}G^{'}(x^{*},x,h^{*}(x^{*}))=h(x)$, which proves that $f^{\Delta\Delta'}\geq h$. ∎

Corollary 4. *A function $f:(0,+\infty)\longrightarrow\overline{\mathbb{R}}$ satisfies $f^{\Delta\Delta'}=f$ if and only if it belongs to $\Gamma_{(0,+\infty)}$ and solves the functional equation* (1).

To characterize the second dual $g^{\Delta'\Delta}$ of a function $g:\mathbb{R}\longrightarrow\overline{\mathbb{R}}$ we will use the following equality, which holds for every $x^{*}\in\mathbb{R}$ by [4, Theorem 3.6] combined with the fact that $G''=G$:

$$g^{\Delta'\Delta}(x^{*})=\sup\{G(x,x^{*},a):G(x,y^{*},a)\leq g(y^{*})\,,\ \ \forall y^{*}\in\mathbb{R}\}\,. \qquad (13)$$

In the next theorem we exclude the trivial case $g\equiv+\infty$, for which $g^{\Delta'\Delta}\equiv+\infty$.

Theorem 2. *For every $g:\mathbb{R}\longrightarrow\overline{\mathbb{R}}$ with $g\not\equiv+\infty$, the second dual $g^{\Delta'\Delta}$ is either the largest nondecreasing nonconstant minorant of g in Γ that solves the functional equation* (4) *or the constant function $-\infty$ if g does not have a minorant with these properties.*

Proof. By (13) the second dual $g^{\Delta'\Delta}$ is a minorant of g, and by Proposition 3 it belongs to Γ and is nondecreasing and nonconstant unless it is identically $-\infty$.

We will next prove that it is the largest minorant of g with these properties if one such minorant exists.

Let $k \in \Gamma$ be a nondecreasing minorant of g such that $k = -k^h(-\cdot)$. For every $x^* \in \mathbb{R}$, by Corollary 1 and Fenchel inequality we have

$$\frac{x^* - k^*(x)}{x} = \frac{x^* - k^{*\diamond}(x)}{x} = \frac{x^* - xk^*\left(\frac{1}{x}\right)}{x} = \frac{x^*}{x} - k^*\left(\frac{1}{x}\right) \leq k(x^*) \quad \forall\, x \in (0, +\infty);$$

hence

$$\begin{aligned} k(x^*) = k^{**}(x^*) &= \sup_{x \in (0,+\infty)} \{x^* x - k^*(x)\} \\ &\leq \sup_{x \in (0,+\infty)} G(x, x^*, k^*(x)) \\ &= \sup_{x \in (0,+\infty)} \max\left\{x^* x - k^*(x), \frac{x^* - k^*(x)}{x}\right\} \\ &\leq \sup_{x \in (0,+\infty)} \max\{x^* x - k^*(x), k(x^*)\} \\ &= \max\left\{\sup_{x \in (0,+\infty)} \{x^* x - k^*(x)\}, k(x^*)\right\} \\ &\leq \max\{k(x^*), k(x^*)\} = k(x^*). \end{aligned}$$

We thus have $\sup_{x \in (0,+\infty)} G(x, x^*, k^*(x)) = k(x^*) \leq g(x^*)$; since x^* is arbitrary this shows that, for each $x \in (0, +\infty)$, $G(x, \cdot, k^*(x))$ is a minorant of g. Therefore, in view of (13), $g^{\Delta'\Delta}(x^*) \geq \sup_{x \in (0,+\infty)} G(x, x^*, k^*(x)) = k(x^*)$, which proves that $g^{\Delta'\Delta} \geq k$.

Finally, if g does not have any nodecreasing nonconstant minorant in Γ then the supremum in (13) is over the empty set and hence $g^{\Delta'\Delta} \equiv -\infty$. ■

Corollary 5. *A function $g : \mathbb{R} \longrightarrow \overline{\mathbb{R}}$ satisfies $g^{\Delta'\Delta} = g$ if and only if it belongs to Γ and is either nondecreasing, nonconstant and a solution to the functional equation* (4) *or one of the constant functions $\pm\infty$.*

From Propositions 3 and 4 and Corollaries 4 and 5 we obtain the first part of the following result.

Corollary 6. *The mapping $\Delta : f \longmapsto f^\Delta$ is a bijection from the set of functions in $\Gamma_{(0,+\infty)}$ that solve the functional equation* (1) *onto the set of functions in Γ that are either nondecreasing and nonconstant and solutions to the functional equation* (4) *or one of the constant functions $\pm\infty$; the inverse bijection is $\Delta' : g \longmapsto g^{\Delta'}$. For solutions f and g to the respective functional equations one actually has $f^\Delta = f^*$ and $g^{\Delta'} = g^*$.*

Proof. The equality $f^\Delta = f^*$ is an immediate consequence of (7); similarly, the equality $g^{\Delta'} = g^*$ follows from (11). ■

On combining the preceding corollary with Propositions 3 and 4, one obtains that the second duals of functions $f : (0, +\infty) \longrightarrow \overline{\mathbb{R}}$ and $g : \mathbb{R} \longrightarrow \overline{\mathbb{R}}$ satisfy $f^{\Delta\Delta'} = f^{\Delta *}$ and $g^{\Delta'\Delta} = g^{\Delta' *}$; moreover, in view of (7) and (11) one has

$$f^{\Delta\Delta'} = \left(\min\left\{f, f^{\diamond}\right\}\right)^{**}$$

and

$$g^{\Delta'\Delta} = \left(\min\left\{g, \left(-g^{h}\left(-\cdot\right)\right)\right\}\right)^{**},$$

respectively.

4. Support Sets

Continuing the abstract convexity approach to the solutions to the functional equation (1), in this section we consider support sets with respect to the dualities Δ and Δ' introduced in the preceding section. Adapting the terminology of [7], we define the Δ-support set of $f : (0, +\infty) \longrightarrow \overline{\mathbb{R}}$ by

$$s\left(f, \Delta\right) = \left\{\left(x^{*}, b\right) \in \mathbb{R}^{2} : G'\left(x^{*}, x, b\right) \leq f\left(x\right), \ \forall \ x \in \left(0, +\infty\right)\right\}.$$

Using (9), a straightforward computation yields

$$s\left(f, \Delta\right) = \left\{\left(x^{*}, b\right) \in epi\, f^{*} : \left(-b, -x^{*}\right) \in epi\ f^{*}\right\}. \tag{14}$$

A set $C \subseteq \mathbb{R}^{2}$ will be called Δ-convex if it is the Δ-support set of some function $f : (0, +\infty) \longrightarrow \overline{\mathbb{R}}$. From (12) it follows that

$$s\left(f, \Delta\right) = s\left(f^{\Delta\Delta'}, \Delta\right); \tag{15}$$

therefore, by Theorem 1, a set is Δ-convex if and only if it is the Δ-support set of a function belonging to $\Gamma_{(0,+\infty)}$ and solving the functional equation (1). One can easily prove that, by Corollary 1, for any such function f the relations $(x^{*}, b) \in epi\, f^{*}$ and $(-b, -x^{*}) \in epi\, f^{*}$ are equivalent; therefore, by Corollary 4, on combining (15) with (14) (with f replaced by $f^{\Delta\Delta'}$), one gets the equality

$$s\left(f, \Delta\right) = epi\, f^{\Delta\Delta' *}, \tag{16}$$

which is valid for an arbitrary function $f : (0, +\infty) \longrightarrow \overline{\mathbb{R}}$.

Our next result provides a characterization of Δ-convex sets.

Theorem 3. *A set $C \subseteq \mathbb{R}^{2}$ is Δ-convex if and only if it is epigraphical, closed, convex and symmetric with respect to the second bisecting line.*

Proof. Let $C = s\left(f, \Delta\right)$ for some $f : (0, +\infty) \longrightarrow \overline{\mathbb{R}}$. By (16), it is clear that C is epigraphical, closed and convex; moreover, from (14) it easily follows that C is symmetric with respect to the second bisecting line.

Conversely, assume that C is an epigraphical closed convex set and is symmetric with respect to the second bisecting line; then $C = epi\, g$ for some $g \in \Gamma$.

By the symmetry property of *epi* g, for every $r \in \mathbb{R}$ one has

$$\begin{aligned} g^h(-r) &= \sup\{s \in \mathbb{R} : g(s) \le -r\} \\ &= \sup\{s \in \mathbb{R} : (s, -r) \in epi\, g\} \\ &= \sup\{s \in \mathbb{R} : (r, -s) \in epi\, g\} \\ &= \sup\{s \in \mathbb{R} : g(r) \le -s\} \\ &= -g(r), \end{aligned}$$

which shows that g satisfies (4). Hence, by Corollaries 1 and 4, $g = g^{**} = g^{*\Delta\Delta'*}$, which, in view of (16), yields $C = epi\ g = epi\ g^{*\Delta\Delta'*} = s(g^*, \Delta)$, thus proving that C is Δ-convex. ■

Corollary 7. *The mapping $f \longmapsto s(f,\Delta)$ is a bijection from the set of functions in $\Gamma_{(0,+\infty)}$ that solve the functional equation* (1) *onto the set of epigraphical closed convex subsets of $\mathbb{R}^2$ that are symmetric with respect to the second bisecting line; the inverse bijection is the mapping that assigns to each C the conjugate g^* of the unique function g such that epi $g = C$.*

Proof. As observed above, a set is Δ-convex if and only if it is the Δ-support set of a function belonging to $\Gamma_{(0,+\infty)}$ and solving the functional equation (1); therefore, by Theorem 3, the mapping $f \longmapsto s(f,\Delta)$, as a mapping between the sets described in the statement, is onto. To see that it is one-to-one, let $f_1, f_2 \in \Gamma_{(0,+\infty)}$ be solutions to the functional equation (1) and assume that $s(f_1,\Delta) = s(f_2,\Delta)$. By (16), we have $f_1^{\Delta\Delta'*} = f_2^{\Delta\Delta'*}$; hence, by Corollary 4, $f_1 = f_1^{**} = f_1^{\Delta\Delta'**} = f_2^{\Delta\Delta'**} = f_2^{**} = f_2$.

To prove the last part of the statement, let C be a Δ-convex set. Then $C = s(f,\Delta)$ for some $f \in \Gamma_{(0,+\infty)}$ satisfying (1). By (16), for $g = f^{\Delta\Delta'*}$ one has $C = epi\ g$; thus it only remains to observe that $g^* = f^{\Delta\Delta'**} = f^{**} = f$, the middle equality being a consequence of Corollary 4. ■

The bijection described in the preceding corollary is strongly related to the correspondence between convex solutions to the functional equation (1) and certain convex subsets of the plane considered in [2]. Indeed, as observed in [2] it is easy to see that, for any function $f : (0,+\infty) \longrightarrow \overline{\mathbb{R}}$, the associated set

$$C_f = \left\{(x^*, b) \in \mathbb{R}^2 : x^*x + by \le yf\left(\frac{x}{y}\right) \text{ for all } x > 0 \text{ and } y > 0\right\}$$

introduced in [2] is nothing but the image of *epi* f^* under the transformation $(x^*, b) \longmapsto (x^*, -b)$ and, on the other hand, if f belongs to $\Gamma_{(0,+\infty)}$ and satisfies (1) then, by (16) and Corollary 4, *epi* $f^* = s(f,\Delta)$. Consequently, the class of Δ-convex sets coincides with the set of images under the above transformation of the sets in the collection considered in [2] of closed convex sets in $\mathbb{R}^2$ whose recession cones contain the nonpositive quadrant $[-\infty, 0]^2$ and are symmetric with respect to the first bisecting line $\left\{(x_1^*, x_2^*) \in \mathbb{R}^2 : x_1^* = x_2^*\right\}$. This assertion can also be easily checked in a direct way by using the characterization of Δ-convex sets provided by Theorem 3.

Since the mapping $\Omega : f \longmapsto s(f, \Delta)$ is a duality from the complete lattice of all functions $f : (0, +\infty) \longrightarrow \overline{\mathbb{R}}$ (endowed with the pointwise ordering) into the complete lattice of subsets of $\mathbb{R}^2$ (endowed with the ordering given by reverse inclusion), i.e., it satisfies

$$s\Big(\inf_{i\in I} f_i, \Delta\Big) = \bigcap_{i\in I} s(f_i, \Delta)$$

(with the convention that the intersection over the empty family is $\mathbb{R}^2$) for every family $\{f_i\}_{i\in I}$ of functions $f_i : (0, +\infty) \longrightarrow \overline{\mathbb{R}}$ and the inverse duality Ω' assigns to each C the function $\inf\big\{f : (0, +\infty) \longrightarrow \overline{\mathbb{R}} : s(f, \Delta) \supseteq C\big\}$, a standard argument in abstract convexity theory yields the following consequence of the preceding corollary on the compositions $\Omega'\Omega$ and $\Omega\Omega'$.

Corollary 8. (i) *For every* $f : (0, +\infty) \longrightarrow \overline{\mathbb{R}}$, *the function*

$$\Omega'\Omega(f) = \inf\big\{h : (0, +\infty) \longrightarrow \overline{\mathbb{R}} : s(h, \Delta) \supseteq s(f, \Delta)\big\}$$

coincides with $f^{\Delta\Delta'}$.

(ii) *For every* $C \subseteq \mathbb{R}^2$, *the set* $\Omega\Omega'(C)$ *is the smallest* Δ*-convex set that contains* C.

In a fully parallel way we next study the Δ'-support set of $g : \mathbb{R} \longrightarrow \overline{\mathbb{R}}$, which we define by

$$s(g, \Delta') = \big\{(x, a) \in \mathbb{R}^2 : G(x, x^*, a) \leq g(x^*) \quad \forall\, x^* \in \mathbb{R}\big\}.$$

One can easily seen, using (5), that

$$s(g, \Delta') = \Big\{ (x, a) \in epi\, g^* : \Big(\frac{1}{x}, \frac{a}{x}\Big) \in epi\, g^* \Big\}. \tag{17}$$

A set $D \subseteq (0, +\infty) \times \mathbb{R}$ will be called Δ'-convex if it is the Δ'-support set of some function $g : \mathbb{R} \longrightarrow \overline{\mathbb{R}}$. From (13) it follows that

$$s(g, \Delta') = s\left(g^{\Delta'\Delta}, \Delta'\right); \tag{18}$$

therefore, by Theorem 2, a nontrivial set (i.e., a set different from the empty set and from $(0, +\infty) \times \mathbb{R}$) in $(0, +\infty) \times \mathbb{R}$ is Δ'-convex if and only if it is the Δ'-support set of a nondecreasing nonconstant function belonging to Γ that solves the functional equation (4). One can easily prove that, by Corollary 1, for any such function g the relations $(x, a) \in epi\ g^*$ and $\left(\frac{1}{x}, \frac{a}{x}\right) \in epi\, g^*$ are equivalent; therefore, by Corollary 5, on combining (18) with (17) (with g replaced by $g^{\Delta'\Delta}$), one gets the following equality

$$s(g, \Delta') = epi\ g^{\Delta'\Delta *}, \tag{19}$$

which is valid for an arbitrary function $g : \mathbb{R} \longrightarrow \overline{\mathbb{R}}$.

To characterize Δ'-convex sets we will use the conical extension of a set $D \subseteq (0,+\infty) \times \mathbb{R}$, which we define by

$$\widetilde{D} = \{(x_1, x_2, x_3) \in (0,+\infty) \times \mathbb{R} \times (0,+\infty) : (x_1, x_2) \in x_3 D\}.$$

We will also consider the mapping σ defined on $(0,+\infty) \times \mathbb{R} \times (0,+\infty)$ by

$$\sigma(x_1, x_2, x_3) = (x_3, x_2, x_1),$$

that is, the restriction to $(0,+\infty) \times \mathbb{R} \times (0,+\infty)$ of the symmetry in $\mathbb{R}^3$ with respect to the plane $x_1 = x_3$.

Theorem 4. *A set $D \subseteq (0,+\infty) \times \mathbb{R}$ is Δ'-convex if and only if it is epigraphical, closed and convex and $\widetilde{D}$ is invariant with respect to σ (i.e., $\sigma(\widetilde{D}) = \widetilde{D}$).*

Proof. Let $D = s(g, \Delta')$ for some $g : \mathbb{R} \longrightarrow \overline{\mathbb{R}}$. By (19), it is clear that D is epigraphical, closed and convex; moreover, from (17) it easily follows that $\widetilde{D}$ is invariant with respect to σ.

Conversely, assume that D is an epigraphical closed convex set and $\widetilde{D}$ is invariant with respect to σ; then $D = epi\ f$ for some $f \in \Gamma_{(0,+\infty)}$. By the symmetry property of $epi\, f$, for every $x \in (0,+\infty)$ one has

$$\begin{aligned} f^{\diamond}(x) &= x f\left(\frac{1}{x}\right) \\ &= x \inf\left\{\lambda \in \mathbb{R} : \left(\frac{1}{x}, \lambda\right) \in epi\, f\right\} \\ &= x \inf\left\{\lambda \in \mathbb{R} : \left(\frac{1}{x}, \lambda, 1\right) \in \widetilde{epi\, f}\right\} \\ &= x \inf\left\{\lambda \in \mathbb{R} : \left(1, \lambda, \frac{1}{x}\right) \in \widetilde{epi\, f}\right\} \\ &= x \inf\left\{\lambda \in \mathbb{R} : (x, \lambda x) \in epi\, f\right\} \\ &= x \inf\left\{\lambda \in \mathbb{R} : f(x) \le \lambda x\right\} \\ &= x \frac{f(x)}{x} \\ &= f(x), \end{aligned}$$

which shows that f satisfies (1). Hence, by Corollaries 1 and 5, $f = f^{**} = f^{*\Delta'\Delta*}$, which, in view of (19), yields $D = epi\ f = epi\ f^{*\Delta'\Delta*} = s(f^*, \Delta')$, thus proving that D is Δ'-convex. ∎

Corollary 9. *The mapping $g \longmapsto s(g, \Delta')$ is a bijection from the set of functions in Γ that are either nondecreasing and nonconstant and solutions to the functional equation (4) or one of the constant functions $\pm\infty$ onto the set of epigraphical closed convex subsets D of $(0,+\infty) \times \mathbb{R}$ whose conical extensions $\widetilde{D}$ are invariant with respect to σ; the inverse bijection is the mapping that assigns to each D the conjugate f^* of the unique function f such that $epi\, f = D$.*

Proof. As observed above, a set is Δ'-convex if and only if it is the Δ'-support set of a function belonging to Γ and solving the functional equation (4); therefore, by Theorem 4, the mapping $g \longmapsto s(g, \Delta')$, as a mapping between the sets described in the statement, is onto. To see that it is one-to-one, let $g_1, g_2 \in \Gamma$ be functions satisfying the properties in the statement and assume that $s(g_1, \Delta') = s(g_2, \Delta')$. By (19), we have $g_1^{\Delta'\Delta*} = g_2^{\Delta'\Delta*}$; hence, by Corollary 5, $g_1 = g_1^{**} = g_1^{\Delta'\Delta**} = g_2^{\Delta'\Delta**} = g_2^{**} = g_2$.

To prove the last part, let D be a Δ'-convex set. Then $D = s(g, \Delta')$ for some function $g \in \Gamma$ satisfying the properties in the statement. By (19), for $f = g^{\Delta'\Delta*}$ one has $D = epi\ f$; thus it only remains to observe that $f^* = g^{\Delta'\Delta**} = g^{**} = g$, the middle equality being a consequence of Corollary 5. ∎

We conclude by presenting a counterpart to Corollary 8.

Corollary 10. (i) *For every* $g : \mathbb{R} \longrightarrow \overline{\mathbb{R}}$, *the function*

$$\Psi'\Psi(f) = \inf\left\{k : \mathbb{R} \longrightarrow \overline{\mathbb{R}} : s(k, \Delta) \supseteq s(g, \Delta')\right\}$$

coincides with $g^{\Delta'\Delta}$.

(ii) *For every* $D \subseteq (0, +\infty) \times \mathbb{R}$, *the set* $\Psi\Psi'(D)$ *is the smallest* Δ'*-convex set that contains* D.

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