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On the Extremal Structure of an OSPF Related Cone

Peter Broström and Kaj Holmberg

*Department of Mathematics, Linköping Institute of Technology,
SE-581 83 Linköping, Sweden*

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Abstract. In a telecommunication network using the OSPF protocol, the routing patterns used are the shortest paths with respect to the link weights. By choosing the link weights, a set of desired shortest paths can often be obtained, but in some cases there are no weights giving the desired shortest paths. We study a polyhedral cone associated with the latter situation, and give a characterization of an important class of extreme rays of the cone. The characterization is based on feasible and valid cycles, which can be found efficiently in practice.

Keywords: OSPF, polyhedral cone, extreme ray, multicommodity, valid cycle.

1. Introduction

Most modern telecommunication networks use Internet Protocol and OSPF (Open Shortest Path First) for determining the routing of the traffic. In a network using the OSPF protocol, the routing patterns used are the shortest paths with respect to certain link weights set by the network operator. The interest in finding good values for these weights has been high since [7]. By appropriate choice of the link weights, a prespecified set of desired shortest paths can often be obtained, but in some cases there are no weights giving the desired shortest paths. A very interesting question is how to characterize, in a practical manner, those sets of shortest paths that are/are not obtainable.

In [3] and [6] we introduced the concept of valid cycles, and showed how to find such cycles efficiently and find that valid cycles seem to exist in almost all non-obtainable instances. Here we will study valid cycles from a more theoretical point of view. We study a polyhedral cone associated with this problem, and give a characterization of some extreme rays of the cone, based on feasible and valid cycles. The immediate objective of the paper is to establish theoretical properties of valid cycles, and the long term goal of the research is to find practical ways of using these structures in network design and routing problems within the telecommunication area.

In Sec. 2, we give the basic mathematical models, one of which is a certain kind of multicommodity network flow model. In Sec. 3, we define feasible and 1-valid cycles, and in Sec. 4 we study the cone which is the feasible set of the multicommodity network flow problem. In Sec. 5 we introduce a cycle representation of the cone, which helps us to show that 1-valid cycles are extreme rays of the cone, which is done in Sec. 6. Section 6 also contains some more general results, which are used in Sec. 7 to show the main result that 1-valid cycles represent all extreme rays using only two commodities. In Sec. 8 we consider solutions using more than two commodities in one cycle and its reverse. We define 2-valid cycles and show that they represent all such solutions. Finally we show that 2-valid cycles exist if and only if there exist 1-valid cycles.

2. The Mathematical Model

We consider a directed graph $G = (N, A)$ with the set N of nodes and the set A of arcs. A number of subsets of arcs, $A_l \subseteq A$ for $l = 1, \dots, m$, called SP-graphs (shortest path graphs), are given. Each set A_l contains an arborescence or a reversed arborescence, and no set A_l contains a directed cycle.

An SP-graph contains a number of paths, and these paths are the desired shortest paths. We wish to find weights w_{ij} for all arcs $(i, j) \in A$, so that the paths in A_l have minimal sum of the weights (i.e. are shortest). Thus if A_l contains a path from node s to node t, this path should have a minimal sum of weights. All paths from s to t not completely in A_l should have larger sums of weights. If A_l contains more than one path from node s to node t, all these paths should have (the same) minimal sum of weights. Such a set of weights is called "compatible", and must be integers greater or equal to 1. See [6] and [3] for an introductory discussion.

In [6] and [3] it is shown that there exist compatible weights if and only if there is a feasible solution to the following constraints.

$$
\text{(P1)}\qquad
\begin{aligned}
w_{ij} + \pi_i^l - \pi_j^l &= 0, \quad \forall (i, j) \in A_l,\ l = 1, \dots, m, && (1.1)\\
w_{ij} + \pi_i^l - \pi_j^l &\geq 1, \quad \forall (i, j) \in A \setminus A_l,\ l = 1, \dots, m, && (1.2)\\
w_{ij} &\geq 1, \quad \forall (i, j) \in A. && (1.3)
\end{aligned}
$$

Substituting $w' = w - 1$ and using LP-duality (or Farkas' lemma) yields the following. P1 has a feasible solution if and only if there is no feasible solution to P2.

$$\sum_{l=1}^{m} \sum_{(i,j)\in A_l} \theta^l_{ij} < 0, \tag{2.0}$$

(P2)
$$\sum_{l=1}^{m} \theta^l_{ij} \le 0, \ \ \forall (i,j) \in A, \tag{2.1}$$

$$\sum_{j:(i,j)\in A} \theta^l_{ij} - \sum_{j:(j,i)\in A} \theta^l_{ji} = 0, \ \ \forall i \in N, \ \forall l, \tag{2.2}$$

$$\theta^l_{ij} \ge 0, \ \ \forall (i,j) \in A \setminus A_l, \ \forall l \tag{2.3}$$

Let $v(\theta)$ denote the left-hand-side of (2.0), i.e. $v(\theta) = \sum_{l=1}^{m} \sum_{(i,j)\in A_l} \theta^l_{ij}$. Let Θ be the polyhedron defined by the constraints (2.1), (2.2) and (2.3). Θ has only one extreme point, the origin, which however does not satisfy (2.0). One may start at the origin and then increase or decrease certain sets of variables in order to fulfill (2.0).

If $\bar{\theta}$ is a feasible solution to P2, then the solution $t\bar{\theta}$ is also a feasible solution for any scalar $t > 0$, so Θ is a polyhedral cone. All constraints satisfied with equality by $\bar{\theta}$ will also be satisfied with equality by $t\bar{\theta}$. We also note that any rational solution can be made integral by letting t be the least common multiple of the denominators.

A polyhedral cone has a finite number of extreme rays, and (2.0) defines an open half space, so the question is whether or not any extreme ray of Θ lies in this half space.

3. Feasible and Valid Cycles

We will interpret P2 as a special kind of multicommodity flow problem. The different commodities are indexed with l (i.e. are connected to the different SP-graphs). Especially we note that some flow may be negative. Due to constraints (2.2) all solutions to P2 will consist of circulating flow, so we will now pay particular attention to cycles. Much of the following is motivated and proved in [6] and [3].

We consider a cycle, r, with arc set $C_r = F_r \cup B_r$, where F_r are the arcs used *forwards*, i.e. in the correct direction, while B_r denotes the arcs used *backwards*, i.e. against the correct direction. If a flow is sent in cycle r, it will be positive in the arcs of F_r and negative in the arcs of B_r.

Definition 1. *A cycle $C_r = F_r \cup B_r$ is called* <u>*1-feasible*</u> *with respect to commodity l if $B_r \subseteq A_l$.*

The reason for this definition is that constraints (2.3) do not allow negative flow in arcs outside A_l.

Lemma 1. *A cycle $C_r = F_r \cup B_r$ can only be used by commodity l if the cycle is 1-feasible with respect to l.*

A cycle that is not 1-feasible for any l can not be used by any commodity, i.e. is not used at all.

The direction of a cycle is a part of its definition, so a cycle can not be used in its reversed direction. (Then it will be another cycle.) We use the notation $rev(r)$ for the reverse of cycle r, so $F_{rev(r)} = B_r$ and $B_{rev(r)} = F_r$. Note that r being 1-feasible with respect to l does not mean that $rev(r)$ is. (Usually it is not.) Cycle $rev(r)$ is 1-feasible with respect to l if $B_{rev(r)} \subseteq A_l$, i.e. if $F_r \subseteq A_l$.

Using a single cycle and a single commodity will, as will be shown later, not give a feasible solution to P2. Therefore we consider two commodities, one using cycle r and one using $rev(r)$.

Definition 2. *A cycle $C_r = F_r \cup B_r$ is called 2-feasible with respect to (l', l'') if it is 1-feasible with respect to l' and $rev(r)$ is 1-feasible with respect to l'', i.e. if $B_r \subseteq A_{l'}$ and $F_r \subseteq A_{l''}$.*

Note that the order of (l', l'') is important. A 2-feasible cycle with respect to (l', l'') is often not 2-feasible with respect to (l'', l'). A 2-feasible cycle clearly satisfies constraints (2.1), (2.2) and (2.3).

Lemma 2. *A 2-feasible cycle represents a point in Θ.*

Now consider a cycle that is not 2-feasible with respect to two commodities l' and l''. According to Definition 2 then either r is not 1-feasible with respect to l' or $rev(r)$ is not 1-feasible with respect to l''. Due to Lemma 1, in the first case, the cycle r can not be used by commodity l', and in the second case, the cycle $rev(r)$ can not be used by commodity l''.

Lemma 3. *For a cycle r that is not 2-feasible with respect to commodities l' and l'', cycle r can not be used by commodity l' and/or the cycle $rev(r)$ can not be used by commodity l''.*

Now consider a cycle r that is not 2-feasible for any pair (l', l''). Assume that the cycle is used by some commodity l'. Then, according to Lemma 3, commodity l'' can not use $rev(r)$. This is true for all l'', so no commodity may use $rev(r)$. This tells us that a cycle that is not 2-feasible for any pair (l', l'') can only be used in one direction.

Let us finally consider the case where there exists no cycle that is 2-feasible for any pair (l', l''). Then every cycle is used only in one direction, and a solution to P2 can not include only two commodities, but must include at least three.

A solution to P2 also needs to satisfy (2.0).

Definition 3. *An arc (i, j) is called eligible if $(i, j) \in (F_r \setminus A_{l'}) \cup (B_r \setminus A_{l''})$.*

Lemma 4. *A 2-feasible cycle is a solution of* P2 *if and only if it contains at least one eligible arc.*

Definition 4. *A cycle $C_r = F_r \cup B_r$ is called 1-valid if there exist two indices*

l' and l'' such that $B_r \subseteq A_{l'}$ and $F_r \subseteq A_{l''}$, while $B_r \not\subseteq A_{l''}$ and/or $F_r \not\subseteq A_{l'}$.

In [6] and [3] a 1-valid cycle is simply called valid, and the following is proved.

Theorem 1. *If there exists a 1-valid cycle, then there exists no compatible set of weights.*

From a practical point of view, in [6] and [5] we present computationally very efficient methods for finding 1-valid cycles. Especially we note that this can be done much quicker than trying to solve P1 as an LP-problem. Furthermore, as shown in [3] and [2], the information obtained from a 1-valid cycle can be used in a more effective way than the information from an infeasible LP-problem, when trying to adjust SP-graphs in order to make them obtainable.

We will also be interested in the case where there is no 1-valid cycle. From the definition we see that there exists no 1-valid cycle if and only if $B_r \subseteq A_{l''}$ and $F_r \subseteq A_{l'}$ for each cycle $C_r = F_r \cup B_r$ and for each l' and l'' such that $B_r \subseteq A_{l'}$ and $F_r \subseteq A_{l''}$. So if B_r is included in some $A_{l'}$ and F_r is included in some other $A_{l''}$, then B_r must be included in $A_{l''}$ and F_r must be included in $A_{l'}$. This means that $C_r \subseteq A_{l'}$ and $C_r \subseteq A_{l''}$, i.e. the cycle lies completely inside both SP-graphs. Such a cycle is called 1-*non-improving*.

Is it possible that $F_r = \emptyset$ for a cycle that is used by some commodity? No, because that would require $C_r = B_r \subseteq A_l$, but A_l contains no directed cycle.

Lemma 5. *If cycle r is used by any commodity, then $F_r \neq \emptyset$.*

In [6], we also show the following. A 1-valid cycle must contain at least three nodes and three arcs. If the SP-graphs $A_{l'}$ and $A_{l''}$ are trees, then any 2-feasible cycle is also 1-valid. If there does *not* exist a 1-valid cycle for SP-graphs that are trees, then there does not exist any 2-feasible cycle.

4. Analyzing the Cone

Let us now return to the cone Θ. Summing up all the constraints in set (2.1) yields

$$s_1(\theta) = \sum_{l=1}^{m} \sum_{(i,j)\in A} \theta_{ij}^l \leq 0,$$

while summing up all the constraints in set (2.3) yields

$$s_2(\theta) = \sum_{l=1}^{m} \sum_{(i,j)\in A\setminus A_l} \theta_{ij}^l \geq 0.$$

Now we find that $s_1(\theta) = v(\theta) + s_2(\theta)$, so the two inequalities $s_1(\theta) \leq 0$ and $s_2(\theta) \geq 0$ yield

$$v(\theta) = s_1(\theta) - s_2(\theta) \leq 0.$$

This means that no point in Θ yields a positive value of $v(\theta)$. Either (2.0) is satisfied, or $v(\theta) = 0$.

Furthermore, assuming that all constraints in sets (2.1) and (2.3) are satisfied with equality, we get $s_1(\theta) = 0$, and $s_2(\theta) = 0$, which together yields $v(\theta) = s_1(\theta) - s_2(\theta) = 0$.

If $v(\theta) = 0$, we must have $s_1(\theta) = s_2(\theta)$. We noted above that for any $\theta \in \Theta$, $s_1(\theta) \leq 0$ and $s_2(\theta) \geq 0$, so both these terms must be equal to zero i.e. $s_1(\theta) = 0$ and $s_2(\theta) = 0$. Furthermore, for any $\theta \in \Theta$, $s_1(\theta)$ is a sum of non-positive terms and $s_2(\theta)$ is a sum of non-negative terms, so each term must be equal to zero in order for the sums to be equal to zero. This means that all constraints of the sets (2.1) and (2.3) must be satisfied with equality. This proves the result below.

Theorem 2. *A point in* Θ *satisfies* (2.0) *if and only if it does not satisfy all constraints in sets* (2.1) *and* (2.3) *with equality.*

Now let Θ_0 be the points in Θ with $v(\theta) = 0$ (i.e. with $s_1(\theta) = 0$ and $s_2(\theta) = 0$). One point in Θ_0 clearly is the origin. Considering other points, we note that equality in constraints (2.1) simply means that the total flow in an arc obtained by summing up the changes of different commodities should cancel out. Equality in constraint set (2.3) means that no flow appears outside the corresponding SP-graph.

In a feasible solution to (2.1), (2.2) and (2.3) with at least one constraint of (2.1) and (2.3) satisfied with inequality, either the total flow in some arc should be negative (2.1) or there should be a positive flow of a commodity in an arc not belonging to the corresponding SP-graph (2.3). It is interesting to note that the more obvious criterion of decreasing the flow in an arc belonging to the SP-graph is replaced by increasing the flow in an arc outside the SP-graph.

5. Cycle Representation

It is well known that any (single commodity) circulating flow can be decomposed into flows on (simple) cycles, see for example [1]. Therefore the flow of each commodity can be decomposed into flow on a number of cycles. The cycles can be represented with the coefficients α the following way

$$\alpha_{ijr} = \begin{cases} 1 & \text{if arc } (i,j) \text{ is used forwards in cycle } r, \\ -1 & \text{if arc } (i,j) \text{ is used backwards in cycle } r, \\ 0 & \text{if arc } (i,j) \text{ is not used by cycle } r. \end{cases}$$

Let λ_r^l be the amount of commodity l sent in cycle r. Letting λ_r^l take negative values corresponds to using the cycle in the opposite direction, and since the cycle in the opposite direction also is present in the whole set of cycles, we can restrict λ_r^l to non-negative values (in order to avoid that each solution is represented in these two different ways). We now have

$$\theta_{ij}^l = \sum_r \alpha_{ijr} \lambda_r^l. \tag{2.5}$$

Since α represents a cycle, we have $\sum_j \alpha_{ijr} = \sum_j \alpha_{jir}$, $\forall i, r$. Therefore

$$\sum_{j:(i,j)\in A} \sum_r \alpha_{ijr}\lambda_r^l - \sum_{j:(j,i)\in A} \sum_r \alpha_{jir}\lambda_r^l = \sum_r \Big(\sum_{j:(i,j)\in A} \alpha_{ijr} - \sum_{j:(j,i)\in A} \alpha_{jir} \Big)\lambda_r^l = 0$$

so constraints (2.2) are automatically satisfied. What remains of P2 is the following

$$\text{(P3)} \quad \begin{aligned} &\sum_{l=1}^{m} \sum_{(i,j)\in A_l} \sum_r \alpha_{ijr}\lambda_r^l < 0, && (3.0)\\ &\sum_{l=1}^{m} \sum_r \alpha_{ijr}\lambda_r^l \le 0, \ \ \forall (i,j) \in A, && (3.1)\\ &\sum_r \alpha_{ijr}\lambda_r^l \ge 0, \ \ \forall (i,j) \in A \setminus A_l, \ \forall l, && (3.2)\\ &\lambda_r^l \ge 0, \ \ \forall r, l. && (3.3) \end{aligned}$$

For completeness sake, we note the following. For any feasible solution to P3, we can calculate θ by (2.5). Since (3.0) is satisfied, (2.0) will be satisfied, since (3.1) is satisfied, (2.1) will be satisfied, since (3.2) is satisfied, (2.3) will be satisfied, and as noted above, (2.2) will be satisfied. For any feasible solution to P2, constraints (2.2) implies, as noted above, that there exists some λ satisfying (2.5). Then (3.0) is satisfied, since (2.0) is satisfied, (3.1) is satisfied, since (2.1) is satisfied, and (3.2) is satisfied, since (2.3) is satisfied. Finally, any solution that would require a negative λ_r^l could instead use a positive $\lambda_{rev(r)}^l$, which takes care of (3.3). We thus have the following.

Lemma 6. P3 *has a feasible solution if and only if* P2 *has a feasible solution.*

We should however note that given θ, the λ-solution is not unique. For any solution to P3, one could add another solution with $\sum_r \alpha_{ijr}\lambda_r^l = 0$ for all (i, j). This corresponds to adding flow in cycles so that the sum over each arc is zero. This is not desirable, so we assume that an arc with $\theta_{ij}^l = 0$ is not contained in any cycle that is used by commodity l. This is called "the non-degeneracy assumption" and is described more formally in Lemma 7 below.

Let us introduce the notation F^l for all arcs in which the flow of commodity l is increased and B^l for all arcs in which the flow of commodity l is decreased. Due to constraints (2.3), we must have $B^l \subseteq A_l$.

The flow of commodity l is made up by flow sent in one or several cycles. Consider a cycle, r, with $C_r = F_r \cup B_r$, used in a solution (meaning $\lambda_r^l > 0$).

Lemma 7. *If the cycle $C_r = F_r \cup B_r$ is used by commodity l, we can assume that $F_r \subseteq F^l$ and $B_r \subseteq B^l$, i.e. $\theta_{ij}^l > 0 \, \forall (i,j) \in F_r$ and $\theta_{ij}^l < 0 \, \forall (i,j) \in B_r$.*

Proof. If, for example, there was some $(i, j) \in B_r$ for which $\theta_{ij}^l > 0$, there has to be another cycle, r_2, used by the same commodity where (i, j) is used forwards and $\lambda_{r_2}^l > \lambda_r^l$. Then it is not necessary to use cycle r. Instead we can send $\lambda^l_{r_2} - \lambda^l_r$ around cycle r_2, and λ^l_r around a new cycle made up by cycle r_2 without arc (i,j) plus cycle r without arc (i,j). Arc (i,j) is after this used in the designated direction, and the situation for all other arcs is not changed at all. A similar procedure can be used to eliminate cycles using arcs in F for which $\theta^l_{ij} < 0$, and any arc with $\theta^l_{ij} = 0$. ■

Lemma 7 deals with the cycle representation of a solution. It tells us that a certain θ-solution does not need to be represented by cycles containing arcs used in the "wrong" direction. Lemma 7 tells us how to change the representation by replacing offending cycles, i.e. verifies that the non-degeneracy assumption can be made. Therefore we can assume that each solution is represented only by arcs used in their designated direction (by each commodity and each cycle).

Let Λ be the polyhedron defined by (3.1), (3.2) and (3.3). Λ is a polyhedral cone, with the origin as its only extreme point. Letting

$$\bar{v}(\lambda) = \sum_{l=1}^{m} \sum_{(i,j)\in A_l} \sum_r \alpha_{ijr}\lambda^l_r,$$

$$\bar{s}_1(\lambda) = \sum_{l=1}^{m} \sum_{(i,j)\in A} \sum_r \alpha_{ijr}\lambda^l_r,$$

$$\bar{s}_2(\lambda) = \sum_{l=1}^{m} \sum_{(i,j)\in A\setminus A_l} \sum_r \alpha_{ijr}\lambda^l_r,$$

we have $\bar{s}_1(\lambda) \leq 0$, $\bar{s}_2(\lambda) \geq 0$ and $\bar{v}(\lambda) = \bar{s}_1(\lambda) - \bar{s}_2(\lambda)$, which yields $\bar{v}(\lambda) \leq 0$. So either P3 has a feasible solution, or

$$\sum_{l=1}^{m} \sum_{(i,j)\in A_l} \sum_r \alpha_{ijr}\lambda^l_r = 0 \text{ for all } \lambda \in \Lambda.$$

Thus we can draw conclusions that are similar to those for P2.

Lemma 8. *A point in* Λ *satisfies* (3.0) *if and only if it does not satisfy all constraints in sets* (3.1) *and* (3.2) *with equality.*

We can draw the conclusions, as for P2, that it is sufficient to search for a point in Λ with at least one of the constraints (3.1) and (3.2) satisfied with strict inequality. We can for example note that 1-valid cycles satisfy all of the constraints (3.1) with equality, but not all of (3.2).

6. Extreme Rays

Let us now consider the polyhedral structure of Θ, which is a convex, polyhedral cone, with the origin as the only extreme point and a finite set of extreme rays. We will study the structure of the extreme rays. In order to study Θ, we will use the strong relations to Λ, which also is a convex, polyhedral cone.

According to Lemma 6, P2 has a feasible solution if and only if P3 has a feasible solution. We also find the following.

Lemma 9. *An extreme ray in* Λ *corresponds to an extreme ray in* Θ*.*

Proof. Consider two feasible rays in Λ, $\lambda^{(1)}$ and $\lambda^{(2)}$, and form $\hat{\lambda} = \lambda^{(1)} + \lambda^{(2)}$. Now assume that $\hat{\lambda}$ is an extreme ray in Λ. Then we must have $\lambda^{(1)} = k\lambda^{(2)}$ for some positive scalar k, so $\hat{\lambda} = k\lambda^{(2)} + \lambda^{(2)} = q\lambda^{(2)}$, where $q = k + 1$.

Let ${\theta^l_{ij}}^{(1)} = \sum_r \alpha_{ijr} {\lambda^l_r}^{(1)}$, ${\theta^l_{ij}}^{(2)} = \sum_r \alpha_{ijr} {\lambda^l_r}^{(2)}$ and form $\hat{\theta} = \theta^{(1)} + \theta^{(2)}$. Then $\hat{\theta}^l_{ij} = \sum_r \alpha_{ijr} {\lambda^l_r}^{(1)} + \sum_r \alpha_{ijr} {\lambda^l_r}^{(2)} = \sum_r \alpha_{ijr} ({\lambda^l_r}^{(1)} + {\lambda^l_r}^{(2)}) = \sum_r \alpha_{ijr} \hat{\lambda}^l_r = \sum_r \alpha_{ijr} q {\lambda^l_r}^{(2)} = q {\theta^l_{ij}}^{(2)}$. Thus $\theta^{(1)} + \theta^{(2)} = q\theta^{(2)}$, i.e. $\theta^{(1)} = k\theta^{(2)}$, so $\hat{\theta}$ is an extreme ray of Θ. ∎

Concerning the non-degeneracy assumption, we find the following. Lemma 9 tells us that a certain extreme ray in Λ corresponds to an extreme ray in Θ. It does not matter if there are other rays (that are not extreme) in Λ that represent the same ray in Θ, since that ray is still extreme in Θ, and our main interest lies in Θ.

Can there be a solution to P2 using only one commodity? Due to constraints (2.1) no variable may be larger than zero. According to Lemma 5, each cycle must contain at least one arc where the flow is increased. This leaves only the origin. Therefore all solutions to P2 must contain flow of at least two commodities.

Can there be a solution to P2 using only one cycle? In view of Lemma 6, we consider P3. For each cycle there will be at least one constraint in set (3.1) that prohibits λ from being positive. We have thus shown the following.

Lemma 10. *A solution to* P2 *must include at least two commodities and two cycles.*

This must obviously also hold for extreme rays of Θ. A 1-valid cycle is a solution of P2 which includes precisely two commodities and two cycles (one being the exact reverse of the other one). We now prove the following.

Theorem 3. *A* 1*-valid cycle represents an extreme ray in* Λ*.*

Proof. Consider two feasible solutions, $\lambda^{(1)}$ and $\lambda^{(2)}$, of P3 and form $\hat{\lambda} = \lambda^{(1)} + \lambda^{(2)}$. Assume that $\hat{\lambda}$ constitutes a 1-valid cycle. This means that $\hat{\lambda}^{l'}_{r'} = 1$, $\hat{\lambda}^{l''}_{r''} = 1$ and $\hat{\lambda}^l_r = 0$ for all other r and l. Furthermore we have $\alpha_{ijr'} = -\alpha_{ijr''}$ for all (i, j). We must thus have $\lambda^{l'(1)}_{r'} + \lambda^{l'(2)}_{r'} = 1$, $\lambda^{l''(1)}_{r''} + \lambda^{l''(2)}_{r''} = 1$ and $\lambda^{l(1)}_r + \lambda^{l(2)}_r = 0$ for all $(r, l) \notin \{(r', l'), (r'', l'')\}$. Due to (3.3), the last condition implies that $\lambda^{l(1)}_r = 0$ and $\lambda^{l(2)}_r = 0$ for all $(r, l) \notin \{(r', l'), (r'', l'')\}$.

$\lambda^{(1)}$ is a feasible solution, so constraints (3.1) yield $\alpha_{ijr'}\lambda^{l'(1)}_{r'} + \alpha_{ijr''}\lambda^{l''(1)}_{r''} \leq 0 \ \forall (i, j) \in A$. Since $\alpha_{ijr'} = -\alpha_{ijr''}$ this yields $\alpha_{ijr'}(\lambda^{l'(1)}_{r'} - \lambda^{l''(1)}_{r''}) \leq 0 \ \forall (i, j) \in$

A. If $\alpha_{ijr'} = 1$ we get $\lambda_{r'}^{l'(1)} - \lambda_{r''}^{l''(1)} \leq 0$ and if $\alpha_{ijr'} = -1$ this yields $\lambda_{r'}^{l'(1)} - \lambda_{r''}^{l''(1)} \geq 0$. A 1-valid cycle may not consist of only forward or only backward arcs, but must contain at least one of each. Therefore we get $\lambda_{r'}^{l'(1)} - \lambda_{r''}^{l''(1)} = 0$, i.e. $\lambda_{r'}^{l'(1)} = \lambda_{r''}^{l''(1)}$. This is true also for $\lambda^{(2)}$, i.e. $\lambda_{r'}^{l'(2)} = \lambda_{r''}^{l''(2)}$.

So we have, for some positive t_1, $\lambda_r^{l\,(1)} = t_1$ for $(r,l) \in \{(r',l'),(r'',l'')\}$, and $\lambda_r^{l\,(1)} = 0$ for all $(r,l) \notin \{(r',l'),(r'',l'')\}$. Also, for some positive t_2, $\lambda_r^{l\,(2)} = t_2$ for $(r,l) \in \{(r',l'),(r'',l'')\}$, and $\lambda_r^{l\,(2)} = 0$ for all $(r,l) \notin \{(r',l'),(r'',l'')\}$. Then, simply letting $k = t_2/t_1 > 0$, we get $\lambda^{(1)} = k\lambda^{(2)}$. This proves that $\hat{\lambda}$ is an extreme ray. ■

Due to Lemma 9, the following holds.

Corollary 1. *A* 1*-valid cycle represents an extreme ray in* Θ*.*

Without the non-degeneracy assumption, there would be other representations of 1-valid cycles in Λ. However, since our main interest lies in Θ, all different rays in Λ representing the same ray in Θ are in our eyes equivalent. Furthermore, we will in practice search for solutions in Θ, not in Λ.

Now consider some arc $(i,j) \in F^{l'}$, i.e. where the flow of commodity l' is increased. Due to constraints (2.1), it is necessary that $(i,j) \in B^l$ for some $l \neq l'$ (i.e. that the flow of some other commodity is decreased). Therefore we must have $F^{l'} \subseteq \bigcup_{l \neq l'} B^l \subseteq \bigcup_{l \neq l'} A_l$. (This also follows from (2.4).) The same is true for F_r. Let us summarize.

Lemma 11. *For all* l'*,* $B^{l'} \subseteq A'_{l'}$ *and* $F^{l'} \subseteq \bigcup\limits_{l \neq l'} A_l$*.*

Lemma 12. *If* $\lambda_r^{l'} > 0$ *then for all* l'*,* $B_r \subseteq A_{l'}$ *and* $F_r \subseteq \bigcup\limits_{l \neq l'} A_l$*.*

We conclude that there is no need to consider cycles using arcs outside $\bigcup_l A_l$.

Corollary 2. *Only cycles within* $\bigcup_l A_l$ *will be used in a solution to* P2*.*

Let us now consider the special case when all cycles lie within $\bigcap_l A_l$. In this case constraints (3.2) disappear. For each cycle r, the cycle $rev(r)$ is also present, so removing (3.3) does not change the set of feasible solutions. Now only constraints (3.1) are needed to define Λ.

In this case there is no difference at all between different commodities. Therefore the partitioning into different commodities is uninteresting. Let $\gamma_{ij} = -\sum_{l=1}^{m} \sum_r \alpha_{ijr} \lambda_r^l$, i.e. γ_{ij} is the total flow on arc (i,j) (with reversed direction). Now (3.1) simply yields $\gamma_{ij} \geq 0$ for all (i,j), so each point in Λ is a non-negative circulating flow.

However, the graph, $\bigcap_l A_l$, contains no directed cycle (since no A_l contains a directed cycle). So there exists no nonzero circulating flow. The only feasible solution is $\gamma_{ij} = 0 \; \forall (i,j)$, so (3.0) can not be satisfied (following Lemma 8). P3 is thus infeasible, and so is P2, due to Lemma 6.

Theorem 4. *There is no solution to* P2 *using only cycles within* $\bigcap_l A_l$.

7. Using Only Two Commodities

We will now study solutions using only two commodities. Let us first describe the solutions with only two commodities that satisfy (2.1) and (2.3) with equality. (2.1) yields $\theta^{l'}_{ij} + \theta^{l''}_{ij} = 0$ for all $(i,j) \in A$, while (2.3) yields $\theta^{l'}_{ij} = 0$ for all $(i,j) \in A \setminus A_{l'}$ and $\theta^{l''}_{ij} = 0$ for all $(i,j) \in A \setminus A_{l''}$. Together they also yield $\theta^{l'}_{ij} = 0$ for all $(i,j) \in A \setminus A_{l''}$ and $\theta^{l''}_{ij} = 0$ for all $(i,j) \in A \setminus A_{l'}$. This means that

$$\Theta_0 = \{\theta : \theta^{l'}_{ij} = 0 \text{ and } \theta^{l''}_{ij} = 0 \\ \forall (i,j) \in A \setminus A_{l'} \cap A_{l''}, \theta^{l'}_{ij} + \theta^{l''}_{ij} = 0 \; \forall (i,j) \in A_{l'} \cap A_{l''}\}.$$

In words, the flow is unchanged outside the intersection of the two SP-graphs, and within the intersection of the SP-graphs the changes of commodity 1 is exactly eliminated by the changes of commodity 2. If the two SP-graphs have no cycle in common, the only solution satisfying this is the origin. However, if there is an intersection of the two SP-graphs that contains one or more cycles, there are non-zero solutions in Θ_0.

Our main interest, however, does not lie in Θ_0 but rather in $\Theta \setminus \Theta_0$. So let us consider such solutions with only two commodities. Our goal is to see if there are any other extreme rays than those represented by 1-valid cycles. For this purpose, we will assume that there are no 1-valid cycles, and under this assumption try to find a feasible solution to P2 using only two commodities.

Constraints (2.1) require that $F^{l'} \subseteq B^{l''}$ and $F^{l''} \subseteq B^{l'}$, while constraints (2.3) require that $B^{l'} \subseteq A_{l'}$ and $B^{l''} \subseteq A_{l''}$, so $F^{l'} \subseteq A_{l''}$ and $F^{l''} \subseteq A_{l'}$. According to Corollary 2 there may be non-zero flow only within $A_{l'} \cup A_{l''}$.

For a 2-feasible cycle, $C = F \cup B$, carrying flow of commodity 1, we have $F \subseteq F^{l'} \subseteq B^{l''} \subseteq A_{l''}$ and $B \subseteq F^{l''} \subseteq B^{l'} \subseteq A_{l'}$. If we assume that there exists no 1-valid cycle, then any such cycle must be 1-non-improving, i.e. $F \subseteq A_{l'}$ and $B \subseteq A_{l''}$. This means that $F \subseteq A_{l'} \cap A_{l''}$ and $B \subseteq A_{l'} \cap A_{l''}$.

A cycle that is not 2-feasible can, according to Lemma 3, not be used by both the two commodities. An arc in $B \setminus A_{l'}$ can not be used by commodity 1. Due to Lemma 7, it may not belong to $B^{l''}$, and due to constraints (2.1), it may not belong to $F^{l''}$. Therefore it is not used by commodity 2 either.

An arc in $F \setminus A_{l''}$ can not be used by commodity 2. Due to Lemma 7, it may not belong to $B^{l'}$, and due to constraints (2.1), it may not belong to $F^{l'}$. Therefore it is not used by commodity 1 either.

Since one of these cases must occur, we conclude that such a cycle will not be used by any of the two commodities.

Lemma 13. *A cycle that is not* 2*-feasible can not be used in a solution to* P2 *with only two commodities.*

This means that a solution to P2 can only contain 2-feasible cycles, i.e. cycles completely within $A_{l'} \cap A_{l''}$. We can now use Theorem 4 to prove the following.

Theorem 5. *There is no solution to* P2 *with only two commodities unless there exists a* 1*-valid cycle.*

We also draw the following conclusion.

Corollary 3. 1*-valid cycles represent all extreme rays of* P2 *using only two commodities.*

8. Using Several Commodities in One Cycle and Its Reverse

Let us now consider the case of using more than two commodities in the same cycle and its reverse. Relating to P3, we use cycle r' and cycle $rev(r')$. Due to the non-degeneracy assumption (Lemma 7) we can assume that $\lambda^l_{r'}$ and/or $\lambda^l_{rev(r')}$ is equal to zero. Therefore we let $\delta_l = \lambda^l_{r'} - \lambda^l_{rev(r')}$, which means that $\delta_l > 0$ if commodity l uses cycle r', and $\delta_l < 0$ if commodity l uses cycle $rev(r')$. By letting $\alpha_{ij} = \alpha_{ijr'}$ $(= -\alpha_{ijrev(r')})$, we get $\theta^l_{ij} = \alpha_{ij}\delta_l$.

Letting $F = \{(i,j) : \alpha_{ij} = 1\}$ and $B = \{(i,j) : \alpha_{ij} = -1\}$, δ_l denotes the flow of commodity l in the cycle $C = F \cup B$ (i.e. the flow is increased by δ_l in arcs in F and decreased by δ_l in arcs in B).

For this type of solutions, constraints (2.2) are clearly satisfied, since each commodity is changed in a cycle. Constraints (2.1) reduce to $\sum_l \delta_l \leq 0$ for all $(i,j) \in F$ and $\sum_l \delta_l \geq 0$ for all $(i,j) \in B$, i.e. $\sum_l \delta_l \leq 0$ if $F \neq \emptyset$ and $\sum_l \delta_l \geq 0$ if $B \neq \emptyset$. Constraints (2.3) become $\delta_l \geq 0$ if $F \setminus A_l \neq \emptyset, \forall l$, and $\delta_l \leq 0$ if $B \setminus A_l \neq \emptyset$ for all l. Furthermore, the left-hand-side of (2.0) becomes

$$v(\delta) = \sum_l \Big(\sum_{(i,j)\in F\cap A_l} \delta_l - \sum_{(i,j)\in B\cap A_l} \delta_l \Big) = \sum_l \delta_l (|F \cap A_l| - |B \cap A_l|).$$

Now let $L^+ = \{l : \delta_l > 0\}$ and $L^- = \{l : \delta_l < 0\}$. $F \setminus A_l \neq \emptyset$ for some $l \in L^-$ would yield a contradiction due to constraints (2.3) $(\delta_l \geq 0)$, so it is necessary that $F \subseteq A_l$ for all $l \in L^-$. Likewise it is necessary that $B \subseteq A_l$ for all $l \in L^+$. These conditions are also sufficient for satisfying constraints (2.3), so constraints (2.3) are completely replaced by $B \subseteq A_l$ for all $l \in L^+$ and $F \subseteq A_l$ for all $l \in L^-$.

Lemma 14. $L^+ \neq \emptyset$ *and* $L^- \neq \emptyset$ *in a solution to* P2.

Proof. Assume that $L^+ = \emptyset$. This yields $\delta_l \leq 0$ for all l. Furthermore, $L^- \neq \emptyset$ (otherwise there is no change at all). Then $F \subseteq A_l$ for all $l \in L^-$. Now Theorem 4 tells us that C must contain at least one arc outside $\bigcap_l A_l$, i.e. $B \setminus A_l \neq \emptyset$ for some l. Clearly then $B \neq \emptyset$, which due to constraints (2.1) yields $\sum_l \delta_l \geq 0$.

This is a contradiction, since it only allows $\delta_l = 0$ for all l, which does not satisfy (2.0).

Assuming that $L^- = \emptyset$ yields a similar contradiction. ■

Using Lemma 5 on some $l \in L^+$ proves that $F \neq \emptyset$, and using it on some $l \in L^-$ proves that $B \neq \emptyset$, which means that constraints (2.1) simply becomes $\sum_l \delta_l = 0$.

P2 thus reduces to the following:

$$\text{(P4)} \qquad \begin{aligned} \sum_{l=1}^{m} \hat{r}_l \delta_l &< 0, && (4.0)\\ \sum_{l} \delta_l &= 0, && (4.1)\\ B \subseteq A_l,&\ \ \forall l \in L^+, && (4.2)\\ F \subseteq A_l,&\ \ \forall l \in L^-, && (4.3) \end{aligned}$$

where $\hat{r}_l = |F \cap A_l| - |B \cap A_l|$. The derivation above tells us the following.

Lemma 15. *Any feasible solution to* P4 *is a feasible solution to* P2.

Therefore, if there exists a feasible solution to P4, there exists no compatible set of weights, see [4] for further details. We may also note that (4.2) and (4.3) are now structural constraints on the cycle used, and do not include the actual values of δ (except the signs).

Now consider, for given L^+ and L^-, the specific solutions

$$\bar{\delta}_l = \frac{1}{|L^+|},\ \forall l \in L^+ \text{ and } \bar{\delta}_l = -\frac{1}{|L^-|},\ \forall l \in L^-.$$

Each possible choice of L^+ and L^- is thus represented by the fixed solution $\bar{\delta}$. We now define a certain kind of cycle.

Definition 5. *A cycle $C = F \cup B$ is called <u>2-valid</u> if there exist two sets $L^+ \neq \emptyset$ and $L^- \neq \emptyset$ such that $F \subseteq A_l$ for all $l \in L^-$, $B \subseteq A_l$ for all $l \in L^+$, and $F \not\subseteq A_l$ for some $l \in L^+$ and/or $B \not\subseteq A_l$ for some $l \in L^-$.*

Lemma 16. $\bar{\delta}$ *is a feasible solution of* P4 *if the cycle is* 2*-valid.*

Proof. Constraints (4.1) are satisfied, since $\displaystyle\sum_{l \in (L^+ \cup L^-)} \delta_l = \sum_{l \in L^+} \frac{1}{|L^+|} - \sum_{l \in L^-} \frac{1}{|L^-|}$ $= 1 - 1 = 0$. Constraints (4.2) and (4.3) are obviously satisfied. Concerning constraints (4.0), we get

$$\begin{aligned} v(\bar{\delta}) &= \sum_{l\in L^+}(|F\cap A_l| - |B\cap A_l|)\frac{1}{|L^+|} - \sum_{l\in L^-}(|F\cap A_l| - |B\cap A_l|)\frac{1}{|L^-|} \\ &= \frac{1}{|L^+|}\sum_{l\in L^+}(|F\cap A_l| - |B\cap A_l|) - \frac{1}{|L^-|}\sum_{l\in L^-}(|F\cap A_l| - |B\cap A_l|) \\ &= \frac{1}{|L^+|}\sum_{l\in L^+}(|F\cap A_l| - |B|) - \frac{1}{|L^-|}\sum_{l\in L^-}(|F| - |B\cap A_l|) \\ &= \frac{1}{|L^+|}\sum_{l\in L^+}|F\cap A_l| - |B| - |F| + \frac{1}{|L^-|}\sum_{l\in L^-}|B\cap A_l| \\ &= \frac{1}{|L^+|}\sum_{l\in L^+}|F\cap A_l| - |F| + \frac{1}{|L^-|}\sum_{l\in L^-}|B\cap A_l| - |B| \\ &= \frac{1}{|L^+|}\sum_{l\in L^+}(|F\cap A_l| - |F|) + \frac{1}{|L^-|}\sum_{l\in L^-}(|B\cap A_l| - |B|) \end{aligned}$$

if the cycle is 2-valid. Obviously $|F\cap A_l| \le |F|$ and $|B\cap A_l| \le |B|$, so this is a sum of non-positive values and therefore less or equal to zero. In order to satisfy (4.0), it is sufficient to make at least one element negative, and this is ensured by the property of 2-valid cycles that $F \not\subseteq A_l$ for some $l \in L^+$ and/or $B \not\subseteq A_l$ for some $l \in L^-$. ∎

If there exists a 2-valid cycle, we thus have a feasible solution to P4, namely $\bar{\delta}$, so we can draw the following conclusion.

Theorem 6. *If there exists a* 2*-valid cycle, then there exists no compatible set of weights.*

Now we consider other solutions than $\bar{\delta}$ to P2.

Lemma 17. *For any given* L^+ *and* L^-*, if* $\bar{\delta}$ *is not feasible in* P4*, there is no feasible solution to* P4.

Proof. First we specify under which conditions $\bar{\delta}$ is not feasible in P4. Constraints (4.1) are clearly always satisfied by $\bar{\delta}$ (regardless of which cycle is chosen). Constraints (4.2) and (4.3) are satisfied by each cycle with $F \subseteq A_l$ for all $l \in L^-$ and $B \subseteq A_l$ for all $l \in L^+$. If there is no such cycle, we are done, so let us assume that there is such a cycle.

Now only constraints (4.0) remain, so in order for $\bar{\delta}$ not to be feasible, it is necessary that $v(\bar{\delta}) = 0$ (since it cannot be positive). From the previous proof, we find that this is possible only if $|F\cap A_l| = |F|$ for all $l \in L^+$ and $|B\cap A_l| = |B|$ for all $l \in L^-$, i.e. $F \subseteq A_l$ for all $l \in L^+$ and $B \subseteq A_l$ for all $l \in L^-$. This together with constraints (4.2) and (4.3) yields $F \subseteq A_l$ for all $l \in (L^+ \cup L^-)$ and $B \subseteq A_l$ for all $l \in (L^+ \cup L^-)$. Thus there can only be cycles that either does not satisfy (4.2) and (4.3) or satisfies $F \subseteq A_l$ for all $l \in (L^+ \cup L^-)$ and $B \subseteq A_l$ for all

$l \in (L^+ \cup L^-)$, i.e. $|F \cap A_l| = |F|$ for all $l \in (L^+ \cup L^-)$ and $|B \cap A_l| = |B|$ for all $l \in (L^+ \cup L^-)$.

Now we consider solutions other than $\bar{\delta}$, taking the above conclusion into account.

$$\begin{aligned} v(\delta) &= \sum_{l \in (L^+ \cup L^-)} (|F \cap A_l| - |B \cap A_l|)\delta_l \\ &= \sum_{l \in (L^+ \cup L^-)} (|F| - |B|)\delta_l \\ &= (|F| - |B|) \sum_{l \in (L^+ \cup L^-)} \delta_l \\ &= 0, \end{aligned}$$

due to constraint (4.1). Thus $v(\delta) = 0$ for all δ-solutions satisfying (4.1), (4.2) and (4.3), and we conclude that there is no feasible solution to P4 for these L^+ and L^-. ■

We conclude that there are only two possibilities for each choice of L^+ and L^-. Either there exists a 2-valid cycle, in which case $\bar{\delta}$ is a feasible solution to P4, and consequently to P2, or there does not exist a 2-valid cycle, in which case $\bar{\delta}$ is not a feasible solution to P4, but, more importantly, there is no feasible solution to P4 for these L^+ and L^-.

Searching over all L^+ and L^-, either there exists a 2-valid cycle, and hence a feasible solution to P4 and P2, or there exists no 2-valid cycle, in which case there exists *no* feasible solution to P4 at all.

Let us now relate 2-valid cycles to 1-valid cycles.

Lemma 18. *There exists a* 2*-valid cycle if and only if there exists a* 1*-valid cycle.*

Proof. Clearly a 1-valid cycle is a special case of a 2-valid cycle, obtained by setting $L^+ = \{l'\}$ and $L^- = \{l''\}$, so there exists a 2-valid cycle if there exists a 1-valid cycle.

Now assume that there exists a 2-valid cycle, and that there exists a $l' \in L^+$ such that $F \not\subseteq A_{l'}$. Then choosing any $l'' \in L^-$ yields a 1-valid cycle, since $B \subseteq A_{l'}$, $F \subseteq A_{l''}$ and $F \not\subseteq A_{l'}$. If there does not exist a $l' \in L^+$ such that $F \not\subseteq A_{l'}$, there must exist a $l'' \in L^-$ such that $B \not\subseteq A_{l''}$, and a 1-valid cycle is obtained by choosing any $l' \in L^+$. Thus there exists a 1-valid cycle if there exists a 2-valid cycle. ■

It follows that any unbounded solution to P2 using only one cycle and its reverse can be detected by comparing two commodities. It is not necessary to include three or more commodities at the same time.

This also means that 2-valid cycles do not represent extreme rays of Θ.

Theorem 7. *All extreme rays of* Θ *that use only one cycle and its reverse use* *only two commodities, and are represented by* 1*-valid cycles.*

9. Conclusion

In this paper we study the dual of a problem for finding compatible weights for OSPF networks, and interpret it as a special multicommodity flow problem. The feasible set is a polyhedral cone, and we study the extremal structure of it.

We find that there is no solution using only one commodity or one cycle. 1-valid cycles use two commodities and two cycles, one being the reverse of the other, and represent extreme rays of the cone. If there is no 1-valid cycle, we find that no solution with only two commodities exists. This means that 1-valid cycles represent all extreme rays involving only two commodities.

Furthermore, we find that if there is no 1-valid cycle, there is no solution, with any number of commodities, using only one cycle and its reverse. The only extreme rays using only one cycle and its reverse are thus represented by 1-valid cycles.

From a practical point of view, computational tests in [6] reveal that of 1423 instances tested, 276 had compatible weights, 1137 had 1-valid cycles and only 10 (0.7%) had neither. We conclude that, for these test problems, the extreme rays corresponding to 1-valid cycles are quite important in practice.

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