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Admissible Transformations and Assignment Problems

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Abstract. We introduce the notion of admissible transformations which is related to the Hungarian method for solving assignment problems. Admissible transformations are stated for linear, quadratic and multi-index assignment problems. Their application to find good lower bounds and/or to solve the problem, respectively, is outlined. Finally it is shown that admissible transformations can also be applied to so-called algebraic objective functions whose cost elements are drawn from a totally ordered semigroup.

Keywords: Combinatorial optimization, assignment problems, quadratic assignment problem, multi-index assignment problem, admissible transformation, algebraic optimization.

1. Assignment Problems

Linear assignment problems count to the classical problems of linear and combinatorial programming. They deal with the question how to assign n jobs to n machines such as to minimize the sum of all processing times. An *assignment* can be described by either a permutation ϕ of the underlying index set $\{1, 2, ..., n\}$ or using linear constraints of the form

$$\sum_{j=1}^{n} x_{ij} = 1 \quad (i = 1, 2, \ldots, n),$$

$$\sum_{i=1}^{n} x_{ij} = 1 \quad (j = 1, 2, \ldots, n),$$

$$x_{ij} \in \{0,1\} \ \ (i,j=1,2,\ldots,n).$$

Here, job i is assigned to machine j if and only if $x_{ij} = 1$. Thus, a linear assignment problem can be stated in the following way. Let c_{ij} be the processing time for job i on machine j. Given the $(n \times n)$ matrix $C = (c_{ij})$ of processing times, find an assignment ϕ of the jobs to the machines such that

$$\sum_{i=1}^{n} c_{i\phi(i)} \tag{1}$$

is minimized. Linear assignment problems can be solved in polynomial time - there are various possibilities for their solution. We shall recall here, in Sec. 3, a dual method based on admissible transformations which was introduced by Burkard, Hahn and Zimmermann [2] in 1977. A transformation T of the cost matrix C to the new cost matrix $\bar{C}$ is called admissible if for all feasible solutions ϕ there is a constant $z(T)$ such that the equation

$$\sum_{i=1}^{n} c_{i\phi(i)} = z(T) + \sum_{i=1}^{n} \bar{c}_{i\phi(i)} \tag{2}$$

holds. Admissible transformations will be discussed in Sec. 2. The solution method using admissible transformations has the advantage that it can be generalized to solve assignment problems with bottleneck and other non-standard objectives. We shall discuss this issue in Sec. 6.

In connection with location problems, *quadratic assignment problems* play an important role. Let us consider the following model: a set of n facilities has to be allocated to a set of n possible locations. We are given two $n \times n$ input matrices: $A = (a_{ik})$ and $B = (b_{jl})$, where a_{ik} is the flow between facility i and facility k and b_{jl} is the distance between location j and location l. We assume that the total cost depend on the flow between facilities multiplied by their distance. Each product $a_{ik}b_{\phi(i)\phi(k)}$ represents the flow between facilities i and k multiplied by their distance when facility i is assigned to location $\phi(i)$ and facility k is assigned to location $\phi(k)$. The objective is to assign each facility to a location such that the total cost is minimized. This model leads to a quadratic assignment problem as considered by Koopmans and Beckmann [6]

$$\min_{\phi} \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} b_{\phi(i)\phi(k)}. \tag{3}$$

Unlike linear assignment problems, quadratic assignment problems are $\mathcal{NP}$-hard and difficult to solve. In Sec. 4 we shall describe how to determine strong lower bounds by means of admissible transformations.

In a time tabling problem the assignment of n courses to n time slots and to n rooms is required. Let c_{ijk} denote the cost for assigning course i to time slot j in room k. We want to find an assignment ϕ of the courses to time slots and an assignment ψ of the courses to the rooms such that the total cost is minimum. This leads to the so-called *(axial) three-index assignment problem*

$$\min_{\varphi,\psi} \sum_{i=1}^{n} c_{i\,\varphi(i)\,\psi(i)}. \tag{4}$$

A similar problem arises if we are looking for *Latin squares*, i.e., square arrays of size n where every position is filled by one of the numbers $1, 2, \ldots, n$ such that every row and column of the square contains all numbers. For example, a Latin square of size 3 may have the form

2	1	3
1	3	2
3	2	1

Latin squares are feasible solutions of so-called *planar 3-index assignment problems*, which can be formulated in the following way: given an $(n \times n \times n)$ array $C = (c_{ijk})$, find n permutations $\phi_1, \phi_2, ..., \phi_n$ which obey for $i \neq j$

$$\phi_i(k) \neq \phi j(k) \quad \text{for all } k = 1, 2, ..., n$$

such that

$$\sum_{i=1}^{n} \sum_{k=1}^{n} c_{ik\phi_i(k)} \tag{5}$$

is minimum. Both the axial and planar 3-index assignment problems are $\mathcal{NP}$-hard. In Sec. 5 we shall describe how to use admissible transformations for finding lower bounds for these three-index assignment problems.

2. Admissible Transformations

In 1971, Khoan Vo-Khac [9] introduced the notion of admissible tranformations in connection with vehicle routing problems. He called a transformation of the costs admissible, if it leaves unchanged the relative order of objective function values of all feasible solution. Motivated by this idea, Burkard, Hahn and Zimmermann [2] introduced (special) admissible transformations for algebraic linear assignment problems. More generally, we can proceed as follows.

Let $E := \{e_1, e_2, ..., e_n\}$ be the ground set of a combinatiorial optimization problem whose feasible solutions $F \subseteq E$ are collected in the class $\mathcal{F}$. The costs of the ground elements $e_1, ..., e_n$ are denoted by $c(e_1), c(e_2), ..., c(e_n) \in \mathbb{R}$. The cost $c(F)$ of a feasible solution F is defined by

$$c(F) := \sum_{e \in F} c(e). \tag{6}$$

A transformation T of the costs $c(e)$, $e \in E$, to new costs $\bar{c}(e)$, $e \in E$, is called *admissible* with *index* $z(T)$, if

$$c(F) = z(T) + \bar{c}(F) \text{ for all } F \in \mathcal{F}. \tag{7}$$

When we perform an admissible transformations T after an admissible transformation S, we get again an admissible transformation. If S and T have the indices $z(S)$ and $z(T)$, respectively, their composition has index $z(S) + z(T)$.

Now let us consider a combinatorial minimization problem of the form

$$\min_{F \in \mathcal{F}} c(F). \tag{8}$$

For this problem we immediately get the following optimality criterion.

Lemma 1. *Let T be an admissible transformation such that there exists a feasible solution F^* with the following properties:*
1. $\bar{c}(e) \geq 0$ *for all* $e \in E$;
2. $\bar{c}(F^*) = 0$.

Then F^* is an optimal solution of (8) with value $z(T)$.

Proof. Let F be an arbitrary feasible solution. According to the properties of admissible transformations we get:

$$c(F) = z(T) + \bar{c}(F) \geq z(T) = z(T) + \bar{c}(F^*) = c(F^*).$$

Therefore F^* is optimal. ∎

This lemma leads to the following *feasibility problem.* Let $E_0 := \{e \mid \bar{c}(e) = 0\}$. The crucial question is to decide whether there exists an $F^* \in \mathcal{F}$ with $F^* \subseteq E_0$ or not. In the first case, an optimal solution is found. In the second case $z(T)$ is at least a lower bound for the optimal objective function value. We shall see in the next section that in the case of linear assignment problems, the feasibility problem can be solved in polynomial time. If there is no feasible solution contained in E_0, a new admissible transformation can be derived such that after at most $O(n^2)$ admissible transformations an optimal solution is obtained.

3. Admissible Transformations and Linear Assignment Problems

We consider a linear assignment problem (1) with an $(n \times n)$ cost matrix $C = (c_{ij})$. The objective function value of a permutation ϕ is denoted by $C(\phi)$. Basic is the following proposition, cf. Burkard, Hahn and Zimmermann [2].

Proposition 1. (Admissible transformations for linear assignment problems) *Let $I, J \subseteq \{1, 2, \ldots, n\}$, $m := |I| + |J| - n \geq 0$, and let c be an arbitrary real. Then the transformation $T = T(I, J; c)$ of the cost coefficients c_{ij} to new cost coefficients $\bar{c}_{ij}$ defined by*

$$\bar{c}_{ij} = \begin{cases} c_{ij} - c\,, & \text{for } i \in I,\ j \in J, \\ c_{ij} + c\,, & \text{for } i \notin I,\ j \notin J, \\ c_{ij}\,, & \text{otherwise} \end{cases} \tag{9}$$

is admissible with $z(T) = mc$.

Proof. Let ϕ be an arbitrary permutation of $\{1, 2, \ldots, n\}$ and let n_0 be the number of pairs $(i, \varphi(i))$ with $i \in I$, $\phi(i) \in J$. Similarly, let n_1 be the number of pairs $(i, \phi(i))$ with $i \in I$, $\phi(i) \notin J$ or $i \notin I$, $\phi(i) \in J$ and let n_2 be the number of pairs $(i, \phi(i))$ with $i \notin I$, $\phi(i) \notin J$. Obviously, $n_0 + n_1 + n_2 = n$ and $2n_0 + n_1 = |I| + |J|$. This implies

$$n_0 - n_2 = |I| + |J| - n = m. \tag{10}$$

As the right hand side in (10) is independent on the particular permutation ϕ, (10) holds for all permutations on $\{1, 2, \ldots, n\}$.

Let $C[i \in I]$ denote the sum of all cost coefficients $c_{i\phi(i)}$ with $i \in I$. Using this notation, the first line in (9) yields for any permutation ϕ

$$C(\phi) = C[i \in I] + C[i \notin I] = n_0 c + \bar{C}[i \in I] + C[i \notin I].$$

Now, the second and third lines of (9) yield

$$n_2 c + C[i \notin I] = \bar{C}[i \notin I].$$

Thus we get for all permutations ϕ

$$C(\phi) = mc + \bar{C}(\phi)$$

which shows that the transformation in Proposition 1 is feasible with index $z(T) = mc$. ∎

If we choose $I := \{k\}$ and $J := \{1, 2, ..., n\}$ we get a *reduction of row* k. In particular, the choice $c := \min\{c_{ij} : i \in I, j \in J\}$ yields that all reduced elements in row k become nonnegative and the row contains at least one reduced element $\bar{c}_{kj} = 0$.

For solving a linear asignment problem, one can proceed as follows. By reducing all rows $i = 1, 2, ..., n$ a transformed cost matrix with nonnegative entries is obtained. Every row contains at least one 0-element. Thus we start now with the feasibility check: to this extent we define a bipartite graph whose vertices correspond to the rows and columns of the given cost matrix. The graph has an edge (i, j), iff $\bar{c}_{ij} = 0$. In this graph we determine a maximum bipartite matching. If this maximum matching has cadinality n, i.e., it is perfect, this matching defines a feasible solution for the assignment problem which is optimal according to Lemma 1. Otherwise, the solution of the matching problem allows us to find a new admissible transformation. According to a famous theorem of König, the cardinality of a maximum matching equals to the cardinality of a minimum vertex cover of the bipartite graph. This means, we get a minimum cover of all 0-elements in the transformed matrix $\bar{C}$ by rows and columns, where the total number of covering rows and columns is less than n (since the matching was not perfect). Let I denote the index set of uncovered rows and let J denote the index set of uncovered columns. Then $|I| + |J| > n$. We perform an admissible transformation with I, J and

$$c := \min\{c_{ij} : i \in I, j \in J\}.$$

This transformation generates at least one new 0-element in the uncovered part of the matrix. We add the corresponding edges to the bipartite graph and determine a new maximum matching. (This can be done by growing alternating trees starting from the previous matching). As every admissible transformation generates a new 0-element, at most $O(n^2)$ admissible transformations are necessary. This leads to a total worst case complexity of $O(n^4)$ for finding an optimal solution. This is not the best complexity known for linear assignment problems, but this solution strategy has its own advantages: it is very simple and it can be used for quite general objective functions (see Sec. 6).

4. Admissible Transformations and Quadratic Assignment Problems

Given a Koopmans–Beckmann problem, we may define

$$d_{ijkl} := a_{ik} b_{jl}$$

and get in this way a *general quadratic assignment problem* (QAP) as introduced by Lawler [7]:

$$\begin{aligned} \min & \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} d_{ijkl} x_{ij} x_{kl} \\ \text{s.t.} \quad & \sum_{j=1}^{n} x_{ij} = 1 \quad (i = 1, 2, \ldots, n), \\ & \sum_{i=1}^{n} x_{ij} = 1 \quad (j = 1, 2, \ldots, n), \\ & x_{ij} \in \{0, 1\} \quad (i, j = 1, 2, \ldots, n). \end{aligned} \tag{11}$$

Let $Y := X \otimes X$ denote the *Kronecker product* of the permutation matrix X which is an $(n^2 \times n^2)$ matrix with the blocks $x_{ij}X$. Thus, the entry y_{ijkl} lies in row $(i-1)n + k$ and column $(j-1) + l$. According to Lawler, the general quadratic assignment problem can be written as

$$\begin{aligned} \min & \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} d_{ijkl} y_{ijkl} \\ \text{s.t.} \quad & Y := X \otimes X, \\ & \sum_{j=1}^{n} x_{ij} = 1 \quad (i = 1, 2, \ldots, n), \\ & \sum_{i=1}^{n} x_{ij} = 1 \quad (j = 1, 2, \ldots, n), \\ & x_{ij} \in \{0, 1\} \quad (i, j = 1, 2, \ldots, n). \end{aligned} \tag{12}$$

We represent the cost coefficients d_{ijkl} as entries of an $(n^2 \times n^2)$ matrix D which is composed from $(n \times n)$ blocks (D^{ij}) where every D^{ij} is the matrix (d_{ijkl}) with fixed indices i and j. For $n = 3$, for example, the cost matrix has the form

$$D = \begin{pmatrix} D^{11} & D^{12} & D^{13} \\ D^{21} & D^{22} & D^{23} \\ D^{31} & D^{32} & D^{33} \end{pmatrix}.$$

Now we get

Proposition 2. (Admissible transformations for the QAP) *Let* $D = (D^{ij})$ *be the cost matrix of a general QAP.*

1. *One can set*

$$d_{ijkl} := \infty \ \text{ for } i = k,\ j \neq l \ \text{ or } i \neq k,\ j = l. \tag{13}$$

   This is an admissible transformation with index 0*.*

2. (Symmetrization) *Set*

$$d_{ijkl} := \begin{cases} d_{ijkl} + d_{klij} & \text{for } k > i, \\ 0 & \text{for } k < i. \end{cases} \tag{14}$$

   This is an admissible transformation with index 0*.*

3. (Type (I)-transformation) *Consider a fixed block* D^{ij} *of* D*. Applying an admissible transformation* $T(I, J; c)$ *with index 0 to matrix* D^{ij} *yields an admissible transformation with index* 0 *for the quadratic assignment problem.*
4. (Type (II)-transformation) *Applying an admissible transformation* $T(I, J; c)$ *with index* $z(T)$ *to the* $(n^2 \times n^2)$ *matrix* D *yields an admissible transformation with index* $z(T)$ *for the quadratic assignment problem.*

Before proving this proposition, let us introduce the notation

$$\langle C, X\rangle := \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij}.$$

Thus we have

$$\langle D^{ij}, X\rangle := \sum_{k=1}^{n}\sum_{l=1}^{n} d_{ijkl}x_{kl}$$

which allows us to rewrite the objective function of a general QAP as

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} d_{ijkl}x_{ij}x_{kl} = \sum_{i=1}^{n}\sum_{j=1}^{n}\langle D^{ij}, X\rangle x_{ij}. \tag{15}$$

Proof. The first transformation follows from the fact that the coefficients d_{ijil} with $j \neq l$ and d_{ijkj} with $i \neq k$ can never occur in the objective function, since ϕ is a one-to-one mapping.
The symmetrization follows from

$$y_{ijkl} = x_{ij}x_{kl} = x_{kl}x_{ij} = y_{klij}.$$

The transformation D^{ij} to $\overline{D}^{ij}$ of Type (I) is admissible, since for every permutation matrix X we get

$$\langle D^{ij}, X\rangle = \langle \overline{D}^{ij}, X\rangle.$$

The result follows now immediately from (15).
Finally, note that the Kronecker product matrix $Y = X \otimes X$ is again a permutation matrix. But a Type (II)-transformation of D yields $\langle D, Y\rangle = z(T) + \langle \bar{D}, Y\rangle$ for all permutation matrices Y. ■

For computing strong lower bounds for the QAP, we shall apply special transformations of Type (I) and Type (II). Note that after applying the first transformation of the above proposition all matrices D^{ij} have a special form: in their i-th row and j-th column there is only one finite element d_{ijij} which we call *the leader* of the matrix D^{ij}. Let $\hat{D}^{ij}$ be the matrix obtained from D^{ij} by deleting the i-th row and j-th column in D^{ij}.

In the bounding procedure we use two different kinds of Type (I)-transformations of a block D^{ij}. In both transformations $T(I,J;c)$ we have $i \notin I,\ j \notin J$ and $|I|+|J|=n$. In the *reduction case* we define c to be the smallest uncovered element, i.e.,

$$c := \min\{d_{ijkl} \mid k \in I, l \in J\}.$$

By these reductions we can achieve that every row and column of $\hat{D}^{ij}$ has at least one 0-element. In the *redistribution case* we set $d_{ijij} := 0$ and add the amount of $\frac{1}{(n-1)}d_{ijij}$ to every element of $\hat{D}^{ij}$. This is the result of $n-1$ admissible transformations $T(\{k\}, J; -d_{ijij}/(n-1))$ for $k = 1, ..., n; k \neq i$ and $J := \{1, 2, ..., j-1, j+1, ..., n\}$.

Concerning Type (II)-transformations we have the following corollary.

Corollary 1. *Let $L := (d_{ijij})$ be the $(n \times n)$ matrix whose entries are the leaders of D. Any admissible transformation of L corresponds to an admissible Type (II)-transformation of D with the same index.*

Proof. Consider an admissible transformation $T(I,J;c)$ of L. The index sets I and J denote the uncovered rows and columns of L, respectively. We construct an admissible Type (II) - transformation $T(\bar{I}, \bar{J}; c)$ of D as follows. Recall that D has rows ik and columns jl, $i, j, k, l = 1, ..., n$.

- $\bar{I} := \{ii \mid i \in I\}$, that is, only rows ii, $i \in I$, are uncovered.
- $\bar{J}$ contains all columns of D except the columns jj with $j \notin J$.

Thus the uncovered rows and columns of D contain only the leaders or ∞-entries. All entries of any $\hat{D}^{ij}$ are covered by a row, but not by a column. Moreover, $|\bar{I}| = |I|$ and $|\bar{J}| = n(n-1) + |J|$. Therefore

$$|\bar{I}| + |\bar{J}| - n^2 = |I| + |J| - n = m.$$

■

For finding a good lower bound of the quadratic assignment problem, one can now proceed as follows. In a prepocessing step, the first transformation of Proposition 2 is applied. Then the following steps are performed:

1. Apply the symmetrization.
2. Solve for all $i, j := 1, 2, ..., n$ the linear assignment problem with cost matrix $\hat{D}^{ij}$ (by admissible transformations) and add the objective function value to the leader d_{ijij}.
3. Collect the leaders in an $(n \times n)$-matrix L. Solve the linear assignment problem with cost matrix L. Let ϕ^* be an optimal solution (the corresponding

permutation matrix is denoted by X_{ϕ^*}) and let z^* be the corresponding optimal objective function value. $\bar{D}$ is the transformed matrix obtained after applying the foregoing admissible transformations to D.

4. (*Feasibility check*). If

$$\sum_{i=1}^{n} \langle \bar{D}^{i\phi^*(i)}, X_{\phi^*} \rangle = 0, \tag{16}$$

then ϕ^* is an optimal solution of the quadratic assignment problem with value z^*. Otherwise, z^* is a lower bound for the optimum value of the quadratic assignment problem.

Remarks.

1. The linear assignment problems in Step 2 can be solved by any algorithm which yields the dual variables $u_k,\ k \neq i$, and $v_l,\ l \neq j$. By setting

$$\overline{d}_{ijij} := d_{ijij} + \sum_{k=1}^{n} u_k + \sum_{l=1}^{n} v_l,$$
$$\overline{d}_{ijkl} := d_{ijkl} - u_k - v_l \text{ for } k \neq i \text{ and } l \neq j$$

we obtain the transformed cost coefficients.

2. The problem to check whether there exists an optimal solution ϕ^* which fulfills (16) is an $\mathcal{NP}$-hard problem.

3. In the case that ϕ^* is not an optimal solution of the quadratic assignment problem, Hahn and Grant [5] suggest to redistribute the transformed leaders in the matrix $\bar{D}$ and reapply the Steps 1–4 above beginning with the symmetrization. They report very good results with such a procedure.

5. Admissible Transformations and Multi-index Assignment Problems

Both the axial three-index assignment problem as well as the planar three-index assignment problem can be described as an intersection problem for partition matroids. This offers a possibility to consider both problems from a unique point of view which easily allows to generalize the ideas for multi-index assignment problems. This section is partially based on the master thesis of Fröhlich [4], see also the extended abstract by Burkard and Fröhlich [1].

We start with the ground set $E = \{(i,j,k) \mid i,j,k = 1,2,...,n\}$. In the case of an axial three-index assignment problem we consider the partitions $\mathcal{P}^i := \{P^i \mid i = 1,2,...,n\}$, $\mathcal{P}^j := \{P^j \mid j = 1,2,...,n\}$ and $\mathcal{P}^k := \{P^k \mid k = 1,2,...,n\}$, where for fixed index i the set P^i is defined by $P^i := \{(i,j,k) \mid j,k = 1,2,...,n\}$. The sets P^j and P^k are defined in an analogue way. A subset $F \subseteq E$ is defined to be a basis of $(E, \mathcal{P}^i)$, if for all $i = 1,2,...,n$

$$|F \cap P^i| = 1.$$

In particular, $|F| = n$. It is well-known that the bases with respect to a partition of E lead to a matroid. Thus, we get three partition matroids on the ground set

E, namely $(E, \mathcal{P}^i)$, $(E, \mathcal{P}^j)$ and $(E, \mathcal{P}^k)$. The *min cost three matroid intersection problem* may be stated as follows: Let c_{ijk} be a real cost for every element (i, j, k) of the ground set. The cost of a set F is defined by $c(F) := \sum_{(i,j,k)\in F} c_{ijk}$. *We want to find a set $F \subset E$ with minimum weight which is a basis in every of the three matroids $(E, \mathcal{P}^i)$, $(E, \mathcal{P}^j)$ and $(E, \mathcal{P}^k)$.*

Obviously, every common basis of the three matroids defined above corresponds to a feasible solution of an axial three-index assignment problem and vice versa.

In the case of planar three-index assignment problems we choose other partitions of the same ground set E, namely $\mathcal{P}^{ij} := \{P^{ij} \,|\, i, j = 1, 2, ..., n\}$, $\mathcal{P}^{ik} := \{P^{ik} \,|\, i, k = 1, 2, ..., n\}$ and $\mathcal{P}^{jk} := \{P^{jk} \,|\, j, k = 1, 2, ..., n\}$, where for fixed indices i and j the set P^{ij} is defined by $P^{ij} := \{(i, j, k) \,|\, k = 1, 2, ..., n\}$. The sets P^{ik} and P^{jk} are defined in an analogue way. A subset $F \subseteq E$ is defined to be a basis of $(E, \mathcal{P}^{ij})$, if for all $i, j = 1, 2, ..., n$

$$|F \cap P^{ij}| = 1.$$

In particular, $|F| = n^2$ holds. We get again three partition matroids on the ground set E, namely $(E, \mathcal{P}^{ij})$, $(E, \mathcal{P}^{ik})$ and $(E, \mathcal{P}^{jk})$. A common basis of these three matroids corresponds in a unique way to a feasible solution of a planar three-index assignment problem (Latin square) and vice versa. Thus the planar three index assignment problem can again be written as min cost three matroid intersection problem.

So, let us consider the following general matroid intersection problem: Let E be a finite ground set and let $\mathcal{P}^A := \{P^a \mid a \in A\}$, $\mathcal{P}^B$ and $\mathcal{P}^C$ three partitions of E. A basis $F \subseteq E$ of the partition matroid $(E, \mathcal{P}^A)$ ($(E, \mathcal{P}^B)$, $(E, \mathcal{P}^C)$, respectively) is defined by

$$|F \cap P^a| = 1 \text{ for all } a \in A.$$

Since we require that F is a basis of all three partition matroids, we have

$$|A| = |B| = |C| =: \nu.$$

In particular, every common basis F has the cardinality $|F| = \nu$.

Before we state admissible transformations for the three matroid intersection problem, let us introduce some notation. We shall consider sets $I \subseteq A$, $J \subseteq B$ and $K \subseteq C$. Moreover, $\overline{I} := A \setminus I$. $\overline{J}$ and $\overline{K}$ are defined analoguously. Finally,

$$P(I, J, K) := \left\{ e \in E \,|\, e \in \left(\bigcup_{a\in I} P^a\right) \cap \left(\bigcup_{b\in J} P^b\right) \cap \left(\bigcup_{c\in K} P^c\right) \right\}.$$

The sets $P(\overline{I}, J, K), ..., P(\overline{I}, \overline{J}, \overline{K})$ are defined in the same way. Now we can state admissible transformations for three matroid partitioning problems:

Theorem 1. (Admissible transformations for three matroid intersection problem) *Let $I \subseteq A$, $J \subseteq B$ and $K \subseteq C$ with $m := (|I| + |J| + |K|) - 2\nu \geq 0$, and* *let c be an arbitrary real. Then the transformation $T = T(I, J, K; c)$ of the cost coefficients c_{ijk} to new cost coefficients $\overline{c}_{ijk}$ defined by*

$$\bar{c}_{ijk} = \begin{cases} c_{ijk} - c\,, & \text{for } (i,j,k) \in P(I,J,K), \\ c_{ijk}\,, & \text{for } P(\overline{I},J,K) \cup P(I,\overline{J},K) \cup P(I,J,\overline{K}), \\ c_{ijk} + c\,, & \text{for } P(\overline{I},\overline{J},K) \cup P(I,\overline{J},\overline{K}) \cup P(\overline{I},J,\overline{K}), \\ c_{ijk} + 2c\,, & \text{for } (i,j,k) \in P(\overline{I},\overline{J},\overline{K}) \end{cases} \tag{17}$$

is admissible with $z(T) = mc$.

Proof. Let F be any common basis of the three matroids $(E, \mathcal{P}^A)$, $(E, \mathcal{P}^B)$ and $(E, \mathcal{P}^C)$. We define

$$\begin{aligned} F_0 &:= F \cap P(I,J,K), \\ F_1 &:= F \cap \left(P(\overline{I},J,K) \cup P(I,\overline{J},K) \cup P(I,J,\overline{K})\right), \\ F_2 &:= F \cap \left(P(\overline{I},\overline{J},K) \cup P(I,\overline{J},\overline{K}) \cup P(\overline{I},J,\overline{K})\right), \\ F_3 &:= F \cap P(\overline{I},\overline{J},\overline{K}). \end{aligned}$$

Since $\{F_0, F_1, F_2, F_3\}$ is a partition of the basis F, we have

$$|F_0| + |F_1| + |F_2| + |F_3| = \nu. \tag{18}$$

Every element c_{ijk} in the basis F with just one index in $\overline{I}$, $\overline{J}$ or $\overline{K}$ lies in F_1 and, with two indices in these sets, it lies in F_2. Finally a basis element c_{ijk} with $(i,j,k) \in \overline{I} \times \overline{J} \times \overline{K}$ lies in F_3. Therefore we get

$$|F_1| + 2|F_2| + 3|F_3| = |\overline{I}| + |\overline{J}| + |\overline{K}| = 3\nu - (|I| + |J| + |K|).$$

By subtracting equation (18) we get

$$|F_2| + 2|F_3| - |F_0| = 2\nu - (|I| + |J| + |K|) = -m$$

and hence

$$|F_0| = m + |F_2| + 2|F_3|. \tag{19}$$

For any feasible solution F we can write the objective function

$$\sum_{(i,j,k)\in F} c_{ijk} = \sum_{(i,j,k)\in F_0} c_{ijk} + \sum_{(i,j,k)\in F_1} c_{ijk} + \sum_{(i,j,k)\in F_2} c_{ijk} + \sum_{(i,j,k)\in F_3} c_{ijk}.$$

Using the transformation rules (17) of Theorem 1 we get

$$\begin{aligned} \sum_{(i,j,k)\in F} c_{ijk} &= \sum_{(i,j,k)\in F_0} \bar{c}_{ijk} + c|F_0| + \sum_{(i,j,k)\in F_1} \bar{c}_{ijk} + \sum_{(i,j,k)\in F_2} \bar{c}_{ijk} - c|F_2| \\ &\quad + \sum_{(i,j,k)\in F_3} \bar{c}_{ijk} - 2c|F_3| \\ &= \sum_{(i,j,k)\in F} \bar{c}_{ijk} - (|F_2| + 2|F_3| - |F_0|)c \\ &= \sum_{(i,j,k)\in F} \bar{c}_{ijk} + mc. \end{aligned}$$

■

Note that this theorem can easily be generalized to the case that we look for a common basis of more than three intersection matroids.

For solving 3-index assignment problems, the value c is chosen as minimum of the elements c_{ijk} with $i \in I, j \in J$ and $k \in K$. By reductions of the form $I := \{a\},\ J := B,\ K := C$ can be achieved that all transformed entries $\bar{c}_{ijk}$ are nonnegative. The problem of finding a feasible solution F with $c(F) = 0$ is in both cases (axial and planar problems) $\mathcal{NP}$-hard (and corresponds to checking whether the hypergraph defined by the 0-entries has the stability number ν). In case of axial 3-index assignment problems we get the transformation index

$$z(T) = ((|I| + |J| + |K|) - 2n)c, \tag{20}$$

in case of planar 3-index assignment problems we get the transformation index

$$z(T) = ((|I| + |J| + |K|) - 2n^2)c. \tag{21}$$

Proposition 1 and the reduction rules of Vlach [8] can be viewed as special cases of Theorem 1.

6. Algebraic Objective Functions

Sometimes applications require to consider other objective function than sums. An important case, for example, are *bottleneck objective functions* of the form

$$\min_{F \in \mathcal{F}} \max_{e \in F} c(e).$$

Those bottleneck functions occur whenever a time is to be minimized. The theorems of the preceding chapter can easily be turned over to admissible transformations for bottleneck problems by a proper choice of the constant c and by replacing the set of 0-elements by the so-called dominated set.

First of all, we can always replace a summation by taking the maximum, i.e., $a+b$ is replaced by $\max(a,b)$. In Proposition 1 and Theorem 1, however, we use subtractions, namely $\bar{c}(e) = c(e) - c$, which can we written as $\bar{c}(e) + c = c(e)$. The last equation reads for bottleneck problems $\max(\bar{c}(e), c) = c(e)$ which is always true, if $c \leqslant c(e)$. We say that, a cost $\bar{c}(e)$ is *dominated* by c, if $\max(\bar{c}(e), c) = c$. The set of zero's is such replaced by the set of elements which are currently dominated by the index of the transformation.

To be more general, let us consider a totally ordered, associative and commutative semigroup $(S, *, \preceq)$ with semigroup operation $*$ and order relation $\preceq$. The semigroup operation $*$ and the order relation $\preceq$ should be compatible which means

$$a \preceq b \text{ implies } a * c \preceq b * c \text{ for all } a, b, c \in S. \tag{22}$$

Moreover, we require the additional axiom

$$\text{For any } a \text{ and } b \text{ with } a \preceq b \text{ there exists } c \in S : \ a * c = b. \tag{23}$$

A semigroup obeying these axioms is sometimes called a *d-monoid*. Examples for d-monoids are

- $(\mathbb{R}, +, \leqslant)$. This system leads to the classical sum objectives.
- $(\mathbb{R} \cup \{-\infty\}, \max, \leqslant)$. This system leads to bottleneck problems.
- $(\mathbb{R}^k, +, \preceq)$ with the lexicographical order $\preceq$. This system leads to vector optimization problems with a lexicographic objective.

For further examples, see the survey paper of Burkard and Zimmermann [3].

Now let us consider the following combinatorial optimization problem with a general objective function: Let $E := \{e_1, e_2, ..., e_n\}$ be the ground set of a combinatorial optimization problem whose feasible solutions $F \subseteq E$ are collected in the class $\mathcal{F}$. The costs of the ground elements $e_1, ..., e_n$ are denoted by $c(e_1), c(e_2), ..., c(e_n) \in S$. The cost $c(F)$ of a feasible solution $F = (e_{r_1}, e_{r_2}, ..., e_{r_k})$ is defined by

$$c(F) := c(e_{r_1}) * c(e_{r_2}) * ... * c(e_{r_k}). \tag{24}$$

A transformation T of the costs $c(e),\ e \in E$, to new costs $\bar{c}(e),\ e \in E$, is called *admissible* with *index* $z(T)$, if

$$c(F) = z(T) * \bar{c}(F) \ \text{ for all } \ F \in \mathcal{F}. \tag{25}$$

For this problem we immediately get as optimality criterion the analogue of Lemma 1.

Lemma 2. *Let T be an admissible transformation with index $z(T)$ such that there exists a feasible solution F^* with the following properties:*

1. $\bar{c}(e) * z(T) \geq z(T)$ *for all* $e \in E$.
2. $\bar{c}(F^*) * z(T) = z(T)$.

Then F^* is an optimal solution of with value $z(T)$.

Proof. Let F be an arbitrary feasible solution. According to the properties of admissible transformations we get:

$$c(F) = z(T) * \bar{c}(F) \geq z(T) = z(T) * \bar{c}(F^*) = c(F^*).$$

Therefore F^* is optimal. ■

As an example we state Proposition 1 for algebraic objective functions. It takes the form

Proposition 3. (Admissible transformations for linear assignment problems with general objective) *Let $I, J \subseteq \{1, 2, \ldots, n\}$, $m := |I| + |J| - n \geq 0$, and let*

$$c := \min\{c_{ij} \mid i \in I,\ j \in J\}.$$

Then the transformation $T = T(I, J; c)$ of the cost coefficients c_{ij} to new cost coefficients $\bar{c}_{ij}$ defined by

$$\bar{c}_{ij} * c = c_{ij} \ \textit{for}\ i \in I, j \in J$$

and

$$\bar{c}_{ij} = \begin{cases} c_{ij} * c\,, & \textit{for } i \notin I, j \notin J, \\ c_{ij}\,, & \textit{otherwise} \end{cases}$$

is admissible with $z(T) = c * c * ... * c$ *(m-times).*

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