Vietnam Journal of Mathematics **36**:1(2008) 1–38

Vietnam Journal
of
MATHEMATICS

Survey

Critical Values of Singularities at Infinity of Complex Polynomials

Ha Huy Vui[1] **and Pham Tien Son**[2]

[1] *Institute of Mathematics, 18 Hoang Quoc Viet Road, 10307 Hanoi, Vietnam*
[2] *Department of Mathematics, University of Dalat,*
1 Phu Dong Thien Vuong, Dalat, Vietnam

In memory of Professor Nguyen Huu Duc

Received October 05, 2007
Revised March 12, 2008

Abstract. This paper is an overview of the theory of critical values at infinity of complex polynomials.

2000 Mathematics Subject Classification: 32S50, 32A20, 32S50.

Keywords: Bifurcation set of complex polynomials; complex affine plane curve; Euler-Poincaré characteristic; link at infinity; Lojasiewicz numbers; Milnor number; resolution, singularities at infinity; splice diagram.

1. Introduction

The study of topology of algebraic and analytic varieties is a very old and fundamental subject. There are three fibrations which appear in this topic: (1) *Lefschetz pencil*, which provides essential tools for treating the topology of projective algebraic varieties (see [75]); (2) *Milnor fibration*, which is a quite powerful instrument in the investigation the local behavior of complex analytic varieties

[1] The first author was supported in part by the National Basic Research Program in Natural Sciences, Vietnam.

[2] The second author was supported by the B2006-19-42.

in a neighborhood of a singular point (see [77]); and recently, (3) *global Milnor fibration*, which is used to study topology of affine algebraic varieties (see, for instance, [4, 10, 14, 17, 25, 40–44, 78–80, 85, 96, 109]). Regarding the status of the first two fibrations there was limited knowledge about global Milnor fibration. This is due, in our opinion, the following difficulties while working with affine algebraic varieties:

(i) affinity (different from projectivity of Lefschetz's theory); and
(ii) globality (different from locality of Milnor's theory).

These two difficulties lead to a new phenomenon, very crucial in the study of the topology of affine algebraic varieties: *it is the singularity at infinity.*

Let $P\colon \mathbb{C}^n \longrightarrow \mathbb{C}$ be a polynomial function. According to Thom [110] (see also [13, 14, 40, 55, 62, 89, 93, 111, 112]), there exists a finite set of points $B \subset \mathbb{C}$ such that the restriction

$$P\colon \mathbb{C}^n \setminus P^{-1}(B) \to \mathbb{C} \setminus B$$

is a C^∞-locally trivial fibration. We call the smallest such set *bifurcation set* and denote by $B(P)$. Naturally, a question arises:

Problem 1. *How to determine the bifurcation set $B(P)$?*

The bifurcation set $B(P)$ includes not only the *set of critical values* $K_0(P)$ of P but may also contain some other values. This is possible because P is not proper for $n \geqslant 2$. Therefore the study of the bifurcation set $B(P)$ is not easy in general. This is a challenging problem of singularity theory. Until now it is solved only in few cases. To understand the problem, let us first look at some examples.

Example 1.1. Let $P\colon \mathbb{C}^2 \to \mathbb{C}, (x, y) \mapsto xy$. Then $(0, 0)$ is the unique critical point with critical value $P(0, 0) = 0$. Now consider the topology of the fiber $P^{-1}(t)$.

If $t = 0$, then

$$P^{-1}(0) = \{(x, y) \in \mathbb{C}^2 \mid xy = 0\} = \mathbb{C} \cup \mathbb{C}.$$

If $t \neq 0$, then

$$P^{-1}(t) = \{(x, y) \in \mathbb{C}^2 \mid xy = t\}.$$

Let us define the map $\varphi\colon \mathbb{C}^* \to P^{-1}(t)$ by $\varphi(x) := (x, t/x)$, where $\mathbb{C}^* := \mathbb{C} - \{0\}$. Then φ is a complex analytic isomorphism. In particular, φ is a homeomorphism.

Therefore

$$H_1(P^{-1}(t), \mathbb{C}) = \begin{cases} H_1(\mathbb{C} \cup \mathbb{C}, \mathbb{C}) = 0, & \text{if} \quad t = 0, \\ H_1(\mathbb{C}^*, \mathbb{C}) = H_1(\mathbb{S}^1, \mathbb{C}) = \mathbb{C}, & \text{if} \quad t \neq 0. \end{cases}$$

In this example, we have $B(P) = K_0(P) = \{0\}$.

Example 1.2. Broughton has remarked (see [14]) that the polynomial $P\colon \mathbb{C}^2 \to \mathbb{C}, (x, y) \mapsto x^2y - x$, has no critical points; however $B(P) = \{0\}$. To explain why $t = 0$ is a special value, let us notice that

$$P^{-1}(0) = \{(x, y) \in \mathbb{C}^2 \mid x^2y - x = 0\} = \mathbb{C} \sqcup \mathbb{C}^* \quad \text{(disjoint union)}.$$

On the other hand, using the parameterization technique as in Example 1.1, we again find that $P^{-1}(t), t \neq 0$, is homeomorphic to $\mathbb{C}^*$. In particular,

$$H_0(P^{-1}(t), \mathbb{C}) = \begin{cases} \mathbb{C}^2, & \text{if} \quad t = 0, \\ \mathbb{C}, & \text{if} \quad t \neq 0. \end{cases}$$

Moreover, it is worth noting that

$$H_1(P^{-1}(t), \mathbb{C}) = \mathbb{C} \quad \text{for all } t \in \mathbb{C}.$$

Hence, the first homology group does not distinguish the special fiber $P^{-1}(0)$ from the generic fiber $P^{-1}(t)$.

To understand what happens in Example 1.2, we have to compare the fibers $P^{-1}(t)$ and $P^{-1}(0)$ in a neighborhood of infinity. Namely, for R sufficiently large we put $\mathbb{B}_R := \{(x, y) \in \mathbb{C}^2 \mid |x|^2 + |y|^2 \leqslant R^2\}$. Then the intersections $P^{-1}(t) \cap \mathbb{B}_R$ and $P^{-1}(0) \cap \mathbb{B}_R$ are homeomorphic, but $P^{-1}(t) \setminus \mathbb{B}_R$ and $P^{-1}(0) \setminus \mathbb{B}_R$ are not. In other words, the topology of fibers $P^{-1}(t)$ of the function P changes (as t tends to 0) in a neighborhood of infinity. This leads us to the following definition.

Definition 1.3. [80] *Let $P\colon \mathbb{C}^n \to \mathbb{C}$ be a polynomial function.*

(i) *A point $t_0 \in \mathbb{C}$ is called a regular value at infinity of P if there is a compact set K in $\mathbb{C}^n$ and a positive number δ such that the restriction*

$$P\colon P^{-1}(D_\delta(t_0)) \setminus K \to D_\delta(t_0) := \{t \in \mathbb{C} \mid |t - t_0| < \delta\}$$

defines a C^∞-trivial fibration.

(ii) *A point $t_0 \in \mathbb{C}$ that is not a regular value at infinity of P is called a critical value at infinity of P.*

We denote by $B_\infty(P)$ the set of critical values at infinity of P. Then one can prove that

$$B(P) = K_0(P) \cup B_\infty(P).$$

Hence, the problem stated above can be reformulated as follows.

Problem 2. *How to determine the set $B_\infty(P)$?*

There is a quite abundant literature on this topic. See, for instance, [4, 5, 7–11, 13–19, 24, 25, 28–34, 36–47, 50, 53–55, 57, 60, 72–74, 78–82, 85–88, 90–94, 96–99, 106–109]. The simplest case $n = 2$ was studied intensively and it is rather well understood: the set $B_\infty(P)$ can be computed effectively [40–44, 101]. In the general case, the set $B_\infty(P)$ can be determined in any dimension only for polynomials which have only "isolated singularities at infinity" (see [85, 86, 96]).

The aim of this paper is to give an overview of the theory of critical values at infinity of complex polynomials. For the sake of harmony we sketch proofs when possible.

2. Two Dimensional Case

We consider the case $n = 2$ separately, because for polynomials in two complex variables we can give concrete information. It should be noticed that, in his paper, Durfee [25] discussed, in a purely local setting near a point on the line at infinity, equivalent ways of defining a critical value at infinity. On the other hand, in this section, we focus topics, with further results, which have not been mentioned in details in [25]. Besides, some other results are also recalled.

2.1. Euler-Poincaré Characteristic

The following first result shows that critical values at infinity of polynomials in two complex variables are characterized by the variation of the Euler-Poincaré number of the fiber.

Theorem 2.1. [40, 101] *Let $P\colon \mathbb{C}^2 \to \mathbb{C}$ be a non-constant polynomial function and let t_0 be a regular value of P. Then the following conditions are equivalent.*
(i) *The value t_0 is a critical value at infinity of P, i.e., $t_0 \in B_\infty(P)$.*
(ii) *$\chi(P^{-1}(t_0)) \neq \chi(P^{-1}(t))$, where $P^{-1}(t)$ is a generic fiber of P and χ denotes the Euler-Poincaré characteristic.*

Before going into the proof, let us begin by calculating the Euler characteristic of a reduced complex affine plane curve.

Let $P\colon \mathbb{C}^2 \to \mathbb{C}$ be a polynomial function of degree $d > 1$. By a linear change of coordinates $(x, y) \in \mathbb{C}^2$ which we can put $P(x, y)$ in the form (this does not change the sets $K_0(P)$ and $B_\infty(P)$)

$$P(x, y) = x^d + a_1(y)x^{d-1} + \cdots + a_d(y), \tag{2.1}$$

where $a_i(y)$ are polynomials in the variable y of degree at most i.

Let $\Delta \in \mathbb{C}[t, y]$ be the resultant (discriminant) obtained by eliminating x of $P(x, y) - t$ and $\frac{\partial P}{\partial x}(x, y)$. By the definition of the resultant, we get easily that

$$\Delta(t, y) = q_0(t)y^s + q_1(t)y^{s-1} + \cdots + q_s(t), \tag{2.2}$$

where $q_i \in \mathbb{C}[t], i = 0, 1, \ldots, s$. We have

Lemma 2.2. (see also [34, Proposition 2.2]) *Suppose that the fiber $P^{-1}(t) \subset \mathbb{C}^2$ is reduced. Then*

$$\chi(P^{-1}(t)) = d - \deg_y \Delta(t, y).$$

Proof. Since the polynomial P is monic in the variable x, for each $t \in \mathbb{C}$, the restriction $\ell|_{C_t}$ of the linear form $\ell\colon \mathbb{C}^2 \to \mathbb{C}$, $(x, y) \mapsto y$, on the fiber $C_t := P^{-1}(t)$ is proper.

Let $\Sigma := \{(x_1, y_1), (x_2, y_2), \ldots, (x_p, y_p)\}$ be the set of critical points of $\ell|_{C_t}$, so that $y_1, y_2, \ldots, y_p$ are critical values of $\ell|_{C_t}$.

Take $y_* \in \mathbb{C} \setminus \{y_1, y_2, \ldots, y_p\}$. Then $\ell|_{C_t}^{-1}(y_*)$ consists of $d = \deg P$ distinct points. In the y-plane, we consider a system of paths $T_1, T_2, \ldots, T_p$ connecting $y_1, y_2, \ldots, y_p$ with y_* such that

(i) each path T_j has no self-intersection points;

(ii) two distinct paths T_i and T_j meet only at the point y_*.

Put

$$\hat{\mathcal{S}}(C_t) := \ell|_{C_t}^{-1}(\cup_{i=1}^p T_i).$$

Then $\hat{\mathcal{S}}(C_t)$ is a union of 1-dimensional curves. Let $\check{\mathcal{S}}(C_t)$ be the set of all curves in $\hat{\mathcal{S}}(C_t) \setminus \ell|_{C_t}^{-1}(y_*)$ which contain a point of Σ.

Since the set $\cup_{i=1}^p T_i$ is a deformation retract of $\mathbb{C}$ and the restriction

$$\ell|_{C_t}\colon C_t \setminus \Sigma \to \mathbb{C} \setminus \{y_1, y_2, \ldots, y_p\}, \quad (x, y) \mapsto y,$$

is a local trivial fibration, $\hat{\mathcal{S}}(C_t)$ is a deformation retract of C_t. Moreover, one can easily see that the set

$$\mathcal{S}(C_t) := \check{\mathcal{S}}(C_t) \cup \ell|_C^{-1}(y_*)$$

is a deformation retract of $\hat{\mathcal{S}}(C_t)$ and also of C_t. Hence,

$$\chi(C_t) = \chi(\mathcal{S}(C_t)).$$

On the other hand, the set $\mathcal{S}(C_t)$ can be identified with a 1-dimensional graph of $d + p$ vertices and $\sum_{j=1}^p m_j$ edges, where m_j is the number of edges incident on the vertex (x_j, y_j), i.e., m_j is equal to the ramification index of $\ell|_{C_t}$ at the point (x_j, y_j). Thus

$$\chi(C_t) \;=\; \chi(\mathcal{S}(C_t)) = d + p - \sum_{j=1}^p m_j \;=\; d - \sum_{j=1}^p (m_j - 1).$$

But, as it is easily seen, the number of ramification points $\sum_{j=1}^p (m_j - 1)$ equals the degree in y of the discriminant $\Delta(t, y)$ of $P(x, y) - t$ with respect to x, i.e.,

$$\sum_{j=1}^p (m_j - 1) = \deg_y \Delta(t, y).$$

This completes the proof of the lemma. ∎

Now we can pass to the proof of Theorem 2.1.

Proof of Theorem 2.1. It is trivial that (ii) $\Rightarrow$ (i).

Conversely, suppose that (ii) fails. Keep all conventions and notations as above. Let $s(t) := \deg_y \Delta(t, y)$ be the degree of the discriminant $\Delta(t, y)$ in the variable y. Lemma 2.2 now shows that $\chi(P^{-1}(t)) = d - s(t)$. By assumption, $s(t) = s$ for all $t \in D_\delta(t_0) := \{t \in \mathbb{C} \mid |t - t_0| < \delta\}$, $0 < \delta \ll 1$. Consequently, there exists a positive number r such that the system of equations

$$\begin{cases} P(x,y) & = t, \\ \dfrac{\partial P}{\partial x}(x,y) & = 0, \end{cases}$$

has no solution on the set $P^{-1}(D_\delta(t_0)) \setminus \{|y| \leqslant r\}$. Thus, we can construct a diffeomorphism which trivializes the fibration

$$P \colon P^{-1}(D_\delta(t_0)) \setminus \{|y| \leqslant r\} \longrightarrow D_\delta(t_0).$$

On the other hand, since P is monic in the variable x, the set $P^{-1}(D_\delta(t_0)) \cap \{|y| \leqslant r\}$ is compact. Therefore, t_0 is a regular value at infinity of P. ■

2.2. Polar Curves

Polar curves play an important role in projective geometry by generic projections, in particular in the study of numerical invariants of projective algebraic varieties, and also in the study of projective duality (Plücker formulas). In the 1970s, local polar curves were used systematically in the study of singularities: it can be used to produce invariants of equisingularity ("topological" invariants of complex analytic singularities) and also to explain why the same invariants appear in apparently unrelated questions (see, for example, [49, 63–69, 76, 102, 103, 105]). Many authors continued this study and found more and more applications of polar curves.

Our aim in this section is to study critical values at infinity with the aid of polar curves (i.e., the critical points of projections).

Let P be a polynomial of the form (2.1) and let us fix the same notation as in Sec. 2.1. Choose a generic linear form, which we shall take as $\ell \colon \mathbb{C}^2 \to \mathbb{C}, (x, y) \mapsto y$. Then by the *polar curve* (respectively, *discriminant locus*) of P with respect to ℓ we mean the set of critical points (respectively, critical values) of the polynomial map

$$(P, \ell) \colon \mathbb{C}^2 \to \mathbb{C}^2, \quad (x, y) \mapsto (P(x, y), y).$$

Clearly, the polar curve is defined by $\frac{\partial P}{\partial x} = 0$. Moreover, the discriminant locus consists of points $(t, y) \in \mathbb{C}^2$ where $\Delta(t, y) = 0$.

On the other hand, it follows from Theorem 2.1 and Lemma 2.2 that $t_0 \in B(P)$ if and only if $s > s(t_0)$. Then Theorem 2.1 can be stated as

Theorem 2.3. [41] (see also [19, 20, 94]) *With notations as above, the following conditions are equivalent*

(i) *t_0 is a critical value at infinity of P.*
(ii) *The line $t - t_0 = 0$ is an asymptote of the discriminant locus, i.e.,*

$$B_\infty(P) = \{t \mid q_0(t) = 0\}.$$

Remark 2.4. It is noted that the set of critical values of P can be characterized in terms of the discriminant locus (see, for example, [90, Theorem 2.3]).

Next, for each $t \in \mathbb{C}$, we consider the restriction

$$\ell|_{P^{-1}(t)} \colon P^{-1}(t) \to \mathbb{C}, \quad (x, y) \mapsto y.$$

We say that a point $(x_*, y_*) \in \mathbb{C}^2$ is a *ramification point* of the fiber $P^{-1}(t)$ with respect to the projection ℓ if it is a critical point of the map $\ell|_{P^{-1}(t)}$, i.e., if the following conditions hold

$$P(x_*, y_*) = t \quad \text{and} \quad \frac{\partial P}{\partial x}(x_*, y_*) = 0.$$

Since P is monic in x, the map $\ell|_{P^{-1}(t)}$ is proper. By Theorem 2.3, we obtain that $t_0 \in B(P)$ if and only if for each $t \in \mathbb{C}$ sufficiently close to t_0 there exists a ramification point $(x(t), y(t))$ of the fiber $P^{-1}(t)$ with respect to the projection ℓ such that $\|(x(t), y(t))\| \to \infty$ as $t \to t_0$. Hence,

Theorem 2.5. [42] *With notions as above, the following conditions are equivalent*
(i) *t_0 is a critical value at infinity of P.*
(ii) *There is at least one ramification point of the fiber $P^{-1}(t)$ (with respect to the projection ℓ) which tends to infinity as $t \to t_0$. Moreover, the number of such ramification points equals $\chi(P^{-1}(t_0)) - \chi(P^{-1}(t))$, where $P^{-1}(t)$ is a generic fiber of P.*

2.3. Semi-cycles Vanishing at Infinity

As is well-known, in the local case of isolated singularities, one can use the middle homology group to distinguish special fibers from the generic one (see [77]). Unfortunately, as shown in Example 1.2, a global version of this result is not true. On the other hand, by introducing the notion of "semi-cycles vanishing at infinity", we show below that a certain relative homology group is sufficient to distinguish special fibers from the generic one.

Let P be, as previously, a polynomial of the form (2.1) and consider the linear form $\ell \colon \mathbb{C} \to \mathbb{C}, (x, y) \mapsto y$. There is no loss of generality in assuming that for each t near a given regular value t_0 of P, the restriction map $\ell|_{C_t}$ of ℓ on the fiber $C_t := P^{-1}(t)$ is simple ($\ell|_{C_t}$ is said to be *simple* if and only if $\ell|_{C_t}^{-1}(y)$ consists of $d-1$ distinguished points for every critical value y of $\ell|_{C_t}$). Then the number of critical points of $\ell|_{C_t}$ is exactly $s(t) := \deg_y \Delta(t, y)$. Let

$$(x_1(t), y_1(t)), \ldots, (x_s(t), y_s(t))$$

be critical points of the map $\ell|_{C_t}, t \notin B(P)$. Now we use the notations as in the proof of Lemma 2.2. Suppose that $y_* \in \mathbb{C}$ is a common regular value of $\ell|_{C_t}$ for all t near t_0 and let $e_j(t) := \ell|_{C_t}^{-1}(T_j) \cap \mathcal{S}(C_t),\ j = 1, \ldots, s$, be the cycles of the relative homology group $H_1(C_t, \ell|_{C_t}^{-1}(y_*))$ corresponding to the critical points $(x_j(t), y_j(t))$. These cycles define a basis of the relative homology group $H_1(C_t, \ell|_{C_t}^{-1}(y_*))$. Then, by Theorem 2.5, $t_0 \in B_\infty(P)$ if and only if there exists an index $j \in \{1, 2, \ldots, s\}$ such that $\|(x_j(t), y_j(t))\| \to \infty$ when $t \to t_0$. Moreover, since the map $\ell|_{C_t}$ is simple, the number of critical points of $\ell|_{C_t}$ tending to infinity when $t \to t_0$ is equal to $s - s(t_0)$. Therefore, we may assume without loss of generality that such critical points are

$$(x_1(t), y_1(t)), \ldots, (x_{s-s(t_0)}(t), y_{s-s(t_0)}(t)).$$

Definition 2.6. [41] *We call* $e_j(t),\ j = 1, 2, \ldots, s - s(t_0)$, *semi-cycles vanishing at infinity when* $t \to t_0$.

As a direct consequence of this definition, we get

Theorem 2.7. [41] (See also [47]) *The following conditions are equivalent*

(i) t_0 *is a critical value at infinity of* P.

(ii) *There exist semi-cycles* $e_j(t)$ *vanishing at infinity when* $t \to t_0$. *The number of such semi-cycles equals* $\chi(P^{-1}(t_0)) - \chi(P^{-1}(t))$.

Remark 2.8. The number of semi-cycles $e_j(t)$ vanishing at infinity can be given in another way as follows

$$\begin{aligned}
\chi(P^{-1}(t_0)) - \chi(P^{-1}(t)) &= s - \deg_y \Delta(t_0, y) \\
&= \left(\{P = t\}, \left\{\frac{\partial P}{\partial x} = 0\right\}\right) - \left(\{P = t_0\}, \left\{\frac{\partial P}{\partial x} = 0\right\}\right) \\
&= \dim_{\mathbb{C}} \frac{\mathbb{C}[x, y]}{(P - t, \frac{\partial P}{\partial x})} - \dim_{\mathbb{C}} \frac{\mathbb{C}[x, y]}{(P - t_0, \frac{\partial P}{\partial x})};
\end{aligned}$$

here (X, Y) stands for the (total) intersection number of complex affine plane curves X and Y.

2.4. Milnor-Lê Number

We will define below an invariant $\lambda_t(P), t \in \mathbb{C}$, as the difference between Milnor numbers at infinity of special fibers and of the generic one. More precisely, let $P\colon \mathbb{C}^2 \to \mathbb{C}$ be a polynomial function of degree d. The homogenization of $P(x, y)$ is the homogeneous complex polynomial $\overline{P}(x, y, z)$ of degree d :

$$\overline{P}(x, y, z) := z^d P\left(\frac{x}{z}, \frac{y}{z}\right).$$

The compactification of the fibers $C_t := P^{-1}(t)$ in the complex projective plane $\mathbb{P}^2$ are the projective curves $\overline{C}_t$ with projective equations $\overline{P}(x, y, z) - tz^d = 0$.

These curves pass through the same points on the line at infinity $\{z = 0\}$, namely through

$$\{a_1, a_2, \ldots, a_p\} = \{\overline{P} = z = 0\}$$

which corresponds to the asymptotic directions of the fibers $P^{-1}(t)$.

For each $a_i, 1 \leqslant i \leqslant p$, let $\mu_{a_i}(\overline{C}_t)$ be the Milnor number of the germ of the analytic curve $\overline{C}_t$ at the point a_i. As $\mu_{a_i}(\overline{C}_t)$ is upper semicontinuous in t [14], there exists an integer

$$\mu_i := \inf_{t \in \mathbb{C}} \mu_{a_i}(\overline{C}_t).$$

Hence, for a generic $t, \mu_{a_i}(\overline{C}_t) = \mu_i$, and for finitely many $t_0, \mu_{a_i}(\overline{C}_{t_0}) > \mu_i$.

Definition 2.9. *We define for any $t \in \mathbb{C}$*

$$\lambda_t(P) := \sum_{i=1}^{p} [\mu_{a_i}(\overline{C}_t) - \mu_i],$$

and we call it the Milnor-Lê number at infinity of the fiber $P^{-1}(t)$.

Remark 2.10. Let us note that $\lambda_t(P)$ and $\sum_{t \in \mathbb{C}} \lambda_t(P)$ are topological invariants of P (see, for example, [5, 14, 17, 34, 40, 86, 96, 101, 107]).

The set of critical values at infinity of polynomials in two variables can be computed by using the λ-invariant, as described below.

Theorem 2.11. [40, 70] *We have*

$$B_\infty(P) = \{t \in \mathbb{C} \mid \lambda_t(P) > 0\}.$$

Proof. Without loss of generality, after a linear change of coordinates, we can suppose that P has the form (2.1).

Let $\overline{D}$ be the compactification of the polar curve $D := \{\frac{\partial P}{\partial x} = 0\}$. By Bezout's theorem,

$$(\overline{C}_t, \overline{D}) = d(d-1) \quad \text{for all} \quad t \in \mathbb{C}.$$

On the other hand,

$$\begin{aligned}
(\overline{C}_t, \overline{D}) &= \sum_{a \in \mathbb{C}^2} (\overline{C}_t, \overline{D})_a + \sum_{a \in \mathbb{P}^2 - \mathbb{C}^2} (\overline{C}_t, \overline{D})_a \\
&= \dim_{\mathbb{C}} \frac{\mathbb{C}[x, y]}{(P - t, \frac{\partial P}{\partial x})} + \sum_{a \in \overline{C}_t \cap \{z=0\}} (\overline{C}_t, \overline{D})_a \\
&= \dim_{\mathbb{C}} \frac{\mathbb{C}[x, y]}{(P - t, \frac{\partial P}{\partial x})} + \sum_{i=1}^{p} (\overline{C}_t, \overline{D})_{a_i},
\end{aligned}$$

where $(\cdot, \cdot)_a$ stands for the local intersection multiplicity of $\overline{C}_t$ with $\overline{D}$ at a. Moreover, since the polynomial P is monic in x, the projective curve $\overline{C}_t$ intersects transversally to the line at infinity $\{z = 0\}$. Then, as it is well known ([104, 22, 6, 35] for instance),

$$(\overline{C}_t, \overline{D})_{a_i} = \mu_{a_i}(\overline{C_t}) + (\overline{C}_t, \{z = 0\})_{a_i} - 1, \quad i = 1, 2, \ldots, p.$$

Hence,

$$d(d-1) = (\overline{C}_t, \overline{D}) = \dim_{\mathbb{C}} \frac{\mathbb{C}[x, y]}{(P - t, \frac{\partial P}{\partial x})} + \sum_{i=1}^{p} [\mu_{a_i}(\overline{C_t}) + (\overline{C}_t, \{z = 0\})_{a_i} - 1].$$

Since the numbers $(\overline{C}_t, \{z = 0\})_{a_i}, i = 1, 2, \ldots, p$, do not depend on t, therefore

$$\dim_{\mathbb{C}} \frac{\mathbb{C}[x, y]}{(P - t, \frac{\partial P}{\partial x})} - \dim_{\mathbb{C}} \frac{\mathbb{C}[x, y]}{(P - t_0, \frac{\partial P}{\partial x})} = \sum_{i=1}^{p} [\mu_{a_i}(\overline{C}_{t_0}) - \mu_{a_i}(\overline{C_t})].$$

Consequently,

$$\lambda_{t_0}(P) = \dim_{\mathbb{C}} \frac{\mathbb{C}[x, y]}{(P - t, \frac{\partial P}{\partial x})} - \dim_{\mathbb{C}} \frac{\mathbb{C}[x, y]}{(P - t_0, \frac{\partial P}{\partial x})} = \chi(C_{t_0}) - \chi(C_t).$$

Combining with Theorem 2.1, this proves the theorem. ∎

Remark 2.12. The number $\lambda_{t_0}(P)$ counts the number of ramification points of the fiber $P^{-1}(t)$ which tend to infinity as t tends to t_0, and then the number $\lambda(P)$ is the total of ramification points which tend to infinity as t varies.

2.5. Resolution

In general, it is easier to deal with proper maps. In the case of complex polynomial functions in two variables, there is a natural way to compactify the function. Let $P: \mathbb{C}^2 \to \mathbb{C}$ be a complex polynomial of degree d and $\overline{P}$ its homogenization. The polynomial P extends to a map

$$\widetilde{P}: \mathbb{P}^2 \cdots \to \mathbb{P}^1 := \mathbb{C} \cup \{\infty\}, \quad (x : y : z) \mapsto (\overline{P}(x, y, z) : z^d),$$

which is undefined at a finite number of points on the line at infinity $\{z = 0\}$. By blowing up these points, one gets a manifold $\mathcal{X}$ and a map $\pi: \mathcal{X} \to \mathbb{P}^2$ such that the map $\hat{P}: \mathcal{X} \to \mathbb{P}^1$ which is the lift of $\widetilde{P}$ is everywhere defined. We call the map π a *resolution* of P. Some interesting results on the structure of resolutions are discovered in [71]. For example, the intersection graph of the divisor $\pi^{-1}(\{z = 0\})$ is a tree $\mathcal{A}$. Moreover, the space $\hat{P}^{-1}(\infty)$ is connected. In other words, $\hat{P}^{-1}(\infty)$ defines a strict connected subtree $\mathcal{A}_\infty$ of $\mathcal{A}$. We shall call the divisor $\pi^{-1}(\{z = 0\})$, the *divisor at infinity* of $\mathcal{X}$, and a component of $\pi^{-1}(\{z = 0\})$ on which $\hat{P}$ is not constant is called *dicritical.*

Following [70, 71], each connected component of $\mathcal{A} - \mathcal{A}_\infty$ is a bamboo which contains a unique dicritical component of $\hat{P}$ and this dicritical component is the only irreducible component of the bamboo which meets $\mathcal{A}_\infty$. Let $\mathcal{B}$ be a bamboo of $\mathcal{A} - \mathcal{A}_\infty$ and let $\mathcal{D}_\mathcal{B}$ be its dicritical component. If $\mathcal{B}$ has more than one component, the components of $\mathcal{B}$ other than $\mathcal{D}_\mathcal{B}$ define a sub-bamboo $\mathcal{B}'$ of $\mathcal{B}$. The restriction to $\mathcal{B}'$ of the function $\hat{P}$ is a finite constant. This value will be called the *atypical value* of the bamboo $\mathcal{B}$.

The next result gives a criterion for critical values at infinity in terms of a resolution.

Theorem 2.13. [70, 71] *Under above notations we have*

$$\begin{aligned}B_\infty(P) = \{&\textit{critical values} \neq \infty \textit{ of the restriction of } \varphi \textit{ to dicritical components}\} \cup \\ &\{\textit{atypical values of } \hat{P} \textit{ on each bamboos of } \mathcal{A} - \mathcal{A}_\infty \textit{ with at least two vertices}\}.\end{aligned}$$

2.6. Lojasiewicz Number at Infinity of the Fiber

Let $P\colon \mathbb{C}^n \to \mathbb{C}$ be a polynomial function. It turns out that for $t \in \mathbb{C}$ the property of being in $B_\infty(P)$ depends on the behavior of the gradient of P near the fiber $P^{-1}(t)$.

Adapting Milnor's definition let us define the gradient of P by

$$\mathrm{grad} P(x) := \left(\overline{\frac{\partial P}{\partial x_1}(x)}, \overline{\frac{\partial P}{\partial x_2}(x)}, \ldots, \overline{\frac{\partial P}{\partial x_n}(x)} \right),$$

where $x := (x_1, x_2, \ldots, x_n)$ and the bar denotes the conjugation.

Definition 2.14. [43] *Let $t \in \mathbb{C}$. By the Lojasiewicz number at infinity of the fiber $P^{-1}(t)$ we mean the number*

$$\mathrm{L}_{\infty,t}(P) := \lim_{\delta \to 0} \lim_{r \to \infty} \frac{\ln \varphi_{\delta,t}(r)}{\ln r},$$

where

$$\varphi_{\delta,t}(r) := \inf\{\|\mathrm{grad} P(x)\| \mid \|x\| = r \text{ and } |P(x) - t| \leqslant \delta\}.$$

An equivalent definition is (*see* [20, 100])

$$\mathrm{L}_{\infty,t}(P) = \inf_{\psi} \frac{\mathrm{val}\,(\overline{\mathrm{grad} P \circ \psi})}{\mathrm{val}\,\psi},$$

where $\psi = (\psi_1, \ldots, \psi_n)$ is a meromorphic map at infinity such that $\lim_{\tau \to 0} \|\psi(\tau)\| = \infty$ and $\lim_{\tau \to 0} P(\psi(\tau)) = t$, here val$\,\lambda$ for λ meromorphic at infinity is defined as follows: if $\lambda(\tau) = \sum_{i=k}^{\infty} a_k \tau^k, a_k \neq 0$, is the Laurent series of λ in a neighborhood of 0 then val$\,\lambda := k$; if $\lambda \equiv 0$ then val$\,\lambda := +\infty$.

The following result shows that critical values at infinity of polynomials in two complex variables are characterised in terms of the Lojasiewicz number at infinity of the fiber.

Theorem 2.15. [43] (see also [25, 59]) *Let $P\colon \mathbb{C}^2 \to \mathbb{C}$ be a polynomial in two complex variables. The following conditions are equivalent*
(i) $t_0 \in B_\infty(P)$.
(ii) $\mathrm{L}_{\infty,t_0}(P) < -1$.
(iii) $\mathrm{L}_{\infty,t_0}(P) < 0$.

To prove the theorem we need additional terminology.

Definition 2.16. *We say that a fractional power series*

$$\varphi(y) = c_1 y^{q_1/N} + c_2 y^{q_2/N} + \cdots, \quad |y| \gg 1, \ \text{with} \quad q_1 > q_2 > \cdots, N \in \mathbb{N},$$

a Puiseux root at infinity of the curve with the equation $P(x,y) = 0$ if the series $\varphi(\tau^N), |\tau| \gg 1$, converges at infinity and $P(\varphi(y), y) \equiv 0$.

Proposition 2.17. *Assume that the polynomial P is of the form* (2.1)*. Then the curve $\{P(x,y) = 0\}$ has exactly d Puiseux roots at infinity. Each root has a representation of the form:*

$$c_i y + \sum_{k=-1}^{-k_0} c_{0k} y^k + c_{10} y^{\frac{m_1}{n_1}} + \sum_{k=-1}^{-k_1} c_{1k} y^{\frac{m_1+k}{n_1}} + \cdots + c_{g0} y^{\frac{m_g}{n_1 n_2 \cdots n_g}} + \sum_{k=-1}^{-\infty} c_{gk} y^{\frac{m_g+k}{n_1 n_2 \cdots n_g}}.$$

Proof. Let $\overline{C} \subset \mathbb{P}^2$ be the compactification of the curve $C := \{P(x,y) = 0\}$. The projective curve $\overline{C}$ meets the line at infinity $\{z = 0\}$ in finitely many points say $a_1, a_2, \ldots, a_p$. Since P is monic in x, we may write

$$a_i = (-c_i : 1 : 0) \in \mathbb{P}^2, \quad i = 1, 2, \ldots, p.$$

Fix $i \in \{1, 2, \ldots, p\}$. Consider the equation

$$z^d P\left(\frac{x}{z}, \frac{1}{z}\right) = 0$$

in a neighborhood of the point $(-c_i, 0) \in \mathbb{C}^2$. We substitute

$$u := x + c_i$$

and obtain

$$f_i(u,z) := z^d P\left(\frac{u - c_i}{z}, \frac{1}{z}\right) = 0.$$

Clearly, $f_i \in \mathbb{C}[u,z]$ has the form

$$f_i(u,z) := g_i(u) + z h_i(u,z),$$

where $g_i \in \mathbb{C}[u], h_i \in \mathbb{C}[u,z]$ and $g(0) = h(0,0) = 0$. By Puiseux's theorem (see [12, 113]), the equation $f_i(u,z) = 0$ (locally around the point $(0,0)$) admits $s_i :=$

$\mathrm{ord} g_i$ roots $u = u_{ij}(z), j = 1, 2, \ldots, s_i$. Note that $\sum_{i=1}^{p} s_i = d$. Moreover, these roots are usually written in the form (omitting the subscripts i, j for simplicity)

$$\sum_{k=1}^{k_0} c_{0k} z^k + c_{10} z^{\frac{m_1'}{n_1'}} + \sum_{k=1}^{k_1} c_{1k} z^{\frac{m_1'+k}{n_1'}} + \cdots + c_{g0} z^{\frac{m_g'}{n_1' n_2' \cdots n_g'}} + \sum_{k=1}^{\infty} c_{gk} z^{\frac{m_g'+k}{n_1' n_2' \cdots n_g'}},$$

with each pair (m_i', n_i') relatively prime and

$$\frac{m_1'}{n_1'} < \frac{m_2'}{n_1' n_2'} < \cdots,$$

and then $(m_1', n_1'), \ldots, (m_g', n_g')$ are called *the Puiseux pairs* for the corresponding root.

Let $\varphi_{ij}'(z) := -c_i + u_{ij}(z)$. Then $x = \varphi_{ij}'(z)$ is a root of the curve $\overline{C}$ with the equation $z^d P\left(\frac{x}{z}, \frac{1}{z}\right) = 0$, locally around the point a_i. Moreover, it is easy to see that the fractional power series $x = y\varphi_{ij}'(1/y)$ satisfies the conditions of the proposition, where

$$\begin{aligned} n_j &= n_j', \\ m_j &= n_1' n_2' \cdots n_j' - m_j', \quad j = 1, 2, \ldots, g. \end{aligned}$$

Hence the proposition follows. ∎

Definition 2.18. *For a fractional power series $x = \varphi(y)$, which is expressed as in Proposition* 2.17*, we call the pairs*

$$(m_1, n_1), (m_2, n_2), \ldots, (m_g, n_g)$$

the Puiseux pairs at infinity of φ.

Now we can pass to the proof of Theorem 2.15.

Proof of Theorem 2.15. This proof is due to [59]. Without loss of generality, we can assume that the polynomial P has the form (2.1).

We shall show that each of conditions (i)-(iii) is equivalent to the following condition

(iv) *There is a Puiseux root at infinity of the polar curve $\{\frac{\partial P}{\partial x} = 0\}$ such that*

$$P(\varphi(y), y)) - t_0 \to 0, \quad \textit{as} \quad y \to \infty.$$

The implication (iv) $\Leftrightarrow$ (i) follows from Theorem 2.5.

(iv) $\Rightarrow$ (ii): In fact, let φ be a Puiseux root at infinity of the polar curve $\{\frac{\partial P}{\partial x} = 0\}$ such that

$$P(\varphi(y), y) - t_0 = cy^{\alpha} + \text{terms of degrees less than } \alpha, \quad \alpha < 0, \quad c \neq 0.$$

An easy computation shows that

$$\mathrm{grad}P(\varphi(y), y) = (0, \alpha c y^{\alpha-1} + \cdots).$$

Hence $|y|\|\mathrm{grad}P(\varphi(y), y)\| \to 0$ as $y \to \infty$.

On the other hand, since $P(x, y)$ is monic in x, so is $\frac{\partial P}{\partial x}$. Consequently, there exist positive numbers c_1, c_2 such that

$$c_1\|(\varphi(y), y)\| \leqslant |y| \leqslant c_2\|(\varphi(y), y)\|, \quad |y| \gg 1.$$

Thus we have (ii).

(ii) $\Rightarrow$ (iii): It is clear.

(iii) $\Rightarrow$ (iv): Assume that $\mathrm{L}_{\infty,t_0}(P) < 0$. This means that there exists a sequence of points $(x_k, y_k) \in \mathbb{C}^2$ such that the following conditions hold:

$$\|(x_k, y_k)\| \to \infty, \quad P(x_k, y_k) \to t_0 \quad \text{and} \quad \|\mathrm{grad}P(x_k, y_k)\| \to 0.$$

Using the Curve Selection Lemma at infinity [79], we can assume that (x_k, y_k) lies on an analytic curve

$$\lambda : x = c_1 s^{n_1} + c_2 s^{n_2} + \ldots, \quad y = s^{-N},$$

where $s \to 0, N > 0, n_1 < n_2 < \cdots$ (n_1 need not be positive). We must have $n_1 + N \geqslant 0$, since x_k/y_k is bounded. We can rewrite λ as a fractional power series

$$\lambda : x = c_1 y^{-n_1/N} + c_2 y^{-n_2/N} + \cdots, \quad -N \leqslant n_1 < n_2 \cdots.$$

Let us apply the change of variables

$$X := x - \lambda(y), Y := y^{-1}$$

to $P(x, y) - t_0$, yielding

$$M(X, Y) := P(X + \lambda(1/Y), 1/Y) - t_0 = \sum c_{ij} X^i Y^{j/N}.$$

For each $c_{ij} \neq 0$, let us plot a dot at $(i, j/N)$, called a *Newton dot*. The set of Newton dots is called the *Newton diagram* of M. Clearly, it has at most finitely many dots lying on or below the X-axis. Moreover, there is one dot at $(d, 0)$ because P is monic in the variable x.

The assumption $P(\lambda(y), y) - t_0 \to 0$ means that $M(0, Y) \to 0$ as $Y \to 0$. As a consequence, all Newton dots of $M(0, Y)$ lie above the X-axis.

If $M(X, Y)$ has no dots on $X = 1$, then

$$0 = \frac{\partial M}{\partial X}(0, Y) = \frac{\partial P}{\partial x}(\lambda(y), y)$$

and we have (iv). If it is not the case, let $(1, h_1)$ denote the lowest Newton dot on $X = 1$. We must have $h_1 > 0$, since otherwise $\frac{\partial M}{\partial X}(0, Y) = \frac{\partial P}{\partial x}(\lambda(y), y)$ does not tend to zero when y tends to infinity.

Now the idea of the proof is to use the Newton dots on or bellow the X-axis to "swallow" the Newton dot $(1, h_1)$. Let us consider this idea on the following example

$$M(X, Y) := Y^3 - 2XY + X^3Y^{-1} + X^4.$$

The dot $(1, 1)$ represents $-2XY$. We use the dot $(3, -1)$ (which represents the monomial X^3Y^{-1}) to swallow $(1, 1)$.

Let us take a root $c \neq 0$ of $\frac{\partial}{\partial z}(z^3 - 2z) = 0$. So $c = \sqrt{\frac{2}{3}}$ and let $\gamma = \sqrt{\frac{2}{3}}Y$. Then, by an easy calculation, we see that the lowest Newton dot on $X = 1$ of $M(X + \sqrt{\frac{2}{3}}Y, Y)$ is higher than $(1, 1)$. On $X = 0$, all dots remain above the X-axis.

In the general case, let $M(X, Y) = \sum c_{ij}X^iY^{j/N}$. The Newton diagram of $\frac{\partial M}{\partial X}$ is obtained from the Newton diagram of M by a shift by one unit to the left. All Newton dots of M on $X = 0$ disappear. Let E_H be the highest Newton edge of $\dfrac{\partial M}{\partial X}$ (i.e., E_H is the compact edge to right of the highest Newton dot of $\frac{\partial M}{\partial X}$). Then let us collect all the terms of $M(X, Y)$ corresponding to E_H :

$$\varphi_H(X, Y) := \sum_{(i-1, j/N) \in E_H} c_{ij}X^iY^{j/N}.$$

Take any root c ($c \neq 0$) of $\frac{\partial}{\partial z}\varphi_H(z, 1) = 0$ and let θ_H be the angle between the edge E_H and the X-axis. In the expansion of $\varphi_H(X + cY^{\tan\theta_H}, Y)$, the term XY^{h_1} has coefficient 0, since $\frac{\partial}{\partial z}\varphi_H(c, 1) = 0$. Let $\gamma_1(Y) := \gamma_0(Y) + cY^{\tan\theta_H}$, here $\gamma_0(Y) := \lambda(1/Y)$. We say that $\gamma_1(Y)$ is the result of the *sliding (at infinity) of* γ_0 *along* $\frac{\partial M}{\partial X}$.

We see that the lowest Newton dot on $X = 1$ of $M_1(X, Y) := M(X + \gamma_1(Y), Y)$ is higher than $(1, h_1)$. On $X = 0$, all dots remain above the X-axis.

A recursive sliding $\gamma_0 \to \gamma_1 \to \gamma_2 \to \cdots$, will then yield a Puiseux root at infinity γ of the polar curve $\frac{\partial P}{\partial x} = \frac{\partial M}{\partial X} = 0$, for which

$$\widetilde{M}(X, Y) := M(X + \gamma(Y), Y)$$

has no dots on $X = 1$, and dots on $X = 0$ all lie above the X-axis.

An easy calculation, using the Chain Rule, yields

$$\begin{aligned} y\frac{\partial P}{\partial x} &= Y^{-1}\frac{\partial \widetilde{M}}{\partial X}, \\ y\frac{\partial P}{\partial y} &= Y\frac{\partial \widetilde{M}}{\partial Y} - Y\gamma'(Y)\frac{\partial \widetilde{M}}{\partial X}, \end{aligned}$$

whence Condition (iv) holds along the curve $(x := \gamma(1/y), y)$ for $|y| \gg 1$.

All is now proven. ■

We end this section with a formula for $\mathrm{L}_{\infty,t_0}(P), t_0 \in B_\infty(P)$, in terms of Puiseux roots at infinity of the curve $\{P(x,y) = t_0\} \subset \mathbb{C}^2$.

Assume that P has the form (2.1). Then, by Proposition 2.17, the curve $\{P(x,y) = t_0\}$ has d Puiseux roots at infinity $\varphi_1, \varphi_2, \ldots, \varphi_d$, say.

For each root φ_i, let

$$\rho_i := \min_{j \neq i} \mathrm{val}(\varphi_i(y) - \varphi_j(y)).$$

Now let $\psi_i(y)$ denote φ_i with its terms of degree ρ_i replaced by ξy^{ρ_i}, ξ a generic number (or an indeterminant), and all lower-order terms omitted.

The following formula is analogous to the formula for the local Lojasiewicz exponent of the gradient of P, given in [58]:

Theorem 2.19. [43, 44] *Let $t_0 \in \mathbb{C}$ be such that the fiber $P^{-1}(t_0)$ is reduced. If t_0 is a critical value at infinity of P, then*

$$L_{\infty,t_0}(P) = \min_{i=1,2,\ldots,d} \mathrm{val}(P(\psi_i(y), y) - t_0) - 1.$$

2.7. Topological Triviality at Infinity, M-tameness and the Lojasiewicz Number at Infinity

Let $P\colon \mathbb{C}^n \to \mathbb{C}$ be a polynomial function. It is important to understand the meaning of $B_\infty(P) = \emptyset$ (hence $B(P) = K_0(P)$). There are some special cases when the polynomial has no critical values at infinity: Pham [89] and Fedoryuk [27] have imposed lower bound conditions for $\|\mathrm{grad}P(x)\|$ for large values $\|x\|$, Kushnirenko has proved this in [61] for convenient polynomials with nondegenerate Newton principal part at infinity, Broughton [13, 14] for "tame" polynomials and Némethi and Zaharia [78, 79] for the larger class of "quasitame" and "M-tame" polynomials. Let us recall the last definition from [78]. Put

$$M(P) := \{x \in \mathbb{C}^n \mid \exists \lambda \in \mathbb{C} \text{ such that } \mathrm{grad}P(x) = \lambda x\}.$$

Geometrically, a point $x \in M(P)$ if and only if either x is a critical point of P, or x is not a critical point of P, but the fiber $P^{-1}(P(x))$ does not intersect transversally, at x, the sphere $\{y \in \mathbb{C}^n \mid \|y\| = \|x\|\}$. The polynomial P is called *M-tame* if for any sequence $\{x^k\} \subset M(P)$ such that $\lim_{k\to\infty} \|x^k\| = \infty$, then $\lim_{k\to\infty} P(x^k) = \infty$. Then the topological triviality at infinity is related to M-tameness as follows.

Proposition 2.20. [78] *Let $P\colon \mathbb{C}^n \to \mathbb{C}$ be a non-constant polynomial. If P is M-tame then $B_\infty(P) = \emptyset$.*

Proof. Indeed, take any $t_0 \in \mathbb{C}$ and let $D_\delta(t_0) := \{t \in \mathbb{C} \mid |t - t_0| < \delta\}$ be a small open disc centered at t_0. Since P is M-tame, we can find a smooth vector field $v(x)$ on $P^{-1}(D_\delta(t_0)) \cap \{\|x\| \gg 1\}$ such that $\langle v(x), x\rangle = 0$ and $\langle v(x), \mathrm{grad} P(x)\rangle = 1$. Now using the solution of the differential equation $\frac{dx}{dt} = v(x)$ we obtain that the restriction

$$P^{-1}(D_\delta(t_0)) \cap \{\|x\| \gg 1\} \to D_\delta(t_0)$$

is a trivial fibration. Hence, t_0 is a regular value at infinity of P. ■

On the other hand, in order to know when the set $B_\infty(P)$ is empty, we consider the Lojasiewicz number at infinity $\mathrm{L}_\infty(P)$ which measures the asymptotic growth at infinity of the gradient of P. Precisely, when P has non-isolated critical points, we let $\mathrm{L}_\infty(P) := -\infty$. If P has only isolated critical points, we define $\mathrm{L}_\infty(P)$ by

$$\mathrm{L}_\infty(P) := \lim_{r \to \infty} \frac{\ln \varphi(r)}{\ln r},$$

where $\varphi(r) := \inf_{\|x\|=r} \|\mathrm{grad} P(x)\|$.

Definition 2.21. *We say that* $\mathrm{L}_\infty(P)$ *is the Lojasiewicz number at infinity of* P.

One can easily show that the number $\mathrm{L}_\infty(P)$ is the smallest upper bound of the set of all real numbers $l > 0$ which satisfy the condition: there exists a positive constant c such that

$$\|\mathrm{grad} P(x)\| \geqslant c\|x\|^l \quad \text{for } \|x\| \gg 1.$$

Moreover, directly from definitions we get easily that

Proposition 2.22. [43] (see also, [20]) *Let* $P\colon \mathbb{C}^n \to \mathbb{C}, n \geqslant 2$, *be a non-constant polynomial function. If* $\mathrm{L}_\infty(P) \leqslant -1$, *then there exists* $t_0 \in \mathbb{C}$ *such that*

$$\mathrm{L}_\infty(P) = \mathrm{L}_{\infty,t_0}(P).$$

The following result gives, in the case $n = 2$, a complete answer of the question mentioned above.

Theorem 2.23. [43] *Let* $P\colon \mathbb{C}^2 \to \mathbb{C}$ *be a non-constant polynomial. Then the following conditions are equivalent*
(i) $B_\infty(P) = \emptyset$.
(ii) *The polynomial* P *is M-tame.*
(iii) $L_\infty(P) > -1$.

Proof. (i) ⇒ (iii) Assuming the contrary and using Proposition 2.22 we can find $t_0 \in \mathbb{C}$ such that

$$\mathrm{L}_{\infty,t_0}(P) = \mathrm{L}_\infty(P) \leqslant -1.$$

By Theorem 2.15, we obtain that $t_0 \in B_\infty(P)$, which is a contradiction.

(iii) $\Rightarrow$ (ii) Assuming the contrary and using Curve Selection Lemma at infinity ([79, 77]) we can find a real analytic curve $(\lambda(\tau), \varphi(\tau)) \colon (0, \epsilon) \to \mathbb{C} \times \mathbb{C}^n$ such that

$$\lim_{\tau \to 0} \|\varphi(\tau)\| = \infty, \ \mathrm{grad} P(\varphi(\tau)) = \lambda(\tau)\varphi(\tau), \quad \text{and} \quad \lim_{\tau \to 0} P(\varphi(\tau)) \in \mathbb{C}.$$

Then it is not hard to see that

$$\lim_{\tau \to 0} \|\varphi(\tau)\| \|\mathrm{grad} P(\varphi(\tau))\| = 0.$$

Hence, it follows from the definition of $\mathrm{L}_\infty(P)$ that $\mathrm{L}_\infty(P) \leqslant -1$, a contradiction.

(ii) $\Rightarrow$ (i) follows from Proposition 2.20. ∎

Remark 2.24. It was proved in [18, 19, 43] that for every polynomial function $P \colon \mathbb{C}^2 \to \mathbb{C}$, the Lojasiewicz number at infinity $\mathrm{L}_\infty(P)$ is either $-\infty$ or a rational number different from -1. Moreover, for each rational number $r \neq -1$, there exists a polynomial function $P \colon \mathbb{C}^2 \to \mathbb{C}$ such that $\mathrm{L}_\infty(P) = r$.

2.8. Links at Infinity

Another way to compute $B_\infty(P)$ is to consider links at infinity of complex affine plane curves. Indeed, let $P \colon \mathbb{C}^2 \to \mathbb{C}$ be a polynomial function and $t \in \mathbb{C}$. The intersection of the fiber $C_t := P^{-1}(t)$ with any sufficiently large sphere $S^3 := \{(x, y) \in \mathbb{C}^2 \mid |x|^2 + |y|^2 = R^2\}$ is transverse, and gives a well-defined link up to isotopy, $\mathcal{L}(C_t, \infty) := (S^3, S^3 \cap C_t)$, called the *link at infinity of C_t*.

In [80] it was shown that the embedded smooth topology of the generic fiber C_t of P is determined by the topological type of its link at infinity. It was conjectured that the same is true for any smooth fiber, but counter-examples have been found by Artal ([2, 3]). Nevertheless, the topology of link at infinity of any reduced fiber C_t of P determines a lot of information about C_t and P. For example, it determines the Euler characteristic of C_t, corrected by the Milnor numbers of the singularities of C_t if C_t is singular. It also determines the polynomial degree of P to extent possible, namely the set of degrees of polynomial maps $P \circ \Phi$ with $\Phi \colon \mathbb{C}^2 \to \mathbb{C}^2$ a polynomial automorphism of $\mathbb{C}^2$ (replacing P by $P \circ \Phi$ does not change the topology of P). We send to [80, 82] for more details.

In this section we will give a relationship between critical values at infinity and "irregular links at infinity". To describe our next main result we need a quick review of splice diagrams for toral links; more details are given in [26, 80].

Let $\overline{C}_t$, as previously, be a compactification of the fiber $C_t := P^{-1}(t)$. The projective curve $\overline{C}_t$ meets the line at infinity $H_\infty := \{z = 0\}$ in $a_1, a_2, \ldots, a_p$. Let D_0 be a 2-disk in H_∞ which contains $\overline{C_t} \cap H_\infty$ and D 4-disk neighborhood of D_0 in $\mathbb{P}^2$ whose boundary $S := \partial D$ meets $\overline{C_t} \cup H_\infty$ transversally. Then $\mathcal{L}_0 := (S, (H_\infty \cup S) \cap \overline{C_t})$ is a link which can be represented by a splice diagram Γ as follows:

Here K_0 is the component $S \cap H_\infty$ and each $\leftarrow\boxed{\Gamma_i}$ is the diagram representing the local link of $H_\infty \cup \overline{C_t}$ at the point a_i.

Let NH_∞ be a closed tubular neighborhood of H_∞ in $\mathbb{P}^2$ whose boundary $S^3 = \partial NH_\infty$ is the sphere at infinity and $\mathcal{L}' := (S^3, S^3 \cap C_t)$ is, but for orientation, the link at infinity that interests us. We may assume that NH_∞ is obtained from D by attaching a 2-handle along K_0, so $\mathcal{L}'$ is obtained from $\mathcal{L}_0$ by $(+1)$-Dehn surgery on $K_0 \subset S = \partial D$.

As in [80], we call a weight in the splice diagram *near* or *far* according as it is on the near or far of its edge, viewed from K_0. As described in [80], the splice diagram for $\mathcal{L}'$ is

where Γ'_i is obtained from Γ_i by replacing each far weight b_v by $b_v - \lambda_v^2 a_v$ with a_v equal to the product of the near weights at vertex v and λ_v the product of the weights adjacent to, but not on, the simple path from v to the vertex corresponding to K_0.

Finally, we must reverse orientation to consider S^3 as a large sphere in $\mathbb{C}^2$ rather than as ∂NH_∞. The effect is to reverse the signs of all near weights. We can then forget the leftmost vertex, which is redundant, to get a diagram Ω for $\mathcal{L}(C_t, \infty)$ as follows:

The leftmost vertex of Ω is called the *root vertex*. Ω is an *(RPI) splice diagram* in the sense of [26] (see also [80]).

We have three types of vertices: *arrowheads* (corresponding to components of $\mathcal{L}(C_t, \infty)$), *leaves* (non-arrowheads of valency (number of incident edges) 1) and *nodes* (non-arrowheads of valency > 2).

There are certain curves in $S^3 - (S^3 \cap C_t)$ associated to the splice diagram Ω for the link $\mathcal{L}(C_t, \infty)$, as follows. To a node v of Ω we associate the curve S_v that would be added to $S^3 \cap C_t$ by adding an additional arrow $\xrightarrow{1}$ at that node. To a leaf v of Ω we associate the curve S_v that would result by replacing this leaf by an arrowhead. These curves S_v are called *virtual components* of the link $\mathcal{L}(C_t, \infty)$. In particular, $S_o = K_0$, where o denotes the root vertex. For any non-arrowhead vertex of Ω, the linking number

$$l_v := \text{link}(S_v, S^3 \cap C_t)$$

(sum of linking numbers of S_v with all components of $\mathcal{L}(C_t, \infty)$) is called the (*total*) *linking coefficient* at vertex v (called "multiplicity" in [80]). The linking coefficient l_o at the root vertex is the degree d of the defining polynomial.

Definition 2.25. [80] Ω *is a* (*RPI*) *regular splice diagram if* $l_v \geqslant 0$ *for all non-arrowhead vertices.*

As proved in [80], the regularity or irregularity of Ω is a topological property of $\mathcal{L}(C_t, \infty)$, and we say the toral link $\mathcal{L}(C_t, \infty)$ is *regular* or *irregular* accordingly. In [80] it also was shown that $\mathcal{L}(C_t, \infty)$ is a regular link if t is a regular value at infinity, and the converse was conjectured. The purpose of this section is to give a proof of this using Puiseux expansions at infinity of the curve with the equation $P(x, y) = t$. We shall prove:

Theorem 2.26. [44, 80] (see also [81, 82]) *Let $P\colon \mathbb{C}^2 \to \mathbb{C}$ be a polynomial function and let $t_0 \in \mathbb{C}$ be a regular value of P. Then the following conditions are equivalent*
(i) *t_0 is a critical value at infinity of P.*
(ii) *$\mathcal{L}(P^{-1}(t_0), \infty)$ is an irregular link at infinity.*

The main idea of the proof of the implication (i) $\Rightarrow$ (ii) comes out from the observations that: (1) t_0 is a critical value at infinity of P if and only if $\mathrm{L}_{\infty,t_0}(P) < -1$ (Theorem 2.15); (2) the number $\mathrm{L}_{\infty,t_0}(P)$ is computed in terms of Puiseux roots at infinity of the curve $C_{t_0} := \{P(x,y) = t_0\}$ (Theorem 2.19); and (3) the splice diagram Ω for the link at infinity $\mathcal{L}(C_{t_0}, \infty)$ can be constructed from Puiseux roots at infinity of C_{t_0}.

In fact, let $x = \varphi(y)$ be a Puiseux root at infinity of C_{t_0}. Then it is well-known (see Proposition 2.17) that $\varphi'(z) := z\varphi(1/z)$ is a Puiseux root of $\overline{C}_{t_0}$ (locally at a certain point on the line at infinity $\{z = 0\}$). Moreover, if

$$(m_1, n_1), (m_2, n_2), \dots, (m_g, n_g)$$

are the Puiseux pairs at infinity of φ, then the Puiseux pairs

$$(m'_1, n'_1), (m'_2, n'_2), \dots, (m'_g, n'_g)$$

of φ' are given by the formulas

$$\begin{aligned} n'_j &= n_j, \\ m'_j &= n_1 n_2 \cdots n_j - m_j, \quad j = 1, 2, \dots, g. \end{aligned}$$

Let $r_i, i = 1, 2, \dots, p$, be the number of irreducible components of the projective curve $\overline{C}_{t_0}$ at the point a_i. Then all roots of the equation $z^d P\left(\frac{x}{z}, \frac{1}{z}\right) - t_0 z^d = 0$ (locally at the point a_i) are divided onto r_i branches, or places, $M'_{il}, l = 1, 2, \dots, r_i$; two roots which differ by a change of variable of the form $z \mapsto \epsilon z$, with ϵ a root of unity, may describe the same branch (see [113, Therorem 4.1, p. 107]).

Correspondingly, all Puiseux roots at infinity of the curve C_{t_0} are also divided onto *branches at infinity* $M_{il}, l = 1, 2, \dots, r_i, i = 1, 2, \dots, p$, say. For a Puiseux root at infinity, φ, of the curve C_{t_0} we rewrite it so as to display its characteristic pairs, as in Proposition 2.17,

$$c_i y + \sum_{k=-1}^{-k_0} c_{0k} y^k + c_{10} y^{\frac{m_1}{n_1}} + \sum_{k=-1}^{-k_1} c_{1k} y^{\frac{m_1+k}{n_1}} + \dots + c_{g0} y^{\frac{m_g}{n_1 n_2 \cdots n_g}} + \sum_{k=-1}^{-\infty} c_{gk} y^{\frac{m_g+k}{n_1 n_2 \cdots n_g}},$$

where the symbols have the following significance: $n_i > 1, \frac{m_1}{n_1} > \frac{m_2}{n_1 n_2} > \cdots$, and each pair (m_i, n_i) relatively prime. The branch M_{il}, containing φ, consists of the following $N := n_1 n_2 \dots n_g$ roots

$$\begin{aligned} c_i \epsilon^{\nu N} y + \sum_{k=-1}^{-k_0} c_{0k} \epsilon^{\nu N} y^k + c_{10} \epsilon^{\frac{\nu N m_1}{n_1}} y^{\frac{m_1}{n_1}} + \cdots \\ + c_{g0} \epsilon^{\frac{\nu N m_g}{n_1 n_2 \cdots n_g}} y^{\frac{m_g}{n_1 n_2 \cdots n_g}} + \cdots, \quad 0 \leqslant \nu < N, \end{aligned}$$

where ϵ is an N^{th} primitive root of unity. Then the pairs $(m_i, n_i), i = 1, 2, \ldots, g$, might well be called the *Puiseux pairs at infinity* of the branch M_{il}. It is worth noting that the number of branches at infinity is exactly the number of components of the link at infinity $\mathcal{L}(C_{t_0}, \infty)$, hence there is a certain component, say S_{il}, associated to a branch at infinity M_{il} of the link $\mathcal{L}(C_{t_0}, \infty)$, and vice versa. We shall say that S_{il} is represented by M_{il} and call N the *order* of S_{il} : $O(S_{il}) := N$.

In [26, Appendix to Chapter 1] the authors described how a splice diagram for an algebraic link may be derived from Puiseux expansions. This method can be adapted to Puiseux expansions at infinity. So we can get a splice diagram Ω' as follows.

Here each Ω'_i is the splice diagram representing the branches at infinity $M_{i1}, M_{i2}, \ldots, M_{ir_i}$.

For example, consider the case $r_i = 1$ (i.e., the curve $\overline{C}_{t_0}$ is irreducible at the point a_i). Then Ω'_i is a graph of the form

where $(m_1, n_1), (m_2, n_2), \ldots, (m_g, n_g)$ are the Puiseux pairs at infinity of the branch M_{il}. Moreover, the numbers α_i above are given by the formulas

$$\alpha_1 = m_1,$$
$$\alpha_{i+1} = m_{i+1} - m_i n_{i+1} + n_i n_{i+1} \alpha_i, \quad i > 1.$$

By induction, one deduces that

Lemma 2.27. *We have for $i > 1$,*

$$\alpha_i = m_i + n_i(n_{i-1} - 1)m_{i-1} + n_i n_{i-1}^2 (n_{i-2} - 1)m_{i-2} + \cdots + n_i n_{i-1}^2 \ldots n_2^2 (n_1 - 1)m_1.$$

Lemma 2.28. *The above splice diagram Ω' is the splice diagram at infinity Ω for the link at infinity of the curve with the equation $P(x,y) = t_0$.*

Proof. Clearly, Ω' has the same shape as Ω. Moreover, it follows directly from the effect of $(+1)-$Dehn surgery on graph links that all weights on these two diagrams are the same (see [80]). ■

We are now ready to prove the main result of this section.

Proof of Theorem 2.26. That (ii) $\Rightarrow$ (i) was proved in [80]. It remains to show that (i) $\Rightarrow$ (ii). Without loss of generality, we can assume that the polynomial P has the form (2.1). Then, by Proposition 2.17, $P(x,y) - t_0$ can be factorized into a product of fractional power series

$$P(x,y) - t_0 = \prod_{i=1}^{d}(x - \varphi_i(y)), \qquad |y| \gg 1,$$

where $\varphi_1, \varphi_2, \ldots, \varphi_d$ are Puiseux roots at infinity of the curve $C_{t_0} := \{P(x,y) = t_0\} \subset \mathbb{C}^2$.

For each root φ_i, let

$$\rho_i := \min_{j \neq i} \operatorname{val}(\varphi_i(y) - \varphi_j(y)).$$

Now let ψ_i denote φ_i with its terms of degree ρ_i replaced by ξy^{ρ_i}, ξ a generic number (or an indeterminant), and all lower-order terms omitted.

Then, since $t_0 \in B_\infty(P)$, it follows from Theorems 2.15 and 2.19 that

$$-1 > \mathrm{L}_{\infty,t_0}(P) = \min_{i=1,2,\ldots,d} \operatorname{val}\,(P(\psi_i(y), y) - t_0) - 1.$$

Thus, without loss of generality, we may assume that

$$0 > \operatorname{val}\,(P(\psi_1(y), y) - t_0).$$

For simplicity, let $M_j, j = 1, 2, \ldots, q$, denote the branches at infinity of C_{t_0}, and for each j, let S_j be the component of link at infinity $\mathcal{L}(C_{t_0}, \infty)$ corresponding to M_j.

By permuting the indices if necessary, we may assume $\varphi_1 \in M_1$. Let v_1 be the node of the splice diagram Ω for the link $\mathcal{L}(C_{t_0}, \infty)$ such that v_1 is adjacent to the arrowhead vertex, which represents the component S_1. Let S_{v_1} denote the virtual component represented by the node v_1.

Assume that we have proved:

Claim 2.29. *For each $j = 1, 2, \ldots, q$, we have*

$$\frac{\operatorname{link}(S_{v_1}, S_j)}{O(S_1)} \leqslant \sum_{\varphi \in M_j} \operatorname{val}(\varphi(y) - \psi_1(y)),$$

where the (finite) sum is taken over all the elements $\varphi \in M_j$.

This, of course, implies that

$$\begin{aligned}
\operatorname{link}(S_{v_1}, S^3 \cap C_{t_0}) &= \sum_{j=1}^{q} \operatorname{link}(S_{v_1}, S_j) \\
&\leqslant \sum_{j=1}^{q} \sum_{\varphi \in M_j} \operatorname{val}(\varphi(y) - \psi_1(y)) O(S_1) \\
&= \operatorname{val}(P(\psi_1(y), y) - t_0) O(S_1) < 0,
\end{aligned}$$

which proves the theorem.

So we are left with proving Claim 2.29.

Take any $j \in \{1, 2, \dots, q\}$. Suppose that the branches at infinity M_1 and M_j correspond to the following two sequences of Puiseux pairs at infinity:

$$M_1 : (m_1, n_1), (m_2, n_2), \dots, (m_g, n_g),$$

$$M_j : (\widetilde{m}_1, \widetilde{n}_1), (\widetilde{m}_2, \widetilde{n}_2), \dots, (\widetilde{m}_{\widetilde{g}}, \widetilde{n}_{\widetilde{g}}).$$

Suppose, moreover, that the numbers α_i (respectively, $\widetilde{\alpha}_i$) are given by Lemma 2.27.

There are two possibilities:

Case 1. *The branches* M_1 *and* M_j *correspond to two distinct points on the line at infinity* $\{z = 0\}$.

It is easy to check, in this case, that

$$\operatorname{val}(\varphi(y) - \psi_1(y)) = 1 \quad \text{for all} \quad \varphi \in M_j.$$

This gives

$$\sum_{\varphi \in M_j} \operatorname{val}(\varphi(y) - \psi_1(y)) = \widetilde{n}_1 \widetilde{n}_2 \dots \widetilde{n}_{\widetilde{g}} = O(S_j).$$

On the other hand, it follows from [80, Lemma 3.2] that

$$\operatorname{link}(S_{v_1}, S_j) = O(S_1) O(S_j).$$

These two equations prove the claim in Case 1.

Case 2. *The branches* M_1 *and* M_j *correspond to the same point on the line at infinity* $\{z = 0\}$.

If $j = 1$, then, by definition, the node v_1 has the form

Hence, by [80, Lemma 2.3] again,

$$\text{link } (S_{v_1}, S_1) = n_g \alpha_g. \tag{2.3}$$

On the other hand, it follows from the definition of M_1 that

- There are exactly n_g elements $\varphi \in M_1$ such that

$$\text{val } (\varphi(y) - \psi_1(y)) = \frac{m_g}{n_1 n_2 \dots n_g}.$$

- For each $i < g$, there are exactly $n_g \dots n_{i+1}(n_i - 1)$ elements $\varphi \in M_1$ such that

$$\text{val } (\varphi(y) - \psi_1(y)) = \frac{m_i}{n_1 n_2 \dots n_i}.$$

Thus

$$\begin{aligned}
\sum_{\varphi \in M_1} \text{val}(\varphi(y) - \psi_1(y)) &= n_g \frac{m_g}{n_1 \dots n_g} + n_g(n_{g-1} - 1) \frac{m_{g-1}}{n_1 \dots n_{g-1}} \\
&\quad + \dots + n_g \dots n_2 (n_1 - 1) \frac{m_1}{n_1} \\
&= \frac{n_g}{n_1 \dots n_g} \Big[m_g + n_g(n_{g-1} - 1) m_{g-1} \\
&\quad + \dots + n_g n_{g-1}^2 \dots n_2^2 (n_1 - 1) m_1 \Big] \\
&= \frac{n_g \alpha_g}{n_1 n_2 \dots n_g}.
\end{aligned}$$

(The last relation follows from Lemma 2.27). This, together with Equation 2.3, implies that

$$\sum_{\varphi \in M_1} \text{val}(\varphi(y) - \psi_1(y)) = \frac{\text{link}(S_{v_1}, S_1)}{O(S_1)},$$

which prove the claim in the case $j = 1$.

We now suppose that $j \neq 1$. Let Γ_{1j} be the splice diagram for the link at infinity with the two components S_1 and S_j. There are several cases to consider (see [26, Appendix to Chapter 1]).

Case 2.1. $\eta = \widetilde{g} < g$ (the case $\eta = g < \widetilde{g}$ is analogous) and the splice diagram Γ_{1j} is of the form

Here $\widetilde{m}_i = m_i, \widetilde{n}_i = n_i$ and $\widetilde{\alpha}_i = \alpha_i$ for $i = 1, 2, \ldots, \eta$.

By [80, Lemma 2.3],

$$\operatorname{link}(S_{v_1}, S_j) = \alpha_{\eta+1} n_{\eta+2} \ldots n_g. \tag{2.4}$$

On the other hand, it is not hard to check that the following statements hold:

• There are at most one element $\varphi \in M_j$ such that

$$\frac{m_\eta}{n_1 n_2 \ldots n_\eta} > \operatorname{val}(\varphi(y) - \psi_1(y)) \geqslant \frac{m_{\eta+1}}{n_1 n_2 \ldots n_{\eta+1}}.$$

• There are exactly $(n_\eta - 1)$ elements $\varphi \in M_j$ such that

$$\operatorname{val}(\varphi(y) - \psi_1(y)) = \frac{m_\eta}{n_1 n_2 \ldots n_\eta}.$$

• For each $i = 1, 2, \ldots, \eta - 1$, there are exactly $n_\eta \ldots n_{i+1}(n_i - 1)$ elements $\varphi \in M_j$ such that

$$\operatorname{val}(\varphi(y) - \psi_1(y)) = \frac{m_i}{n_1 n_2 \ldots n_i}.$$

These relations imply that

$$\begin{aligned}
\sum_{\varphi \in M_j} \operatorname{val}(\varphi(y) - \psi_1(y)) \geqslant & \frac{m_{\eta+1}}{n_1 \ldots n_{\eta+1}} + (n_\eta - 1)\frac{m_\eta}{n_1 \ldots n_\eta} \\
& + \cdots + n_\eta \ldots n_2 (n_1 - 1)\frac{m_1}{n_1} \\
= & \frac{\alpha_{\eta+1}}{n_1 n_2 \ldots n_{\eta+1}} \\
= & \frac{\alpha_{\eta+1} n_{\eta+2} \ldots n_g}{n_1 n_2 \ldots n_g}.
\end{aligned}$$

Together with the Equation 2.4, we get the claim in this case.

Case 2.2. $\eta < g, \eta < \widetilde{g}$ and the splice diagram Γ_{1j} is of the form

Here $\widetilde{m}_i = m_i, \widetilde{n}_i = n_i$ and $\widetilde{\alpha}_i = \alpha_i$ for $i = 1, 2, \ldots, \eta$. By [80, Lemma 3.2], we have

$$\operatorname{link}(S_{v_1}, S_1) = \widetilde{n}_{\widetilde{g}} \ldots \widetilde{n}_{\eta+2} n_g \ldots n_{\eta+1} \widetilde{\alpha}_{\eta+1}.$$

Then the claim follows directly from the following statements:

• There are at most $\widetilde{n}_{\widetilde{g}} \ldots \widetilde{n}_{\eta+1}$ elements $\varphi \in M_j$ such that

$$\frac{\widetilde{m}_\eta}{\widetilde{n}_1 \widetilde{n}_2 \ldots \widetilde{n}_\eta} > \operatorname{val}(\varphi(y) - \psi_1(y)) \geqslant \frac{\widetilde{m}_{\eta+1}}{\widetilde{n}_1 \widetilde{n}_2 \ldots \widetilde{n}_\eta \widetilde{n}_{\eta+1}}.$$

• For each $i = 1, 2, \ldots, \eta$, there are exactly $\widetilde{n}_{\widetilde{g}} \ldots \widetilde{n}_{i+1}(\widetilde{n}_i - 1)$ elements $\varphi \in M_j$ such that

$$\operatorname{val}(\varphi(y) - \psi_1(y)) = \frac{\widetilde{m}_i}{\widetilde{n}_1 \widetilde{n}_2 \ldots \widetilde{n}_i}.$$

Case 2.3. $\eta < g, \eta < \widetilde{g}$ and the splice diagram Γ_{1j} has the form

The proof is similar for Case 2.1 and we will leave to the reader to verify this fact.

The proof of Theorem 2.26 is now complete. ■

3. n-Dimensional Case

In this section we shall study the general case of polynomials $P: \mathbb{C}^n \to \mathbb{C}$ for arbitrary $n \geqslant 2$. We shall see that in this case the "nice" properties of polynomials from $\mathbb{C}^2$ to $\mathbb{C}$ established in Sec.2 are no longer valid.

First of all, the following example shows that the constancy of Euler-Poincaré characteristic does not imply topological triviality.

Example 3.1. [5, 107] Let $P := x + x^2yz$. The polynomial function $P: \mathbb{C}^3 \to \mathbb{C}$ has no critical points. Moreover, homotopically, the fiber $P^{-1}(0)$ is the disjoint union of $\mathbb{C}^2$ with a torus $\mathbb{C}^* \times \mathbb{C}^*$, whereas the fiber $P^{-1}(t)$, for $t \neq 0$, is the union of the torus $\mathbb{C}^* \times \mathbb{C}^*$ with $\{1\} \times \mathbb{C}$. Therefore, the Euler-Poincaré characteristic of all fibers is equal to one. On the other hand, it is clear that the fiber $P^{-1}(0)$ is not topologically equivalent to any other fiber.

Let $P: \mathbb{C}^n \to \mathbb{C}$ be a polynomial function. In order to examine the set $B_\infty(P)$, one often constructs larger sets in which it is easier to study. There is a relation between such sets and the asymptotic growth at infinity of the gradient of P. For instance, let

$$\widetilde{K}_\infty(P) := \{t \in \mathbb{C} | \text{there exists a sequence } x^k \to \infty \text{ such that } P(x^k) \to t \text{ and } \|\mathrm{grad} P(x^k)\| \to 0\}.$$

If $t \notin \widetilde{K}_\infty(P)$, then we say that P satisfies *Fedoryuk's condition* at t (see [27]). If one looks for a weaker condition then it is natural to consider the set

$$K_\infty(P) := \{t \in \mathbb{C} | \text{there exists a sequence } x^k \to \infty \text{ such that } P(x^k) \to t \text{ and } \|x^k\| \|\mathrm{grad} P(x^k)\| \to 0\}.$$

If $t \notin K_\infty(P)$ then it is usual to say that P satisfies *Malgrange's condition* at t (see [62, 89]).

It is well-known (see, for example, [60]) that the set $K_\infty(P)$ is always finite. However, the set $\widetilde{K}_\infty(P)$ may be equal to $\mathbb{C}$, see the example below

Example 3.2. (see also, [60, 86]) Consider the homogeneous polynomial $P(x, y, z) := x^2y - xz^2 \in \mathbb{C}[x, y, z]$ and the curve

$$\varphi: (0, 1) \to \mathbb{C}^3, \quad \tau \mapsto (\tau^2, \frac{1}{2}\tau^{-4}, \tau^{-1}).$$

By an easy computation, we get

$$\lim_{\tau \to 0} \|\varphi(\tau)\| = \infty, \quad \lim_{\tau \to 0} P(\varphi(\tau)) = -\frac{1}{2} \quad \text{and} \quad \lim_{\tau \to 0} \|\mathrm{grad} P(\varphi(\tau))\| = 0.$$

Hence, $-\frac{1}{2} \in \widetilde{K}_\infty(P)$. Then, by virtue of the homogeneity of the polynomial P, we find that $\widetilde{K}_\infty(P) = \mathbb{C}$. Moreover, it is not difficult to see that $K_\infty(P) = \{0\}$.

The next result seems to be well-known, see [79]. Its proof can be easily obtained, by assuming the contrary.

Proposition 3.3. *For $n \geqslant 2$, let $P\colon \mathbb{C}^n \to \mathbb{C}$ be a polynomial function. If P satisfies Malgrange's condition for any $t \in \mathbb{C}$, then P is M-tame, and, in particular, $B_\infty(P) = \emptyset$.*

Let us pass to characterizations of the sets $\widetilde{K}_\infty(P)$ and $K_\infty(P)$ in terms of the Lojasiewicz number at infinity of the fiber $\mathrm{L}_{\infty,t}(P)$:

Proposition 3.4. [20, 43, 95] *The following relations hold*

$$\widetilde{K}_\infty(P) = \{t \in \mathbb{C} \mid L_{\infty,t}(P) < 0\},$$

$$K_\infty(P) = \{t \in \mathbb{C} \mid L_{\infty,t}(P) < -1\}.$$

Proof. It is an immediate consequence of definitions. ∎

As an application, Theorem 2.15 can be translated in the following way:

Corollary 3.5. *Let $P\colon \mathbb{C}^2 \to \mathbb{C}$ be a polynomial in two complex variables. Then*

$$B_\infty(P) = K_\infty(P) = \widetilde{K}_\infty(P).$$

In the general case, by standard arguments, we obtain

Theorem 3.6. [55, 60, 78, 85] *Let $P\colon \mathbb{C}^n \to \mathbb{C}$ be a polynomial function. Then*

$$B_\infty(P) \subset K_\infty(P) \subset \widetilde{K}_\infty(P).$$

Remark 3.7. It may happen that the above inclusions are strict for $n > 2$ (see, for instance, Examples 3.2, 3.8 and 3.9).

The following example shows that the characterization of critical values at infinity of polynomials in two variables in terms of the Lojasiewicz numbers is no longer valid in dimensions $n > 2$.

Example 3.8. [87] For $p, q \in \mathbb{N} - \{0\}$ we consider the polynomial functions

$$P_{p,q}\colon \mathbb{C}^3 \to \mathbb{C}, \quad (x, y, z) \mapsto x - 3x^{2p+1}y^{2q} + 2x^{3p+1}y^{3q} + yz.$$

Then we have

(i) There exists a polynomial automorphism $\Phi\colon \mathbb{C}^3 \to \mathbb{C}^3$ such that $P_{p,q} \circ \Phi(x, y, z) = x$. In particular, the fibers $P^{-1}(t)$ are smooth and $B_\infty(P_{p,q}) = \emptyset$.

(ii) $\mathrm{L}_\infty(P_{p,q}) = -\frac{p}{q}$.

(iii) Suppose that $p > q$. Then $\mathrm{L}_\infty(P_{p,q}) < -1$. This shows that Theorem 2.23 cannot be extended to the case of a polynomial function $P\colon \mathbb{C}^n \to \mathbb{C}$, when $n \geqslant 3$. Moreover, by Proposition 2.22, there exists $t_0 \in \mathbb{C}$ such that

$\mathrm{L}_{\infty,t_0}(P_{p,q}) = \mathrm{L}_\infty(P_{p,q}) < -1$. Hence, Theorem 2.15 is also no longer true for polynomials in $n \geqslant 3$ variables.

(iv) If $p = q$ then $\mathrm{L}_\infty(P_{p,q}) = -1$. (Compare Remark 2.24).

(v) $K_\infty(P) \neq \emptyset$ if and only if $p > q$. (Compare Corollary 3.5) .

The next example shows that the property of being M-tame depends on the algebraic coordinate system of $\mathbb{C}^n$. This contrasts deeply to the case $n = 2$, where, by Theorem 2.23, being M-tame is independent of the coordinate system.

Example 3.9. [88] Let $P\colon \mathbb{C}^4 \to \mathbb{C}$ be defined by

$$P(x, y, z, u) = x + y - 2x^2y^3 + x^3y^6 + zy^3 - z^2y^5 + uy^5.$$

Then the following statements hold (compare Theorem 2.23)

(i) The polynomial P is a fibration over $\mathbb{C}$; that is $K_0(P) = B_\infty(P) = \emptyset$.

(ii) The polynomial P is not M-tame, and thus $K_\infty(P) \neq \emptyset$; in fact $K_\infty(P) = \{0\}$.

As we shall see below that the results of Sec.2 are valid in any dimension for polynomials which have only "isolated singularities at infinity". We first need some preliminaries.

Let $P\colon \mathbb{C}^n \to \mathbb{C}$ be a polynomial of degree d and let $P = P_d + P_{d-1} + \cdots + P_0$, where P_j is homogeneous of degree j. Consider the homogenization of P :

$$\overline{P}(x_0, x_1, \ldots, x_n) := x_0^d P\left(\frac{x_1}{x_0}, \frac{x_2}{x_0}, \ldots, \frac{x_n}{x_0}\right)$$

and the hypersurface in $\mathbb{P}^n \times \mathbb{C}$ defined by

$$X := \{(x, t) \in \mathbb{P}^n \times \mathbb{C} \mid \overline{P}(x) - tx_0^d = 0\}.$$

Let H_∞ be the hyperplane at infinity of $\mathbb{P}^n$ defined by $\{x_0 = 0\}$. Recall after [23] or [85] that the singular locus of X is precisely $A \times \mathbb{C}$, where

$$A := \{x \in H_\infty \mid \frac{\partial P_d}{\partial x_1} = \cdots = \frac{\partial P_d}{\partial x_n} = P_{d-1} = 0\}.$$

The class of polynomials on which we want to mention is defined as follows:

Definition 3.10. [85, 86, 96] *We say that P has isolated singularities at infinity if A is a finite set.*

Remark 3.11. If P is a non-constant polynomial function of two variables, then P has always isolated singularities at infinity.

We have the following generalization of main results in Sec.2:

Theorem 3.12. [85, 86, 96] *Let $P\colon \mathbb{C}^n \to \mathbb{C}$ be a polynomial function with isolated singularities A at infinity. Let t_0 be a regular value of P. Then the following conditions are equivalent*

(i) *The value t_0 is a critical value at infinity of P.*

(ii) $\chi(P^{-1}(t_0)) \neq \chi(P^{-1}(t))$, *where $P^{-1}(t)$ is a generic fiber of P.*

(iii) *There exists a point $a \in A$ such that the family of isolated singularities $(\overline{P^{-1}(t)}, a \times t)$ is not μ-constant for t sufficiently close to t_0; here $\overline{P^{-1}(t)}$ denotes the compactification of the fiber $P^{-1}(t)$ in the complex projective space $\mathbb{P}^n$.*

(iv) *The polynomial P does not satisfy Malgrange's condition at t_0, i.e., $L_{\infty,t_0}(P) < -1$.*

From the above theorem, Proposition 3.3 and Remark 2.24 (ii) we obtain a generalization of Theorem 2.23:

Corollary 3.13. *Let $P\colon \mathbb{C}^n \to \mathbb{C}$ be a polynomial function with isolated singularities at infinity. Then the following are equivalent*

(i) $B_\infty(P) = \emptyset$;

(ii) *P is M-tame;*

(iii) $L_\infty(P) \geqslant -1$.

4. Final Remarks

The study of singularities at infinity might be useful while approaching the following:

4.1. Jacobian Conjecture

Let $P, Q \in \mathbb{C}[x, y]$ be such that the Jacobian

$$J(P,Q) := \begin{vmatrix} \dfrac{\partial P}{\partial x} & \dfrac{\partial P}{\partial y} \\ \dfrac{\partial Q}{\partial x} & \dfrac{\partial Q}{\partial y} \end{vmatrix}$$

is a nonzero constant. Then the map $(P, Q)\colon \mathbb{C}^2 \to \mathbb{C}^2$ is a polynomial automorphism, i.e., invertible with polynomial inverse.

This question, in a somewhat restricted form, was first formulated by Keller [56] in 1939, and has yet to receive a definite answer despite substantial amount of work that has been devoted to its solution.

The following is a characterization of the polynomial automorphisms of $\mathbb{C}^2$, in terms of singularities at infinity:

Proposition 4.1. *Let $P, Q \in \mathbb{C}[x, y]$. The map $(P, Q)\colon \mathbb{C}^2 \to \mathbb{C}^2$ is a polynomial automorphism if and only if*

(i) *the Jacobian $J(P, Q)$ is a nonzero constant; and*

(ii) *the polynomial P has no critical values at infinity*[3].

Proof. Indeed, the condition $\lambda := J(P,Q) \in \mathbb{C}^*$ implies that P has no critical points. This, together with the condition (ii), implies that the polynomial function P is a locally trivial fibration on $\mathbb{C}$; so that the generic fibers of P are isomorphic to the complex line. Therefore, in this situation, the Embedding Theorem of Abhyankar and Moh [1] shows that we can choose coordinates on $\mathbb{C}^2$ such that $P(x,y) \equiv x$. For this choice,

$$J(P,Q) = \begin{vmatrix} 1 & 0 \\ \dfrac{\partial Q}{\partial x} & \dfrac{\partial Q}{\partial y} \end{vmatrix} = \frac{\partial Q}{\partial y} = \lambda.$$

By integration, hence $Q(x,y) = \lambda y + R(x)$, where R is a complex polynomial function. This implies that the map $(P,Q)\colon \mathbb{C}^2 \to \mathbb{C}^2$ defined as $(x,y) \mapsto (x, \lambda y + R(x))$ is an automorphism of the complex plane, as announced. ∎

The Jacobian Conjecture speculates that the condition (i) suffices in Proposition 4.1, or, equivalently, that it implies the condition (ii). This suggests yet another approach to proving or disproving the Jacobian Conjecture. One may consult, for example, [18, 19, 21, 40, 51, 52, 70, 71, 83, 84, 96] for more details. More importantly, Proposition 4.1 and the results in Sec.2 provide a fairly simple and verifiable characterization of polynomial automorphisms irrespective of the truth of the Jacobian Conjecture.

Let us end this paper by posing some open questions.

Question 1. *Let $X \subset \mathbb{C}^n$ be an affine algebraic surface and let $P\colon X \to \mathbb{C}$ be a polynomial function. Find a criterion to decide whether a noncritical value is critical value of singularities at infinity or not.*

Question 2. *Let $P\colon \mathbb{C}^n \to \mathbb{C}^{n-1}$ be a polynomial map and $t \in \mathbb{C}^{n-1}$. Give necessary and sufficient conditions on t for it to be a critical value at infinity of P.*

It should be noticed, in the above two problems, that the fibers $P^{-1}(t)$ are of dimension 1; hence, we think that all results mentioned in this paper for polynomial maps in two complex variables could extend correspondingly for these cases.

Remark 4.2. Recently, Theorem 2.1 has been extended by the first author and Nguyen Tat Thang for complex polynomial functions on smooth affine algebraic surfaces, under the assumption that there is a so-called "very good projection".

We next denote by $\mathrm{Aut}(\mathbb{C}^n)$ the set of algebraic automorphisms of the affine space $\mathbb{C}^n$ of dimension n and let $P\colon \mathbb{C}^n \to \mathbb{C}$ be a complex polynomial function.

[3] Or, of course Q.

Then, it is well-known that if $B_\infty(P) \neq \emptyset$ then $B_\infty(P \circ \Phi) \neq \emptyset$ for all $\Phi \in \mathrm{Aut}(\mathbb{C}^n)$; and conversely, if $B_\infty(P) = \emptyset$ then $B_\infty(P \circ \Phi) = \emptyset$ for all $\Phi \in \mathrm{Aut}(\mathbb{C}^n)$. On the other hand, the Lojasiewicz number at infinity $\mathrm{L}_\infty(P)$ of P behaves only well relative to a fixed linear structure on $\mathbb{C}^n$; changing the polynomial P by a non-linear automorphism of $\mathbb{C}^n$ generally changes the Lojasiewicz number at infinity. These lead us to the following definition. Set

$$\mathrm{L}_{\infty,\mathrm{int}}(P) := \sup_{\Phi \in \mathrm{Aut}(\mathbb{C}^n)} \mathrm{L}_\infty(P \circ \Phi),$$

and we call it the *intrinsic Lojasiewicz number at infinity* for P after [80] and [15].

Question 3. *Let P be a complex polynomial function. Are the following statements equivalent?*
(i) $B_\infty(P) = \emptyset$.
(ii) $L_{\infty,\mathrm{int}}(P) > -1$.

References

1. S. Abhyankar and T. Moh, Embeddings of the line in the plane, *J. Reine Angrew. Math.* **267** (1975) 148-166.
2. E. Artal-Bartolo, Combinatoire et type topologique des applications polynomiales de $\mathbb{C}^2$ dans $\mathbb{C}$, *J. Enseign. Math., II.* **39** (1993) 211-224.
3. E. Artal-Bartolo, Sur les couples de Zariski, *J. Algebr. Geom.* **3** (1994) 223-247.
4. E. Artal Bartolo, P. CassouNogués, and A. Dimca, Sur la topologie des polynômes complexes, *Singularities (Oberwolfach, 1996), Progr. Math., Birkhäuser, Basel,* **162** (1998) 317-343.
5. E. Artal-Bartolo, I. Luengo, and A. Melle-Hernéndez, Milnor number at infinity, topology and Newton boundary of a polynomial function, *Math. Z.* **233**(2000) 679-696.
6. R. Benedetti and J. J. Risler, *Real Algebraic and Semi-algebraic Sets,* Hermann, 1990.
7. A. Bodin, Milnor fibration and fibred links at infinity, *Internat. Math. Res. Notices* **11** (1999) 615-621.
8. A. Bodin, Classification of polynomials from $\mathbb{C}^2$ to $\mathbb{C}$ with one critical value, *Math. Z.* **242** (2002) 303-322.
9. A. Bodin, Invariance of Milnor numbers and topology of complex polynomials, *Comment. Math. Helv.* **78** (2003) 134-152.
10. A. Bodin and M. Tibăr, Topological equivalence of complex polynomials, *Adv. Math.* **199** (2006) 136-150.
11. M. Boileau and L. Fourrier, Knot theory and plane algebraic curves, *Chaos Solitons Fractals* **9**, (1998) 779-792.
12. E. Brieskorn and H. Knorrer, *Plane Algebraic Curves,* Birkhauser, 1986.
13. S. A. Broughton, On the topology of polynomial hypersurfaces, Singularities, Summer Inst., Arcata Calif. 1981; *Proc. Symp. Pure Math.* **40** (1983) 167-178.

14. S. A. Broughton, Milnor numbers and the topology of polynomial hypersurfaces, *Invent. Math.* **92** (1988) 217-242.
15. P. Cassou-Noguès and Ha Huy Vui, Sur le nombre de Lojasiewicz à l'infini d'un polynôme, *Ann. Pol. Math.* **62** (1995) 23-44.
16. Cassou-Noguès P. and Ha Huy Vui, Théorème de Kuiper-Kuo-Bochnak-Lojasiewicz à l'infini, *Ann. Fac. Sci. Toulouse Math.* **5** (1996) 387-406.
17. P. Cassou-Noguès and A. Dimca, Topology of complex polynomials via polar curves, *Kodai Math. J.* **22** (1999) 131-139.
18. J. Chadzyński and T. Krasiński, On the Lojasiewicz exponent at infinity for polynomial mappings of $\mathbb{C}^2$ into $\mathbb{C}^2$ and components of polynomial automorphisms of $\mathbb{C}^2$, *Annales Polonici Mathematici,* **57** (1992) 291-302.
19. J. Chadzyński and T. Krasiński, Sur l'exposant de Lojasiewicz à l'infini pour les applications polynomiales de $\mathbb{C}^2$ dans $\mathbb{C}^2$ et les composantes des automorphismes polynomiaux de $\mathbb{C}^2$, *C. R. Acad. Sci., Paris, Série. I.* **315** (1992) 1399-1402.
20. J. Chadzyński and T. Krasiński, The gradient of a polynomial at infinity, *Kodai Math. Journal,* **26** (2003) 317-339.
21. A. D. R. Choudary, Topology of complex polynomials and Jacobian Conjecture, *Topology and its Applications,* **123** (2002) 69-72.
22. A. Dimca, *Topics on Real and Complex Singularities. An Introduction,* Advanced Lectures in Mathematics. Friedr. Vieweg & Sohn, Braunschweig, 1987.
23. A. Dimca, *Singularities and Topology of Hypersurfaces,* Springer-Verlag. xvi, (1992).
24. A. Dimca and A. Némethi, On the monodromy of complex polynomials, *Duke Math. J.* **108** (2001) 199-209.
25. A. Durfee, Five definitions of critical points at infinity, In: *Singularities, The Brieskorn Anniversary Volume, Progress in Math. 162*, Birkhäuser Verlag, (1998) 345-360.
26. D. Eisenbud and W. Neumann, *Three-dimensional Link Theory and Invariants of Plane Curve Singularities, Annals of Mathematics Studies, No. 110. Princeton, New Jersey*: Princeton University Press. VII, (1985).
27. M. V. Fedoryuk, The asymptotics of a Fourier transform of the exponential function of a polynomial, *Soviet Math. Dokl.* **17** (1976) 486-490.
28. L. Fourrier, Classification topologique à l'infini des polynômes de deux variables complexes, *C. R. Acad. Sci., Paris, Série I* **318** (1994) 461-466.
29. Fourrier L. Topologie d'un polynôme de deux variables complexes au voisinage de l'infini, *Ann. Inst. Fourier* **46** (1996) 645-687.
30. M. Françoise and C. Weber, On the monodromies of a polynomial map from $\mathbb{C}^2$ to $\mathbb{C}$, *Topology* **40** (2001) 1217-1240.
31. T. Gaffney, Fibers of polynomial mappings at infinity and Generalized Malgrange Condition, *Compositio Math.* **119** (1999) 157-167.
32. E. R. García Barroso and A. Płoski, Pinceaux de coubes planes et invariants polaires, *Ann. Pol. Math.* **83** (2004) 113-128.
33. R. García López and A. Némethi, On the monodromy at infinity of a polynomial map I, II, *Compos. Math.* **100** (1996) 205-231; and *Compos. Math.* **115** (1999) 1-20.
34. L. Gavrilov, *On the Topology of Polynomials in Two Complex Variables,* Preprint 45 Laboratoire de Topologie et Géométrie, Université de Toulouse III, 1994.

35. G. M. Greuel, C. Lossen and E. Shustin, *Introduction to Singularities and Deformations,* Springer Monographs in Mathematics. Springer, Berlin, 2007.
36. J. Gwoździewicz and A. Płoski, Formulae for the singularities at infinity of plane algebraic curves, *Zesz. Nauk. Uniw. Jagiell. 1255, Univ. Iagell. Acta Math.* **39** (2001) 109-133.
37. J. Gwoździewicz and A. Płoski, On the singularities at infinity of plane algebraic curves, *Rocky Mountain J. Math.* **32** (2002) 139-148.
38. J. Gwoździewicz and M. Sękalski, A polynomial with $2k$ critical values at infinity, *Ann. Pol. Math.* **84** (2004) 41-44.
39. J. Gwoździewicz and A. Płoski, Lojasiewicz exponents and singularities at infinity of polynomials in two complex variables, *Colloq. Math.* **103** (2005) 47-60.
40. Ha Huy Vui and Le Dung Trang, Sur la topologie des polynômes complexes, *Acta Math. Vietnam.* **9** (1984) 21-32.
41. Ha Huy Vui, Sur la fibration globale des polynômes de deux variables complexes, *C. R. Acad. Sci., Série I. Math.* **309** (1989) 231-234.
42. Ha Huy Vui and Nguyen Le Anh, Le comportement géoométrique à l'infini des polynômes de deux variables complexes, *C. R. Acad. Sci., Paris, Série I,* **309** (1989) 183-186.
43. Ha Huy Vui, Nombres de Lojasiewicz et singularités à l'infini des polynômes de deux variables complexes, *C. R. Acad. Sci. Paris, Série I,* **311** (1990) 429-432.
44. Ha Huy Vui, Sur l'irrégularité du diagramme splice pour l'entrelacement à l'infini des courbes planes, *C. R. Acad. Sci., Paris, Série I,* **313**, (1991) 277-280.
45. Ha Huy Vui, A version at infinity of the Kuiper-Kuo theorem, *Acta Math. Vietnam.* **19** (1994) 3-12.
46. Ha Huy Vui, and A. Zaharia, Families of polynomials with total Milnor number constant, *Math. Ann.* **304** (1996) 481-488.
47. Ha Huy Vui and Pham Tien Son Topology of families of affine plane curves, *Ann. Pol. Math.* **71** (1999) 129-139.
48. Ha Huy Vui and Nguyen Tat Thang, *On the Topology of Polynomial Functions on Algebraic Surfaces in* $\mathbb{C}^n$, manuscript 2007.
49. H. A. Hamm and Le Dung Trang, Un théorème de Zariski du type de Lefschetz, *Ann. Sci. École Norm. Sup.* **6** (1973) 317-355.
50. M. Ishikawa, The bifurcation set of a complex polynomial function of two variables and the Newton polygons of singularities at infinity, *J. Math. Soc. Japan* **54** (2002) 161-196.
51. Z. Jelonek, The Jacobian conjecture and the extensions of polynomial embeddings, *Math. Ann.* **294** (1992) 289-293.
52. Z. Jelonek, Testing sets for properness of polynomial mappings, *Math. Ann.* **315** (1999) 1-35.
53. Z. Jelonek, On bifurcation points of a complex polynomial, *Proc. Amer. Math. Soc.* **131** (2002) 1361-1367.
54. Z. Jelonek and K. Kurdyka, On asymptotic critical values of a complex polynomial, *Journal Fur Die Reine Und Angewandte Mathematik,* **565** (2003) 1-11.
55. Z. Jelonek, On the generalized critical values of a polynomial mapping, *Manuscripta math.* **110** (2003) 145-157.

56. O. H. Keller, *Ganze Cremona transformationen,* Monats. Math. Physik, **47** (1939), 299-306.
57. T. Krasiński, On branches at infinity of a pencil of polynomials in two complex variables, *Ann. Pol. Math.* **55** (1991) 213-220.
58. T. C. Kuo and Y. C. Lu, On analytic function germs of two complex variables, *Topology,* **16** (1977) 299-310.
59. T. C. Kuo and A. Parusinski, Newton polygon relative to an arc, *Real and Complex Sigularities. Eds. J. W. Bruce and F. Tari.* (1998) 76-93.
60. K. Kurdyka, P. Orro and S. Simon, Semialgebraic Sard theorem for generalized critical values, *J. Diff. Geom.* **56** (2000) 67-92.
61. A. G. Kushnirenko, Polyèdres de Newton et nombres de Milnor, *Invent. math.* **32** (1976) 1-31.
62. B. Malgrange, Méthode de la phase stationnaire et sommation de Borel, *Complex Analysis, Microlocal Calculus and Relativistic Quantum Theory, Proc. Colloq., Les Houches 1979, Lect. Notes Phys.* **126** (1980) 170-177.
63. Le Dung Trang, Calcul du nombre de cycles évanouissants dune hypersurface complexe, *Ann. Inst. Fourier,* **23** (1973) 261-270.
64. Le Dung Trang, Topological use of polar curves, *Algebraic Geom., Proc. Symp. Pure Math.* **29** (1975) 507-512.
65. Le Dung Trang, La monodromie n'a pas de points fixes, *J. Fac. Sci. Univ. Tokyo Sec. 1A Math.* **22** (1975) 409-427.
66. Le Dung Trang, Some remarks on the relative monodromy, in *Real and complex singularities, Oslo, 1976,* Sijthoff and Nordhoff, Alphen a. d. Rijn, (1977) 397-403.
67. Le Dung Trang and B. Teissier, Cycles evanescents, sections planes et conditions de Whitney. II, *Singularities, Summer Inst., Arcata/Calif. 1981, Proc. Symp. Pure Math.* **40** (1983) 65-103.
68. Le Dung Trang, M. Françoise and C. Weber, Courbes polaires et topologie des courbes planes, *Ann. Sci. Éc. Norm. Sup. Série 4,* **24** (1991) 141-169.
69. Le Dung Trang, Complex analytic functions with isolated singularities, *J. Algebraic Geometry,* **1** (1992) 83-100.
70. Le Dung Trang and C. Weber, A geometrical approach to the Jacobian conjecture for $n = 2$, *Kodai Math. J.* **17** (1994) 374-381.
71. Le Dung Trang and C. Weber, Polynômes à fibres rationnelles et conjecture jacobienne à 2 variables, *C. R. Acad. Sci., Paris, Sér. I* **320** (1995) 581-584.
72. Le Van Thanh, Affine polar quotients and singularity at infinity of an algebraic plane curve, in *Singularity theory. Proceedings of the symposium, Trieste, Italy, August 19-September 6, 1991.* Singapore: World Scientific. (1995) 336-344.
73. Le Van Thanh and M. Oka, Estimation of the number of the critical values at infinity of a polynomial function $f\colon \mathbb{C}^2 \to \mathbb{C}$, *Publ. Res. Inst. Math. Sci.,* **31** (1995) 577-598.
74. A. Lenarcik, On the Lojasiewicz exponent of the gradient of a polynomial function, *Ann. Pol. Math.* **71** (1999) 211-239.
75. S. Lefschetz, *L'analysis Situs et la Géométrie Algébrique,* Gauthier-Villars, Paris, 1924.
76. M. Merle, Invariants polaires de courbes planes, *Invent. Math.* **41** (1977) 103-111.

77. J. Milnor, *Singular Points of Complex Hypersurfaces,* Annals of Mathematics Studies **61**, Princeton University Press, 1968.
78. A. Némethi and A. Zaharia, On the bifurcation set of a polynomial function and Newton boundary, *Publ. RIMS Kyoto Univ.* **26** (1990) 681-689.
79. A. Némethi and A. Zaharia, Milnor fibration at infinity, *Indag. Math.* **3** (1992) 323-335.
80. W. D. Neumann, Complex algebraic plane curves via their links at infinity, *Invent. Math.* **98** (1989) 445-489.
81. W. D. Neumann and Le Van Thanh, On irregular links at infinity of algebraic plane curve, *Math. Ann.* **295** (1993) 239-244.
82. W. D. Neumann, Irregular links at infinity of complex affine plane curves, *Quart J. Math., Oxford II. Ser.* **50** (1999) 301-320.
83. Nguyen Van Chau and C. Gutierrez, Properness and the Jacobian conjecture in $\mathbb{R}^2$, *Vietnam J. Math.* **31** (2003) 421-427.
84. Nguyen Van Chau, Note on the Jacobian condition and the non-proper value set, *Ann. Pol. Math.* **84** (2004) 203-210.
85. A. Parusiński, On the bifurcation set of a complex polynomial with isolated singularities at infinity, *Compositio mathematica,* **97** (1995) 369-384.
86. A. Parusiński, A note on singularities at infinity of complex polynomials, *Banach Center Publication,* **39** (1997) 131-141.
87. L. Paunescu and A. Zaharia, On the Lojasiewicz exponent at infinity for polynomial functions, *Kodai Math. J.* **20** (1997) 269-274.
88. L. Paunescu and A. Zaharia, Remarks on the Milnor fibration at infinity, *manuscripta math.* **103** (2000) 351-361.
89. Phạm Frederic, Vanishing homologies and the n variables saddlepoint method, *Proc. Symp. Pure Math.* **40** (1983) 310-333.
90. Pham Tien Son, Geometric monodromy of polynomials of two complex variables, *Acta Math. Vietnam.* **24** (1999) 283-310.
91. A. Płoski, A note on the Lojasiewicz exponent at infinity, *Bull. Soc. Sci. Lettres Lodź* **44** (1994) 11-15.
92. A. Płoski, Polar quotients and singularities at infinity of polynomials in two complex variables, *Ann. Pol. Math.* **78** (2002) 49-58.
93. P. J. Rabier, Ehresmanns fibrations and Palais-Smale conditions for morphisms of Finsler manifolds, *Annals of Math.* **146**, (1997) 647-691.
94. P. J. Rabier, On the Malgrange condition for complex polynomials of two variables, *manuscripta math.* **109** (2002) 493-509.
95. T. Rodak, The Fedoryuk condition and the Lojasiewicz exponent near a fibre of a polynomial, *Bull. Pol. Acad. Sci., Math.* **52** (2004) 227-229.
96. D. Siersma and M. Tibăr, Singularities at infinity and their vanishing cycles, *Duke Math. J.* **80** (1995) 771-783.
97. D. Siersma and M. Tibăr, Topology of polynomial functions and monodromy dynamics, *C. R. Acad. Sci., Paris, Série I,* **327** (1998) 655-660.
98. D. Siersma and M. Tibăr, Singularities at infinity and their vanishing cycles, II. Monodromy, *Publ. Res. Inst. Math. Sci.,* **36** (2000) 659-67.
99. D. Siersma and J. Smeltnik, Classification of singularities at infinity of polynomials of degree 4 in two variables, *Georgian Math. J.* **7** (2000) 179-190.

100. G. Skalski, On the Lojasiewicz exponent near the fibre of a polynomial, *Bull. Acad. Polon. Sci. Math.* **52** (2004) 231-236.
101. M. Suzuki, Propriétés topologiques des polynômes de deux variables complexes, et automorphismes algébriques de l'espace $\mathbb{C}^2$, *J. Math. Soc. Japan,* **26** (1974) 241-257.
102. B. Teissier, Cycles évanescents, sections planes et conditions de Whitney, in: *Singularités à Cargèse 1972,* Asterisque, Nos. 7 et 8, Soc. Math. France, Paris, (1973) 285-362.
103. B. Teissier, Varietés polaires I - Invariants polaires des singularités d'hypersufaces, *Invent. Math.* **40** (1977) 267-293.
104. B. Teissier, *The hunting of invariants in the geometry of discriminants,* Real and complex singularities (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976), 565-678. Sijthoff and Noordhoff, Alphen aan den Rijn, 1977.
105. B. Teissier, Variétés polaires. II. Multiplicités polaires, sections planes, et conditions de Whitney, in: *Algebraic geometry (La Rbida, 1981), Lecture Notes in Mathematics,*, Springer, Berlin, **961** (1982) 314-491.
106. M. Tibăr, On the monodromy fibration of polynomial functions with singularities at infinity, *C. R. Acad. Sci. Paris* **324** (1997) 1031-1035.
107. M. Tibăr, Asymptotic equisingularity and topology of complex hypersurfaces, *Int. Math. Res. Not.* **18** (1998) 979-990.
108. M. Tibăr, Topology at infinity of polynomial maps and Thom regularity condition, *Compositio Math.* **111** (1998) 89-109.
109. M. Tibăr, *Polynomials and Vanishing Cycles,* Cambridge University Press, 2007.
110. R. Thom, Ensembles et morphismes stratifié, *Bull. Amer. Math. Soc,* **75** (1969) 249-312.
111. A. N. Varchenko, Theorems on topolgycal equisingularity of families of algebraic varieties and families of polynomial maps, *Izvestiya Akad. Nauk.* **36** (1972) 957-1019.
112. J. L. Verdier, Stratifications de Whitney et théorème de Bertini-Sard, *Inventiones Mathematicae,* **36** (1976) 295-312.
113. R. J. Walker, *Algebraic Curves,* Princeton University, 1950.