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On Some New Explicit Evaluations of Class Invariants

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Abstract. Ramanujan recorded the values for 107 class invariants or polynomials satisfied by them at scattered places of his first notebook. On pages 294-299 in his second notebook, Ramanujan gave a table of values for 77 class invariants. In this paper, we establish some connecting formulas for G_n and g_n. We also establish several new explicit evaluations of Ramanujan-Weber type class invariants using the modular equations and several explicit values of singular moduli.

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1. Introduction

Ramanujan's class invariants are defined by

$$G_n := 2^{-\frac{1}{4}} q^{-\frac{1}{24}} \chi(q) \qquad \text{and} \qquad g_n := 2^{-\frac{1}{4}} q^{-\frac{1}{24}} \chi(-q), \tag{1.1}$$

where

$$\chi(q) := (-q; q^2)_\infty, \qquad q = e^{-\pi\sqrt{n}}$$

and

$$(a; q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n), \qquad |q| < 1.$$

On pages 294-299, Ramanujan recorded table of values for 77 class invariants or monic irreducible polynomials. In [19, 20], Watson proved 24 of Ramanujan's class invariants from Ramanujan's paper [17]. Watson also wrote further four papers [21–24] on the calculation of class invariants. In [7], Chan has used class field theory, Galois theory and Kronecker's limit formula to justify Watson's assumptions and calculated some values of G_n. In [2], Baruah has established the value of G_{217} using modular equations of degrees 7 and 31.

The ordinary hypergeometric series ${}_2F_1(a, b; c; x)$ is defined by

$$ {}_2F_1(a, b; c; x) := \sum_{i=0}^{\infty} \frac{(a)_i (b)_i}{(c)_i \; i!} x^i, $$

where $(a)_0 = 1$, $(a)_i = (a)(a+1)...(a+i-1)$ for $i \geq 1$ and $|x| < 1$.

The complete elliptic integral of the first kind $K = K(k)$ associated with the modulus k, $0 < k < 1$, is defined by

$$ K(k) := \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} = \frac{\pi}{2} \, {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right), $$

where the latter representation is achieved by expanding the integrand in a binomial series and integrating termwise. If we put $\alpha := k^2$ in the above integral we see that Ramanujan's function $F(\alpha)$ is $K(k)$:

$$ F(\alpha) = K(k), $$

where

$$ F(\alpha) = 1 + \left(\frac{1}{2}\right)^2 \alpha + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 \alpha^2 + \dots $$

The number k is called the modulus of K. The number k', defined by $k' := \sqrt{1 - k^2}$, is called the complementary modulus and $K' := K(k')$ is called the complementary integral. $F(\alpha)$ converges for $-1 \leq \alpha < 1$. Moreover, as α increases from 0 to 1, $F(\alpha)$ increases from 1 to infinity and $F(1-\alpha)$ decreases from infinity to 1, and therefore

$$ \frac{F(1-\alpha)}{F(\alpha)} $$

decreases monotonically from infinity to zero as α increases from 0 to 1. Thus there exists a unique real number α_n with $0 < \alpha_n < 1$ which satisfies

$$ \frac{F(1-\alpha_n)}{F(\alpha_n)} = \sqrt{n}, $$

and we call α_n the singular modulus (for n).

For example, Ramanujan's famous singular modulus [18, p. 320]:

$$ \begin{aligned} k_{210}^2 = {} & \left(4 - \sqrt{15}\right)^4 \left(8 - 3\sqrt{7}\right)^2 \left(2 - \sqrt{3}\right)^2 \left(6 - \sqrt{35}\right)^2 \\ & \times \left(\sqrt{10} - 3\right)^4 \left(\sqrt{7} - \sqrt{6}\right)^4 \left(\sqrt{15} - \sqrt{14}\right)^2 \left(\sqrt{2} - 1\right)^4. \end{aligned} $$

Let p denote a fixed positive integer and suppose that

$$p\frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-k^2)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;k^2)}=\frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-l^2)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;l^2)}, \tag{1.2}$$

where $0<k,\,l<1$. Then a modular equation of degree p is a relation between the moduli k and l which is implied by (1.2). Following Ramanujan, we put $\alpha=k^2$ and $\beta=l^2$. We often say that β has degree p over α. The multiplier m is defined by

$$m:=\frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;\alpha)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;\beta)}.$$

In [8], Mahadeva Naika has proved the connecting formulas G_{κ^2n} with G_n and g_{κ^2n} with g_n for $\kappa=3,5,7,11$. One can use the class invariants to find the explicit evaluations of Ramanujan's remarkable product of theta functions, cubic continued fraction, Ramanujan-Selberg continued fraction etc. For more details see [1, 5, 9-15].

In this paper, we establish several new explicit evaluations of G_n for the even values of n and g_n for odd values of n using Ramanujan's modular equations. We also establish several explicit evaluations of singular moduli.

2. Preliminary Results

In this section, we collect some identities which are useful in proving our main results.

Lemma 2.1. *We have*

$$\chi(e^{-y})=2^{\frac{1}{6}}\{x(1-x)e^{y}\}^{-\frac{1}{24}}, \tag{2.1}$$

$$\chi(-e^{-y})=2^{\frac{1}{6}}(1-x)^{\frac{1}{12}}(xe^{y})^{-\frac{1}{24}}. \tag{2.2}$$

Proof. For the proofs of (2.1)- (2.2), see [3, Entry 12 (v),(vi), Ch.17, p.124]. ∎

We restate Ramanujan's class invariants (1.1) using Lemma 2.1 replacing q by e^{-y} and x by α as follows:

$$G_n:=(4\alpha(1-\alpha))^{-\frac{1}{24}} \tag{2.3}$$

and

$$g_n:=\left(4\alpha(1-\alpha)^{-2}\right)^{-\frac{1}{24}},\qquad \alpha=\alpha(e^{-\pi\sqrt{n}}). \tag{2.4}$$

In the following lemma, we state several Ramanujan's modular equations.

Lemma 2.2. *If β is of degree* 3 *over α, then*

$$(\alpha\beta)^{\frac{1}{4}}+\{(1-\alpha)(1-\beta)\}^{\frac{1}{4}}=1, \tag{2.5}$$

$$m^2 = 1 + 4\left(\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}\right)^{\frac{1}{8}}, \tag{2.6}$$

$$\frac{9}{m^2} = 1 + 4\left(\frac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}\right)^{\frac{1}{8}}, \tag{2.7}$$

$$Q - \frac{1}{Q} = 2\left(P - \frac{1}{P}\right), \tag{2.8}$$

where $P = (\alpha\beta)^{\frac{1}{8}}$ *and* $Q = \left(\frac{\beta}{\alpha}\right)^{\frac{1}{4}}$.

Proof. For the proofs of (2.5)- (2.8), see [3, Entry 5(ii), (v), (vi), (xiii), Ch.19, pp.230-231]. ∎

Lemma 2.3.

$$g_{30} = \left(2+\sqrt{5}\right)^{\frac{1}{6}}\left(3+\sqrt{10}\right)^{\frac{1}{6}}, \tag{2.9}$$

and

$$G_{15} = 2^{\frac{1}{4}}\left(\frac{1+\sqrt{5}}{2}\right)^{\frac{1}{3}}. \tag{2.10}$$

Proof. For a proof of (2.9), see[4, p.200] and for a proof of (2.10), see [4, p.190]. ∎

3. Main Theorems

In this section, we establish several connecting formulas for G_n with g_n, G_n with G_{9n}, g_n with g_{9n} and g_{4n} with G_n.

Theorem 3.1. *If* β *is of degree* 3 *over* α, *then*

$$y = \frac{1+\sqrt{1+2u^6}}{2}, \tag{3.1}$$

where $u = (g_n g_{9n})^{-1}$ *and* $y = \{(1-\alpha)(1-\beta)\}^{-\frac{1}{4}}$.

Proof. Using (2.4) and (2.5), we obtain the required result. ∎

Theorem 3.2. *If* β *is of degree* 3 *over* α, *then*

$$x = \frac{1+\sqrt{1-2v^6}}{2}, \tag{3.2}$$

where $v = (G_n G_{9n})^{-1}$ *and* $x = \{(1-\alpha)(1-\beta)\}^{\frac{1}{4}}$.

Proof. Using (2.3) and (2.5), we obtain the required result. ∎

Theorem 3.3. *We have*

$$\frac{G_n}{g_n} = \frac{1}{(1-\alpha_n)^{\frac{1}{8}}}. \tag{3.3}$$

Proof. Using (1.1), (2.1) and (2.2), we obtain the required result. ∎

Theorem 3.4. *We have*

$$\alpha_n = (G_n g_{4n})^{-8}. \tag{3.4}$$

Proof. Using (1.1), (2.1) and (2.2), we obtain the required result. ∎

Theorem 3.5. *We have*

$$4G_n^{16}g_n^8 - 4G_n^8 g_n^{16} - 1 = 0, \tag{3.5}$$

$$4g_{4n}^8 G_n^8 - g_{4n}^{16}G_n^{-8} - 4 = 0, \tag{3.6}$$

$$g_{4n}^{16} - 4g_{4n}^8 g_n^{16} - 4g_n^8 = 0, \tag{3.7}$$

$$g_{4n}^2 g_{36n}^2 = g_n^4 g_{9n}^4 + g_n g_{9n}\sqrt{g_n^6 g_{9n}^6 + 2}, \tag{3.8}$$

$$G_n^2 G_{9n}^2 = \frac{g_n^3 g_{9n}^3 + \sqrt{g_n^6 g_{9n}^6 + 2}}{2}, \tag{3.9}$$

$$2\sqrt{2}\left[g_n^3 g_{9n}^3 + \frac{1}{g_n^3 g_{9n}^3}\right] = \frac{g_{9n}^6}{g_n^6} - \frac{g_n^6}{g_{9n}^6} \tag{3.10}$$

and

$$2\sqrt{2}\left[G_n^3 G_{9n}^3 - \frac{1}{G_n^3 G_{9n}^3}\right] = \frac{G_{9n}^6}{G_n^6} + \frac{G_n^6}{G_{9n}^6}. \tag{3.11}$$

Proof. Using (2.3), (3.4) and the fact that $g_{4n} = 2^{\frac{1}{4}}G_n g_n$, we obtain the required identity (3.5).

Identities (3.6) and (3.7) are obtained by using (3.5) and the fact $g_{4n} = 2^{\frac{1}{4}}G_n g_n$.

Identities (3.8) and (3.9) are obtained by using (3.3) and (3.4) in (2.5).

Identities (3.10) and (3.11) are proved by using (2.1) and (2.2) in (2.6) and (2.7). ∎

Remark. The identities (3.10) and (3.11) are recorded by Ramanujan in [18]. The different proofs of the identities (3.8), (3.10) and (3.11) can be found in [8]. The identities (3.5), (3.6) and (3.7) seem to be new.

4. Computation of G_n for even values of n

In this section, we establish some explicit evaluations of class invariant G_n for even values of n by using Ramanujan's modular equations and the class invariant g_n.

Theorem 4.1. *We have*

$$G_{30} = \left(\sqrt{5}+2\right)^{\frac{1}{6}} \left(\sqrt{10}+3\right)^{\frac{1}{6}} \left(\frac{1+2\sqrt{6}-\sqrt{15}}{2}\right)^{\frac{1}{4}}$$
$$\left(\sqrt{\frac{-39-12\sqrt{10}+18\sqrt{6}+9\sqrt{15}}{2}} - \sqrt{\frac{-41-12\sqrt{10}+18\sqrt{6}+9\sqrt{15}}{2}}\right)^{\frac{1}{4}} \tag{4.1}$$

and

$$G_{\frac{10}{3}} = \left(\sqrt{5}-2\right)^{\frac{1}{6}} \left(\sqrt{10}+3\right)^{\frac{1}{6}} \left(\frac{1+2\sqrt{6}-\sqrt{15}}{2}\right)^{\frac{1}{4}}$$
$$\left(\sqrt{\frac{-39-12\sqrt{10}+18\sqrt{6}+9\sqrt{15}}{2}} + \sqrt{\frac{-41-12\sqrt{10}+18\sqrt{6}+9\sqrt{15}}{2}}\right)^{\frac{1}{4}}. \tag{4.2}$$

Proof. Using (2.9) in (3.10) with $n = \frac{10}{3}$, we deduce that

$$g_{\frac{10}{3}} = \left(\sqrt{5}-2\right)^{\frac{1}{6}} \left(3+\sqrt{10}\right)^{\frac{1}{6}}. \tag{4.3}$$

From (2.9) and (4.3), we find that

$$u = \left(\sqrt{10}+3\right)^{-\frac{1}{3}}. \tag{4.4}$$

Using (4.4) in (3.1), we deduce that

$$y = \frac{1+\sqrt{39-12\sqrt{10}}}{2}. \tag{4.5}$$

Changing α to $1-\beta$ and β to $1-\alpha$ in (2.8) and then using (4.5) in the resultant equation, we find that

$$Q = \left(\sqrt{\frac{-39-12\sqrt{10}+18\sqrt{6}+9\sqrt{15}}{2}} - \sqrt{\frac{-41-12\sqrt{10}+18\sqrt{6}+9\sqrt{15}}{2}}\right). \tag{4.6}$$

Using (4.5) and (4.6), we deduce that

$$(1-\alpha_{30})^{-\frac{1}{8}} = \left(\frac{1+2\sqrt{6}-\sqrt{15}}{4}\right)^{\frac{1}{4}}$$
$$\left(\sqrt{\frac{-39-12\sqrt{10}+18\sqrt{6}+9\sqrt{15}}{2}} - \sqrt{\frac{-41-12\sqrt{10}+18\sqrt{6}+9\sqrt{15}}{2}}\right)^{\frac{1}{4}}$$

(4.7)

and

$$(1-\alpha_{\frac{10}{3}})^{-\frac{1}{8}} = \left(\frac{1+2\sqrt{6}-\sqrt{15}}{4}\right)^{\frac{1}{4}}$$
$$\left(\sqrt{\frac{-39-12\sqrt{10}+18\sqrt{6}+9\sqrt{15}}{2}}+\sqrt{\frac{-41-12\sqrt{10}+18\sqrt{6}+9\sqrt{15}}{2}}\right)^{\frac{1}{4}}. \tag{4.8}$$

Using (4.7) and (2.9) in (3.3), we obtain the result (4.1). Similarly using (4.8) and (4.3) in (3.3), we obtain (4.2). Hence we complete the proof. ■

Theorem 4.2. *We have*

$$G_{42} = \left(2\sqrt{2}+\sqrt{7}\right)^{\frac{1}{6}}\left(\frac{\sqrt{7}+\sqrt{3}}{2}\right)^{\frac{1}{2}}\left(\frac{1-4\sqrt{3}+3\sqrt{7}}{2}\right)^{\frac{1}{4}}$$
$$\left(\sqrt{\frac{-111+60\sqrt{3}+45\sqrt{7}-24\sqrt{21}}{2}}-\sqrt{\frac{-113+60\sqrt{3}+45\sqrt{7}-24\sqrt{21}}{2}}\right)^{\frac{1}{4}}, \tag{4.9}$$

$$G_{\frac{14}{3}} = \left(2\sqrt{2}-\sqrt{7}\right)^{\frac{1}{6}}\left(\frac{\sqrt{7}+\sqrt{3}}{2}\right)^{\frac{1}{2}}\left(\frac{1-4\sqrt{3}+3\sqrt{7}}{2}\right)^{\frac{1}{4}}$$
$$\left(\sqrt{\frac{-111+60\sqrt{3}+45\sqrt{7}-24\sqrt{21}}{2}}+\sqrt{\frac{-113+60\sqrt{3}+45\sqrt{7}-24\sqrt{21}}{2}}\right)^{\frac{1}{4}}, \tag{4.10}$$

$$\begin{aligned} G_{78} = &\left(\frac{\sqrt{13}+3}{2}\right)^{\frac{1}{2}}\left(\sqrt{26}+5\right)^{\frac{1}{6}}\left(\frac{1+15\sqrt{3}-4\sqrt{39}}{2}\right)^{\frac{1}{4}} \\ &\Bigg(\sqrt{\frac{-1299+765\sqrt{3}+204\sqrt{39}-360\sqrt{13}}{2}} \\ &-\sqrt{\frac{-1301+765\sqrt{3}+204\sqrt{39}-360\sqrt{13}}{2}}\Bigg)^{\frac{1}{4}}, \end{aligned} \tag{4.11}$$

$$\begin{aligned}G_{\frac{26}{3}} =&\Big(\frac{\sqrt{13}+3}{2}\Big)^{\frac{1}{2}}\Big(\sqrt{26}-5\Big)^{\frac{1}{6}}\Big(\frac{1+15\sqrt{3}-4\sqrt{39}}{2}\Big)^{\frac{1}{4}}\\ &\Big(\sqrt{\frac{-1299+756\sqrt{3}+204\sqrt{39}-360\sqrt{13}}{2}}\\ &+\sqrt{\frac{-1301+756\sqrt{3}+204\sqrt{39}-360\sqrt{13}}{2}}\Big)^{\frac{1}{4}},\end{aligned} \tag{4.12}$$

$$\begin{aligned}G_{102} =&\Big(\sqrt{2}+1\Big)^{\frac{1}{2}}\Big(3\sqrt{2}+\sqrt{17}\Big)^{\frac{1}{3}}\Big(\frac{1+7\sqrt{51}-20\sqrt{6}}{2}\Big)^{\frac{1}{4}}\\ &\Big(\sqrt{\frac{-4899-840\sqrt{34}+693\sqrt{51}+1980\sqrt{6}}{2}}\\ &-\sqrt{\frac{-4901-840\sqrt{34}+693\sqrt{51}+1980\sqrt{6}}{2}}\Big)^{\frac{1}{4}}\end{aligned} \tag{4.13}$$

and

$$\begin{aligned}G_{\frac{34}{3}} =&\Big(\sqrt{2}-1\Big)^{\frac{1}{2}}\Big(3\sqrt{2}+\sqrt{17}\Big)^{\frac{1}{3}}\left(\frac{1+7\sqrt{51}-20\sqrt{6}}{2}\right)^{\frac{1}{4}}\\ &\Big(\sqrt{\frac{-4899-840\sqrt{34}+693\sqrt{51}+1980\sqrt{6}}{2}}\\ &+\sqrt{\frac{-4901-840\sqrt{34}+693\sqrt{51}+1980\sqrt{6}}{2}}\Big)^{\frac{1}{4}}.\end{aligned} \tag{4.14}$$

The proof of Theorem 4.2 is similar to that of Theorem 4.1, hence we omit the details. ■

5. Computation of g_n for Odd Values of n

In this section, we establish some explicit evaluations of class invariant g_n for odd values of n by using Ramanujan's modular equations and the class invariant G_n.

Theorem 5.1. *We have*

$$g_{15} = \left(\frac{\sqrt{5}+1}{2}\right)^{\frac{1}{3}}\left(\frac{\sqrt{3}+1}{2}\right)^{\frac{1}{2}}\left(\sqrt{\frac{30-15\sqrt{3}}{4}}+\sqrt{\frac{26-15\sqrt{3}}{4}}\right)^{\frac{1}{4}} \tag{5.1}$$

and

$$g_{\frac{5}{3}}=\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{1}{3}}\left(\frac{\sqrt{3}+1}{2}\right)^{\frac{1}{2}}\left(\sqrt{\frac{30-15\sqrt{3}}{4}}-\sqrt{\frac{26-15\sqrt{3}}{4}}\right)^{\frac{1}{4}}. \tag{5.2}$$

Proof. Using (2.10) in (3.11) with $n=\frac{5}{3}$, we find that

$$G_{\frac{5}{3}}=2^{\frac{1}{4}}\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{1}{3}}. \tag{5.3}$$

From (2.10) and (5.3), we find that

$$v=\frac{1}{\sqrt{2}}. \tag{5.4}$$

Using (5.4) in (3.2), we deduce that

$$x=\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)^{2}. \tag{5.5}$$

Changing α to $1-\beta$ and β to $1-\alpha$ in (2.8) and then using (5.5) in the resultant equation, we deduce that

$$Q=\left(\sqrt{\frac{30-15\sqrt{3}}{4}}+\sqrt{\frac{26-15\sqrt{3}}{4}}\right). \tag{5.6}$$

Using (5.5) and (5.6), we find that

$$(1-\alpha_{15})^{\frac{1}{8}}=\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)^{\frac{1}{2}}\left(\sqrt{\frac{30-15\sqrt{3}}{4}}-\sqrt{\frac{26-15\sqrt{3}}{4}}\right)^{\frac{1}{4}} \tag{5.7}$$

and

$$(1-\alpha_{\frac{5}{3}})^{\frac{1}{8}}=\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)^{\frac{1}{2}}\left(\sqrt{\frac{30-15\sqrt{3}}{4}}+\sqrt{\frac{26-15\sqrt{3}}{4}}\right)^{\frac{1}{4}}. \tag{5.8}$$

Using (5.7) and (2.10) in (3.3), we obtain the result (5.1). Similarly using (5.8) and (5.3) in (3.3), we obtain (5.2). Hence we complete the proof. ■

Theorem 5.2. *We have*

$$g_{21}=\left(\frac{\sqrt{7}+\sqrt{3}}{2}\right)^{\frac{1}{4}}\left(\frac{3+\sqrt{7}}{\sqrt{2}}\right)^{\frac{1}{6}}\left(\frac{1+\sqrt{6\sqrt{7}-15}}{2}\right)^{\frac{1}{4}}\times$$

$$\times\left(\sqrt{\frac{15+6\sqrt{7}-9\sqrt{5+2\sqrt{7}}}{2}}+\sqrt{\frac{13+6\sqrt{7}-9\sqrt{5+2\sqrt{7}}}{2}}\right)^{\frac{1}{4}}, \tag{5.9}$$

$$g_{\frac{7}{3}}=\left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^{\frac{1}{4}}\left(\frac{3+\sqrt{7}}{\sqrt{2}}\right)^{\frac{1}{6}}\left(\frac{1+\sqrt{6\sqrt{7}-15}}{2}\right)^{\frac{1}{4}}$$

$$\left(\sqrt{\frac{15+6\sqrt{7}-9\sqrt{5+2\sqrt{7}}}{2}}-\sqrt{\frac{13+6\sqrt{7}-9\sqrt{5+2\sqrt{7}}}{2}}\right)^{\frac{1}{4}}, \tag{5.10}$$

$$\begin{aligned}g_{33}=&\left(\frac{\sqrt{11}+3}{\sqrt{2}}\right)^{\frac{1}{6}}\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)^{\frac{1}{2}}\left(\frac{1+\sqrt{-51+30\sqrt{3}}}{2}\right)^{\frac{1}{4}}\\&\Big(\sqrt{\frac{51+30\sqrt{3}-3\sqrt{561+330\sqrt{3}}}{2}}\\&+\sqrt{\frac{49+30\sqrt{3}-3\sqrt{561+330\sqrt{3}}}{2}}\Big)^{\frac{1}{4}},\end{aligned} \tag{5.11}$$

$$\begin{aligned}g_{\frac{11}{3}}=&\left(\frac{\sqrt{11}-3}{\sqrt{2}}\right)^{\frac{1}{6}}\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)^{\frac{1}{2}}\left(\frac{1+\sqrt{-51+30\sqrt{3}}}{2}\right)^{\frac{1}{4}}\\&\Big(\sqrt{\frac{51+30\sqrt{3}-3\sqrt{561+330\sqrt{3}}}{2}}\\&-\sqrt{\frac{49+30\sqrt{3}-3\sqrt{561+330\sqrt{3}}}{2}}\Big)^{\frac{1}{4}},\end{aligned} \tag{5.12}$$

$$\begin{aligned}g_{39}=&2^{\frac{1}{4}}\left(\frac{\sqrt{13}+3}{2}\right)^{\frac{1}{6}}\left(\sqrt{\frac{5+\sqrt{13}}{8}}+\sqrt{\frac{-3+\sqrt{13}}{8}}\right)\left(\frac{2\sqrt{2}+\sqrt{-3+3\sqrt{13}}}{4\sqrt{2}}\right)^{\frac{1}{4}}\\&\Big(\sqrt{\frac{348+96\sqrt{13}-9\sqrt{2906+806\sqrt{13}}}{8}}\\&+\sqrt{\frac{340+96\sqrt{13}-9\sqrt{2906+806\sqrt{13}}}{8}}\Big)^{\frac{1}{4}},\end{aligned} \tag{5.13}$$

$$
\begin{aligned}
g_{\frac{13}{3}} =& 2^{\frac{1}{4}} \left(\frac{\sqrt{13}+3}{2}\right)^{\frac{1}{6}} \left(\sqrt{\frac{5+\sqrt{13}}{8}} - \sqrt{\frac{-3+\sqrt{13}}{8}}\right) \left(\frac{2\sqrt{2}+\sqrt{-3+3\sqrt{13}}}{4\sqrt{2}}\right)^{\frac{1}{4}} \\
&\Bigg(\sqrt{\frac{348+96\sqrt{13}-9\sqrt{2906+806\sqrt{13}}}{8}} \\
&- \sqrt{\frac{340+96\sqrt{13}-9\sqrt{2906+806\sqrt{13}}}{8}}\Bigg)^{\frac{1}{4}},
\end{aligned} \tag{5.14}
$$

$$
\begin{aligned}
g_{57} =& \left(\frac{3\sqrt{19}+13}{\sqrt{2}}\right)^{\frac{1}{6}} \left(2+\sqrt{3}\right)^{\frac{1}{4}} \left(\frac{1+\sqrt{78\sqrt{19}-339}}{2}\right)^{\frac{1}{4}} \\
&\Bigg(\sqrt{\frac{339+78\sqrt{19}-45\sqrt{113+26\sqrt{19}}}{2}} \\
&+ \sqrt{\frac{337+78\sqrt{19}-45\sqrt{115+26\sqrt{19}}}{2}}\Bigg)^{\frac{1}{4}},
\end{aligned} \tag{5.15}
$$

$$
\begin{aligned}
g_{\frac{19}{3}} =& \left(\frac{3\sqrt{19}+13}{\sqrt{2}}\right)^{\frac{1}{6}} \left(2-\sqrt{3}\right)^{\frac{1}{4}} \left(\frac{1+\sqrt{78\sqrt{19}-339}}{2}\right)^{\frac{1}{4}} \\
&\Bigg(\sqrt{\frac{339+78\sqrt{19}-45\sqrt{113+26\sqrt{19}}}{2}} \\
&- \sqrt{\frac{337+78\sqrt{19}-45\sqrt{115+26\sqrt{19}}}{2}}\Bigg)^{\frac{1}{4}},
\end{aligned} \tag{5.16}
$$

$$
\begin{aligned}
g_{93} =& \left(\frac{39+7\sqrt{31}}{\sqrt{2}}\right)^{\frac{1}{6}} \left(\frac{\sqrt{31}+3\sqrt{3}}{2}\right)^{\frac{1}{4}} \left(\frac{1+\sqrt{-3039+546\sqrt{31}}}{2}\right)^{\frac{1}{4}} \\
&\Bigg(\sqrt{\frac{3039+546\sqrt{31}-135\sqrt{1013+182\sqrt{31}}}{2}} \\
&+ \sqrt{\frac{3037+546\sqrt{31}-135\sqrt{1013+182\sqrt{31}}}{2}}\Bigg)^{\frac{1}{4}},
\end{aligned} \tag{5.17}
$$

$$g_{\frac{31}{3}} = \left(\frac{39+7\sqrt{31}}{\sqrt{2}}\right)^{\frac{1}{6}} \left(\frac{\sqrt{31}-3\sqrt{3}}{2}\right)^{\frac{1}{4}} \left(\frac{1+\sqrt{-3039+546\sqrt{31}}}{2}\right)^{\frac{1}{4}}$$
$$\Bigg(\sqrt{\frac{3039+546\sqrt{31}-135\sqrt{1013+182\sqrt{31}}}{2}}$$
$$-\sqrt{\frac{3037+546\sqrt{31}-135\sqrt{1013+182\sqrt{31}}}{2}}\Bigg)^{\frac{1}{4}}, \tag{3.18}$$

$$g_{177} = \left(\frac{3\sqrt{59}+23}{\sqrt{2}}\right)^{\frac{1}{6}} \left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)^{\frac{3}{2}} \left(\frac{1+\sqrt{-140451+81090\sqrt{3}}}{2}\right)^{\frac{1}{4}}$$
$$\Bigg(\sqrt{\frac{140451+81090\sqrt{3}-69\sqrt{8286609+4784310\sqrt{3}}}{2}}$$
$$+\sqrt{\frac{140449+81090\sqrt{3}-69\sqrt{8286609+4784310\sqrt{3}}}{2}}\Bigg)^{\frac{1}{4}} \tag{5.19}$$

and

$$g_{\frac{59}{3}} = \left(\frac{3\sqrt{59}-23}{\sqrt{2}}\right)^{\frac{1}{6}} \left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)^{\frac{3}{2}} \left(\frac{1+\sqrt{-140451+81090\sqrt{3}}}{2}\right)^{\frac{1}{4}}$$
$$\Bigg(\sqrt{\frac{140451+81090\sqrt{3}-69\sqrt{8286609+4784310\sqrt{3}}}{2}}$$
$$-\sqrt{\frac{140449+81090\sqrt{3}-69\sqrt{8286609+4784310\sqrt{3}}}{2}}\Bigg)^{\frac{1}{4}}. \tag{5.20}$$

The proof of Theorem 5.2 is similar to that of Theorem 5.1, hence we omit the details. ■

6. Some New Explicit Evaluations of g_n

In this section, we establish several new explicit evaluations of the class invariants g_n using the values in Theorems 4.1, 4.2, 5.1, 5.2 and the formula $g_{4n} = 2^{\frac{1}{4}} G_n g_n$.

Theorem 6.1. *We have*

$$g_{60} = 2^{\frac{1}{2}} \left(\frac{\sqrt{5}+1}{2}\right)^{\frac{2}{3}} \left(\frac{\sqrt{3}+1}{2}\right)^{\frac{1}{2}} \left(\sqrt{\frac{30-15\sqrt{3}}{4}}+\sqrt{\frac{26-15\sqrt{3}}{4}}\right)^{\frac{1}{4}}, \tag{6.1}$$

$$g_{\frac{20}{3}} = 2^{\frac{1}{2}} \left(\frac{\sqrt{5}-1}{2}\right)^{\frac{2}{3}} \left(\frac{\sqrt{3}+1}{2}\right)^{\frac{1}{2}} \left(\sqrt{\frac{30-15\sqrt{3}}{4}} - \sqrt{\frac{26-15\sqrt{3}}{4}}\right)^{\frac{1}{4}}, \quad (6.2)$$

$$\begin{aligned} g_{84} =& 2^{\frac{1}{4}} \left(\frac{\sqrt{7}+\sqrt{3}}{2}\right)^{\frac{1}{2}} \left(\frac{3+\sqrt{7}}{\sqrt{2}}\right)^{\frac{1}{3}} \left(\frac{1+\sqrt{-15+6\sqrt{7}}}{2}\right)^{\frac{1}{4}} \\ & \left(\sqrt{\frac{15+6\sqrt{7}-9\sqrt{5+2\sqrt{7}}}{2}} + \sqrt{\frac{13+6\sqrt{7}-9\sqrt{5+2\sqrt{7}}}{2}}\right)^{\frac{1}{4}}, \end{aligned} \quad (6.3)$$

$$\begin{aligned} g_{\frac{28}{3}} =& 2^{\frac{1}{4}} \left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^{\frac{1}{2}} \left(\frac{3+\sqrt{7}}{\sqrt{2}}\right)^{\frac{1}{3}} \left(\frac{1+\sqrt{-15+6\sqrt{7}}}{2}\right)^{\frac{1}{4}} \\ & \left(\sqrt{\frac{15+6\sqrt{7}-9\sqrt{5+2\sqrt{7}}}{2}} - \sqrt{\frac{13+6\sqrt{7}-9\sqrt{5+2\sqrt{7}}}{2}}\right)^{\frac{1}{4}}, \end{aligned} \quad (6.4)$$

$$\begin{aligned} g_{120} =& 2^{\frac{1}{4}} (\sqrt{5}+2)^{\frac{1}{3}} (\sqrt{10}+3)^{\frac{1}{3}} \left(\frac{1+2\sqrt{6}-\sqrt{15}}{2}\right)^{\frac{1}{4}} \\ & \left(\sqrt{\frac{-39-12\sqrt{10}+18\sqrt{6}+9\sqrt{15}}{2}}\right. \\ & \left. - \sqrt{\frac{-41-12\sqrt{10}+18\sqrt{6}+9\sqrt{15}}{2}}\right)^{\frac{1}{4}}, \end{aligned} \quad (6.5)$$

$$\begin{aligned} g_{\frac{40}{3}} =& 2^{\frac{1}{4}} (\sqrt{5}-2)^{\frac{1}{3}} (\sqrt{10}+3)^{\frac{1}{3}} \left(\frac{1+2\sqrt{6}-\sqrt{15}}{2}\right)^{\frac{1}{4}} \\ & \left(\sqrt{\frac{-39-12\sqrt{10}+18\sqrt{6}+9\sqrt{15}}{2}}\right. \\ & \left. + \sqrt{\frac{-41-12\sqrt{10}+18\sqrt{6}+9\sqrt{15}}{2}}\right)^{\frac{1}{4}}, \end{aligned} \quad (6.6)$$

$$\begin{aligned} g_{132} =& 2^{\frac{1}{4}} \left(\frac{\sqrt{11}+3}{\sqrt{2}}\right)^{\frac{1}{3}} \left(\frac{\sqrt{3}+1}{\sqrt{2}}\right) \left(\frac{1+\sqrt{-51+30\sqrt{3}}}{2}\right)^{\frac{1}{4}} \\ & \left(\sqrt{\frac{51+30\sqrt{3}-3\sqrt{561+330\sqrt{3}}}{2}}\right. \\ & \left. + \sqrt{\frac{49+30\sqrt{3}-3\sqrt{561+330\sqrt{3}}}{2}}\right)^{\frac{1}{4}}, \end{aligned} \quad (6.7)$$

$$
\begin{aligned}
g_{\frac{44}{3}} =& 2^{\frac{1}{4}} \left(\frac{\sqrt{11}-3}{\sqrt{2}}\right)^{\frac{1}{3}} \left(\frac{\sqrt{3}+1}{\sqrt{2}}\right) \left(\frac{1+\sqrt{-51+30\sqrt{3}}}{2}\right)^{\frac{1}{4}} \\
& \Big(\sqrt{\frac{51+30\sqrt{3}-3\sqrt{561+330\sqrt{3}}}{2}} \\
& - \sqrt{\frac{49+30\sqrt{3}-3\sqrt{561+330\sqrt{3}}}{2}}\Big)^{\frac{1}{4}},
\end{aligned} \tag{6.8}
$$

$$
\begin{aligned}
g_{156} =& 2^{\frac{3}{4}} \left(\frac{\sqrt{13}+3}{2}\right)^{\frac{1}{3}} \left(\sqrt{\frac{5+\sqrt{13}}{8}} + \sqrt{\frac{-3+\sqrt{13}}{8}}\right)^{2} \left(\frac{2\sqrt{2}+\sqrt{-3+3\sqrt{13}}}{4\sqrt{2}}\right)^{\frac{1}{4}} \\
& \Big(\sqrt{\frac{348+96\sqrt{13}-9\sqrt{2906+806\sqrt{13}}}{8}} \\
& + \sqrt{\frac{340+96\sqrt{13}-9\sqrt{2906+806\sqrt{13}}}{8}}\Big)^{\frac{1}{4}},
\end{aligned} \tag{6.9}
$$

$$
\begin{aligned}
g_{\frac{52}{3}} =& 2^{\frac{3}{4}} \left(\frac{\sqrt{13}+3}{2}\right)^{\frac{1}{3}} \left(\sqrt{\frac{5+\sqrt{13}}{8}} - \sqrt{\frac{-3+\sqrt{13}}{8}}\right)^{2} \left(\frac{2\sqrt{2}+\sqrt{-3+3\sqrt{13}}}{4\sqrt{2}}\right)^{\frac{1}{4}} \\
& \Big(\sqrt{\frac{348+96\sqrt{13}-9\sqrt{2906+806\sqrt{13}}}{8}} \\
& - \sqrt{\frac{340+96\sqrt{13}-9\sqrt{2906+806\sqrt{13}}}{8}}\Big)^{\frac{1}{4}},
\end{aligned} \tag{6.10}
$$

$$
\begin{aligned}
g_{168} =& 2^{\frac{1}{4}} (2\sqrt{2}+\sqrt{7})^{\frac{1}{3}} \left(\frac{\sqrt{7}+\sqrt{3}}{2}\right) \left(\frac{1-4\sqrt{3}+3\sqrt{7}}{2}\right)^{\frac{1}{4}} \\
& \Big(\sqrt{\frac{-111+60\sqrt{3}+45\sqrt{7}-24\sqrt{21}}{2}} \\
& - \sqrt{\frac{-113+60\sqrt{3}+45\sqrt{7}-24\sqrt{21}}{2}}\Big)^{\frac{1}{4}},
\end{aligned} \tag{6.11}
$$

$$g_{\frac{56}{3}} = 2^{\frac{1}{4}}(2\sqrt{2}-\sqrt{7})^{\frac{1}{3}}\left(\frac{\sqrt{7}+\sqrt{3}}{2}\right)\left(\frac{1-4\sqrt{3}+3\sqrt{7}}{2}\right)^{\frac{1}{4}}$$
$$\Big(\sqrt{\frac{-111+60\sqrt{3}+45\sqrt{7}-24\sqrt{21}}{2}}$$
$$+\sqrt{\frac{-113+60\sqrt{3}+45\sqrt{7}-24\sqrt{21}}{2}}\Big)^{\frac{1}{4}}, \tag{6.12}$$

$$g_{228} = 2^{\frac{1}{4}}(2+\sqrt{3})^{\frac{1}{2}}\left(\frac{3\sqrt{19}+13}{\sqrt{2}}\right)^{\frac{1}{3}}\left(\frac{1+\sqrt{-339+78\sqrt{19}}}{2}\right)^{\frac{1}{4}}$$
$$\Big(\sqrt{\frac{339+78\sqrt{19}-45\sqrt{113+26\sqrt{19}}}{2}}$$
$$+\sqrt{\frac{337+78\sqrt{19}-45\sqrt{113+26\sqrt{19}}}{2}}\Big)^{\frac{1}{4}}, \tag{6.13}$$

$$g_{\frac{76}{3}} = 2^{\frac{1}{4}}(2-\sqrt{3})^{\frac{1}{2}}\left(\frac{3\sqrt{19}+13}{\sqrt{2}}\right)^{\frac{1}{3}}\left(\frac{1+\sqrt{-339+78\sqrt{19}}}{2}\right)^{\frac{1}{4}}$$
$$\Big(\sqrt{\frac{339+78\sqrt{19}-45\sqrt{113+26\sqrt{19}}}{2}}$$
$$-\sqrt{\frac{337+78\sqrt{19}-45\sqrt{113+26\sqrt{19}}}{2}}\Big)^{\frac{1}{4}}, \tag{6.14}$$

$$g_{312} = 2^{\frac{1}{4}}(\sqrt{26}+5)^{\frac{1}{3}}\left(\frac{\sqrt{13}+3}{2}\right)\left(\frac{1+15\sqrt{3}-4\sqrt{39}}{2}\right)^{\frac{1}{4}}$$
$$\Big(\sqrt{\frac{-1299+765\sqrt{3}+204\sqrt{39}-360\sqrt{13}}{2}}$$
$$-\sqrt{\frac{-1301+765\sqrt{3}+204\sqrt{39}-360\sqrt{13}}{2}}\Big)^{\frac{1}{4}}, \tag{6.15}$$

$$
\begin{aligned}
g_{\frac{104}{3}} =& 2^{\frac{1}{4}}(\sqrt{26}-5)^{\frac{1}{3}}\left(\frac{\sqrt{13}+3}{2}\right)\left(\frac{1+15\sqrt{3}-4\sqrt{39}}{2}\right)^{\frac{1}{4}} \\
&\Bigg(\sqrt{\frac{-1299+765\sqrt{3}+204\sqrt{39}-360\sqrt{13}}{2}} \\
&+\sqrt{\frac{-1301+765\sqrt{3}+204\sqrt{39}-360\sqrt{13}}{2}}\Bigg)^{\frac{1}{4}},
\end{aligned} \tag{6.16}
$$

$$
\begin{aligned}
g_{372} =& 2^{\frac{1}{4}}\left(\frac{39+7\sqrt{31}}{\sqrt{2}}\right)^{\frac{1}{3}}\left(\frac{\sqrt{31}+3\sqrt{3}}{2}\right)^{\frac{1}{2}}\left(\frac{1+\sqrt{-3039+546\sqrt{31}}}{2}\right)^{\frac{1}{4}} \\
&\Bigg(\sqrt{\frac{3039+546\sqrt{31}-135\sqrt{1013+182\sqrt{31}}}{2}} \\
&+\sqrt{\frac{3037+546\sqrt{31}-135\sqrt{1013+182\sqrt{31}}}{2}}\Bigg)^{\frac{1}{4}},
\end{aligned} \tag{6.17}
$$

$$
\begin{aligned}
g_{\frac{124}{3}} =& 2^{\frac{1}{4}}\left(\frac{39+7\sqrt{31}}{\sqrt{2}}\right)^{\frac{1}{3}}\left(\frac{\sqrt{31}-3\sqrt{3}}{2}\right)^{\frac{1}{2}}\left(\frac{1+\sqrt{-3039+546\sqrt{31}}}{2}\right)^{\frac{1}{4}} \\
&\Bigg(\sqrt{\frac{3039+546\sqrt{31}-135\sqrt{1013+182\sqrt{31}}}{2}} \\
&-\sqrt{\frac{3037+546\sqrt{31}-135\sqrt{1013+182\sqrt{31}}}{2}}\Bigg)^{\frac{1}{4}},
\end{aligned} \tag{6.18}
$$

$$
\begin{aligned}
g_{408} =& 2^{\frac{1}{4}}(\sqrt{2}+1)\left(3\sqrt{2}+\sqrt{17}\right)^{\frac{2}{3}}\left(\frac{1+7\sqrt{51}-20\sqrt{6}}{2}\right)^{\frac{1}{4}} \\
&\Bigg(\sqrt{\frac{-4899-840\sqrt{34}+693\sqrt{51}+1980\sqrt{6}}{2}} \\
&-\sqrt{\frac{-4901-840\sqrt{34}+693\sqrt{51}+1980\sqrt{6}}{2}}\Bigg)^{\frac{1}{4}}
\end{aligned} \tag{6.19}
$$

and

$$
\begin{aligned}
g_{\frac{136}{3}} =& 2^{\frac{1}{4}}(\sqrt{2}-1)\left(3\sqrt{2}+\sqrt{17}\right)^{\frac{2}{3}}\left(\frac{1+7\sqrt{51}-20\sqrt{6}}{2}\right)^{\frac{1}{4}} \\
&\Bigg(\sqrt{\frac{-4899-840\sqrt{34}+693\sqrt{51}+1980\sqrt{6}}{2}} \\
&+\sqrt{\frac{-4901-840\sqrt{34}+693\sqrt{51}+1980\sqrt{6}}{2}}\Bigg)^{\frac{1}{4}}.
\end{aligned}
\tag{6.20}
$$

7. Explicit Evaluations of $\alpha_n := \alpha(e^{-\pi\sqrt{n}})$

In this section, we establish the following explicit evaluations of the singular moduli α_n using the values of the class invariants obtained in Theorems 4.1, 4.2 and 6.1 in Equation (3.4).

Theorem 7.1. *We have*

$$
\alpha_{15} = 2^{-4}\left(\frac{\sqrt{5}-1}{2}\right)^{8}\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^{4}\left(\sqrt{\frac{30-15\sqrt{3}}{4}}-\sqrt{\frac{26-15\sqrt{3}}{4}}\right)^{2}, \tag{7.1}
$$

$$
\alpha_{\frac{5}{3}} = 2^{-4}\left(\frac{\sqrt{5}+1}{2}\right)^{8}\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^{4}\left(\sqrt{\frac{30-15\sqrt{3}}{4}}+\sqrt{\frac{26-15\sqrt{3}}{4}}\right)^{2}, \tag{7.2}
$$

$$
\begin{aligned}
\alpha_{21} =& \left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^{6}\left(\frac{3-\sqrt{7}}{\sqrt{2}}\right)^{4}\left(\frac{1-\sqrt{-15+6\sqrt{7}}}{2}\right)^{2}\left(8+3\sqrt{7}\right)^{2} \\
&\left(\sqrt{\frac{15+6\sqrt{7}-9\sqrt{5+2\sqrt{7}}}{2}}-\sqrt{\frac{13+6\sqrt{7}-9\sqrt{5+2\sqrt{7}}}{2}}\right)^{2},
\end{aligned}
\tag{7.3}
$$

$$
\begin{aligned}
\alpha_{\frac{7}{3}} =& \left(\frac{\sqrt{7}+\sqrt{3}}{2}\right)^{6}\left(\frac{3-\sqrt{7}}{\sqrt{2}}\right)^{4}\left(\frac{1-\sqrt{-15+6\sqrt{7}}}{2}\right)^{2}\left(8+3\sqrt{7}\right)^{2} \\
&\left(\sqrt{\frac{15+6\sqrt{7}-9\sqrt{5+2\sqrt{7}}}{2}}+\sqrt{\frac{13+6\sqrt{7}-9\sqrt{5+2\sqrt{7}}}{2}}\right)^{2},
\end{aligned}
\tag{7.4}
$$

$$\alpha_{30} = \left(\sqrt{5}-2\right)^4 \left(\sqrt{10}-3\right)^4 \left(\frac{-1+\sqrt{15}+2\sqrt{6}}{\sqrt{2}}\right)^4 \left(4-\sqrt{15}\right)^4$$
$$\Big(\sqrt{\frac{-39-12\sqrt{10}+18\sqrt{6}+9\sqrt{15}}{2}} + \sqrt{\frac{-41-12\sqrt{10}+18\sqrt{6}+9\sqrt{15}}{2}}\Big)^4, \tag{7.5}$$

$$\alpha_{\frac{10}{3}} = \left(\sqrt{5}+2\right)^4 \left(\sqrt{10}-3\right)^4 \left(\frac{-1+\sqrt{15}+2\sqrt{6}}{\sqrt{2}}\right)^4 \left(4-\sqrt{15}\right)^4$$
$$\Big(\sqrt{\frac{-39-12\sqrt{10}+18\sqrt{6}+9\sqrt{15}}{2}} - \sqrt{\frac{-41-12\sqrt{10}+18\sqrt{6}+9\sqrt{15}}{2}}\Big)^4, \tag{7.6}$$

$$\alpha_{33} = \left(\frac{\sqrt{11}-3}{\sqrt{2}}\right)^4 \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^{12} \left(\frac{1-\sqrt{-51+30\sqrt{3}}}{2}\right)^2 \left(26+15\sqrt{3}\right)^2$$
$$\Big(\sqrt{\frac{51+30\sqrt{3}-3\sqrt{561+330\sqrt{3}}}{2}}$$
$$-\sqrt{\frac{49+30\sqrt{3}-3\sqrt{561+330\sqrt{3}}}{2}}\Big)^2, \tag{7.7}$$

$$\alpha_{\frac{11}{3}} = \left(\frac{\sqrt{11}+3}{\sqrt{2}}\right)^4 \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^{12} \left(\frac{1-\sqrt{-51+30\sqrt{3}}}{2}\right)^2 \left(26+15\sqrt{3}\right)^2$$
$$\Big(\sqrt{\frac{51+30\sqrt{3}-3\sqrt{561+330\sqrt{3}}}{2}}$$
$$+\sqrt{\frac{49+30\sqrt{3}-3\sqrt{561+330\sqrt{3}}}{2}}\Big)^2, \tag{7.8}$$

$$\alpha_{39} = \left(\frac{\sqrt{13}-3}{2}\right)^4 \left(\sqrt{\frac{5+\sqrt{13}}{8}} - \sqrt{\frac{-3+\sqrt{13}}{8}}\right)^{24}$$
$$\left(\frac{11+3\sqrt{13}}{2}\right)^2 \left(\frac{2\sqrt{2}-\sqrt{-3+3\sqrt{13}}}{4\sqrt{2}}\right)^2$$
$$\Big(\sqrt{\frac{348+96\sqrt{13}-9\sqrt{2906+806\sqrt{13}}}{8}}$$
$$-\sqrt{\frac{340+96\sqrt{13}-9\sqrt{2906+806\sqrt{13}}}{8}}\Big)^2, \tag{7.9}$$

$$\alpha_{\frac{13}{3}} = \left(\frac{\sqrt{13}-3}{2}\right)^4 \left(\sqrt{\frac{5+\sqrt{13}}{8}} + \sqrt{\frac{-3+\sqrt{13}}{8}}\right)^{24}$$
$$\left(\frac{11+3\sqrt{13}}{2}\right)^2 \left(\frac{2\sqrt{2}-\sqrt{-3+3\sqrt{13}}}{4\sqrt{2}}\right)^2$$
$$\Big(\sqrt{\frac{348+96\sqrt{13}-9\sqrt{2906+806\sqrt{13}}}{8}}$$
$$+\sqrt{\frac{340+96\sqrt{13}-9\sqrt{2906+806\sqrt{13}}}{8}}\Big)^2, \tag{7.10}$$

$$\alpha_{42} = \left(2\sqrt{2}-\sqrt{7}\right)^4 \left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^{12} \left(\frac{1+4\sqrt{3}+3\sqrt{7}}{\sqrt{2}}\right)^4 \left(8-3\sqrt{7}\right)^4$$
$$\Big(\sqrt{\frac{-111+60\sqrt{3}+45\sqrt{7}-24\sqrt{21}}{2}}$$
$$+\sqrt{\frac{-113+60\sqrt{3}+45\sqrt{7}-24\sqrt{21}}{2}}\Big)^4, \tag{7.11}$$

$$\alpha_{\frac{14}{3}} = \left(2\sqrt{2}+\sqrt{7}\right)^4 \left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^{12} \left(\frac{1+4\sqrt{3}+3\sqrt{7}}{\sqrt{2}}\right)^4 \left(8-3\sqrt{7}\right)^4$$
$$\Big(\sqrt{\frac{-111+60\sqrt{3}+45\sqrt{7}-24\sqrt{21}}{2}}$$
$$-\sqrt{\frac{-113+60\sqrt{3}+45\sqrt{7}-24\sqrt{21}}{2}}\Big)^4, \tag{7.12}$$

$$\alpha_{57} = \left(2-\sqrt{3}\right)^6 \left(\frac{3\sqrt{19}-13}{\sqrt{2}}\right)^4 \left(\frac{1-\sqrt{-339+78\sqrt{19}}}{2}\right)^2 \left(170+39\sqrt{19}\right)^2$$
$$\Big(\sqrt{\frac{339+78\sqrt{19}-45\sqrt{113+26\sqrt{19}}}{2}}$$
$$-\sqrt{\frac{337+78\sqrt{19}-45\sqrt{113+26\sqrt{19}}}{2}}\Big)^2, \tag{7.13}$$

$$\alpha_{\frac{19}{3}} = \left(2+\sqrt{3}\right)^6 \left(\frac{3\sqrt{19}-13}{\sqrt{2}}\right)^4 \left(\frac{1-\sqrt{-339+78\sqrt{19}}}{2}\right)^2 \left(170+39\sqrt{19}\right)^2 \left(\sqrt{\frac{339+78\sqrt{19}-45\sqrt{113+26\sqrt{19}}}{2}} + \sqrt{\frac{337+78\sqrt{19}-45\sqrt{113+26\sqrt{19}}}{2}}\right)^2, \tag{7.14}$$

$$\alpha_{78} = \left(\frac{\sqrt{13}-3}{2}\right)^{12} \left(\sqrt{26}-5\right)^4 \left(\frac{-1+15\sqrt{3}+4\sqrt{39}}{\sqrt{2}}\right)^4 \left(25-4\sqrt{39}\right)^4 \left(\sqrt{\frac{-1299+765\sqrt{3}+204\sqrt{39}-360\sqrt{13}}{2}} + \sqrt{\frac{-1301+765\sqrt{3}+204\sqrt{39}-360\sqrt{13}}{2}}\right)^4, \tag{7.15}$$

$$\alpha_{\frac{26}{3}} = \left(\frac{\sqrt{13}-3}{2}\right)^{12} \left(\sqrt{26}+5\right)^4 \left(\frac{-1+15\sqrt{3}+4\sqrt{39}}{\sqrt{2}}\right)^4 \left(25-4\sqrt{39}\right)^4 \left(\sqrt{\frac{-1299+765\sqrt{3}+204\sqrt{39}-360\sqrt{13}}{2}} - \sqrt{\frac{-1301+765\sqrt{3}+204\sqrt{39}-360\sqrt{13}}{2}}\right)^4, \tag{7.16}$$

$$\alpha_{93} = \left(\frac{39-7\sqrt{31}}{\sqrt{2}}\right)^4 \left(\frac{\sqrt{31}-3\sqrt{3}}{2}\right)^6 \left(\frac{1-\sqrt{-3039+546\sqrt{31}}}{2}\right)^2 \left(1520+273\sqrt{31}\right)^2 \left(\sqrt{\frac{3039+546\sqrt{31}-135\sqrt{1013+182\sqrt{31}}}{2}} - \sqrt{\frac{3037+546\sqrt{31}-135\sqrt{1013+182\sqrt{31}}}{2}}\right)^2, \tag{7.17}$$

$$\begin{aligned}\alpha_{\frac{31}{3}} = {} & \left(\frac{39-7\sqrt{31}}{\sqrt{2}}\right)^4 \left(\frac{\sqrt{31}+3\sqrt{3}}{2}\right)^6 \\ & \left(\frac{1-\sqrt{-3039+546\sqrt{31}}}{2}\right)^2 \left(1520+273\sqrt{31}\right)^2 \\ & \Bigg(\sqrt{\frac{3039+546\sqrt{31}-135\sqrt{1013+182\sqrt{31}}}{2}} \\ & +\sqrt{\frac{3037+546\sqrt{31}-135\sqrt{1013+182\sqrt{31}}}{2}}\Bigg)^2, \end{aligned} \tag{7.18}$$

$$\begin{aligned}\alpha_{102} = {} & \left(\sqrt{2}-1\right)^{12} \left(3\sqrt{2}-\sqrt{17}\right)^8 \left(\frac{1+7\sqrt{51}+20\sqrt{6}}{\sqrt{2}}\right)^4 \left(50-7\sqrt{51}\right)^4 \\ & \Bigg(\sqrt{\frac{-4899-840\sqrt{34}+693\sqrt{51}+1980\sqrt{6}}{2}} \\ & +\sqrt{\frac{-4901-840\sqrt{34}+693\sqrt{51}+1980\sqrt{6}}{2}}\Bigg)^4 \end{aligned} \tag{7.19}$$

and

$$\begin{aligned}\alpha_{\frac{34}{3}} = {} & \left(\sqrt{2}+1\right)^{12} \left(3\sqrt{2}-\sqrt{17}\right)^8 \left(\frac{1+7\sqrt{51}+20\sqrt{6}}{\sqrt{2}}\right)^4 \left(50-7\sqrt{51}\right)^4 \\ & \Bigg(\sqrt{\frac{-4899-840\sqrt{34}+693\sqrt{51}+1980\sqrt{6}}{2}} \\ & -\sqrt{\frac{-4901-840\sqrt{34}+693\sqrt{51}+1980\sqrt{6}}{2}}\Bigg)^4. \end{aligned} \tag{7.20}$$

Remark. The identities (6.1), (6.3), (6.5), (6.7), (6.11), (6.15) and (6.19) are recorded by Ramanujan in his notebooks [18, pp. 80, 214, 288, 289, 310, 312, 313, 345, 346]. The different proofs of these identities can also be found in [16], [6, pp. 139, 151] and [4, pp. 281, 282, 291, 292].

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